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On the Nucleonic Components producing Large Cosmic-Ray Bursts under Thick Shield

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(Received November 20, 1951)

By analysing the size-frequency distributions of large cosmic ray bursts obtained at three stations by SRI ion chambers, it is concluded that: a) about 40% of the total bursts at sea level are those induced by N-rays, and the portion of them decreases with increasing size, b) absorption mean free path of burst-producing N-rays is 120 g/cm^2 in air, and decreases with increasing size, approaching to collision mean free path of air. Moreover, from the comparison of our results with others, it is shown that the absorption mean free path of burst-producing N-rays near sea level is not same as that at high altitude, but it elongates near sea level. Some discussions on burst production are presented in connection with our results.

§ 1. Introduction

The large cosmic ray bursts under thick shield have been studied by many investigators with ion chambers of various geometries. Especially, since the importance of the study on large cosmic ray bursts was pointed out by Christy and Kusaka¹⁾ in connection with meson bremsung, Lapp²⁾ elucidated much of the nature of cosmic ray bursts.

However, recent studies on nucleonic components in cosmic ray particles, observed in nuclear emulsions and cloud chambers, made it clear that some of the dense electron cascade showers are initiated by nuclear events, and Rossi et al³⁾ clarified that some of the bursts observed at high altitude are initiated by the passage of electrons induced by nuclear events.

Concerning the nucleonic origin of large cosmic ray bursts, Hayakawa and Fujimoto⁴⁾ suggested theoretically that most of them at mountain altitude are initiated by nuclear events, and they succeeded qualitatively in the explanation of the variation of the ratio of burst frequencies at mountain to those at sea level with burst size. Thereby, as other investigators, they assumed that all bursts at sea level are meson induced bursts (μ -bursts), and compared their results derived theoretically with experimental data.

However, it must be noted that there has been no proof that the bursts at sea level are all initiated by μ -mesons. Thus, when comparing the experimental data with the theoretical results, it is so important to know whether at sea level some bursts produced by nucleonic components (N-bursts) exist or not, and to know the absorption mean free pathes (hereafter denoted by absorption MFP for abbreviation) of the burst-producing rays in air and other materials, if there are some N-bursts.

From the viewpoint above mentioned, we compared the size-frequency curves of bursts obtained at three places, that is, at mountain altitude (2930 m elevation), at sea level, and at basement (under 170 g/cm^2 concrete at sea level). The ion chambers used are the

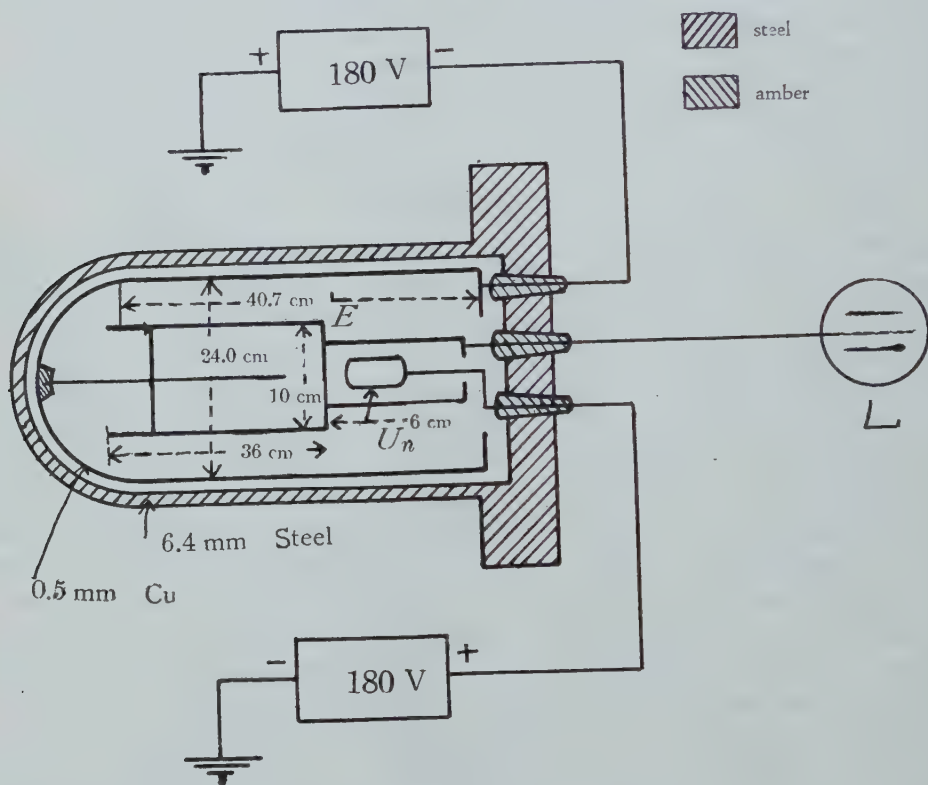
same one at sea level and at basement, and that used at mountain is the same geometrical size as the above one, though the pressure of gas filled in the chamber is lower than that of the former one. Thus, after carrying out some corrections, the comparison of the data at three different places became sufficiently meaningful and reliable, and made it possible to deduce some informations on the nature of burst production.

§ 2. Apparatus and experimental data

a) Apparatus

The ion chambers used were made at Scientific Research Institute in order to measure the cosmic ray intensities continuously, and details of the structure and characteristics of them were reported by Dr. Ishii⁽⁵⁾.

Fig. 1.



Structure of SRI ion chamber

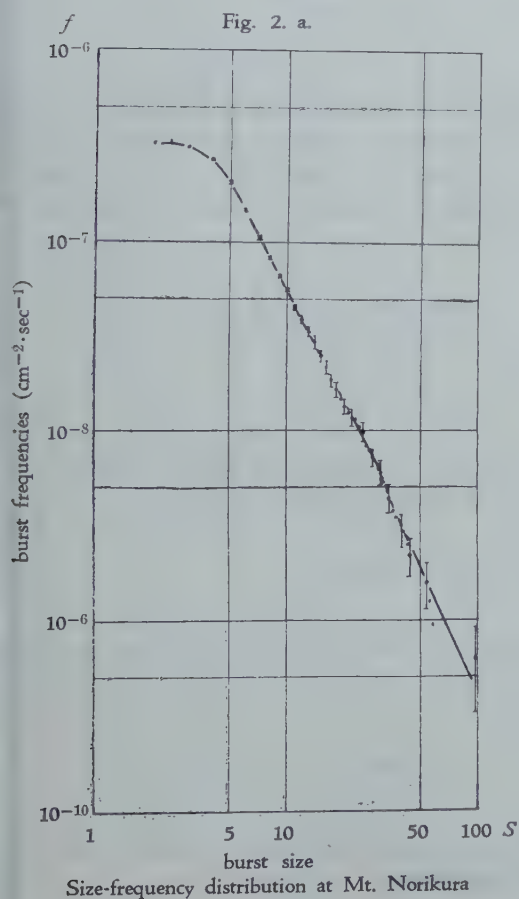
As illustrated in Fig. 1. the chamber is a cylindrical vessel made of steel and uran compensation type. Its effective volume and horizontal cross-sectional area are 21.5 litres and 1120 cm^2 respectively, and details of the geometrical sizes are shown in the figure. The chamber is filled with argon of 95% purity to a pressure of about 40 atm. usually. The inner electrode E (which consists of five rods) is connected to Lindemann electrometer L , and the deflection of its needle is photographed continuously on the film by

camera system. In operation, the potential difference of 180 volts is given between electrodes, and earth contact of the inner electrode and calibration of sensitivity of the electrometer are automatically photographed periodically. All measurements from which the following data of bursts were obtained were carried out with 10 cm thickness of lead shield surrounding all sides of the chamber.

b). Experimental data

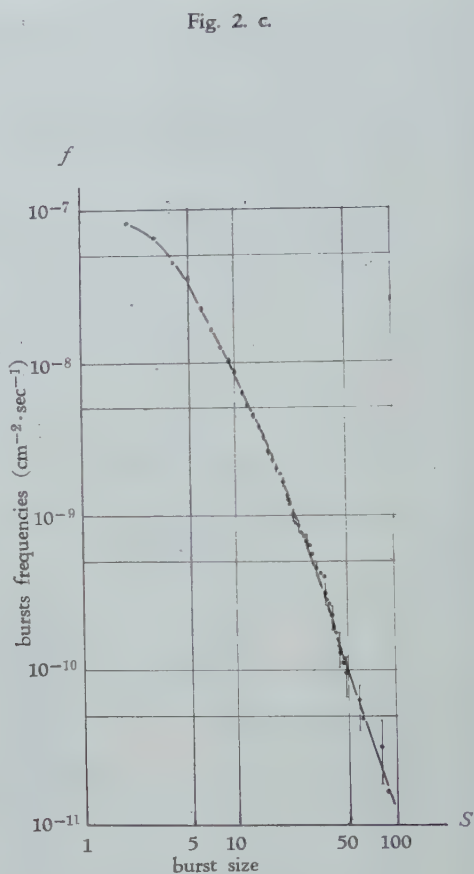
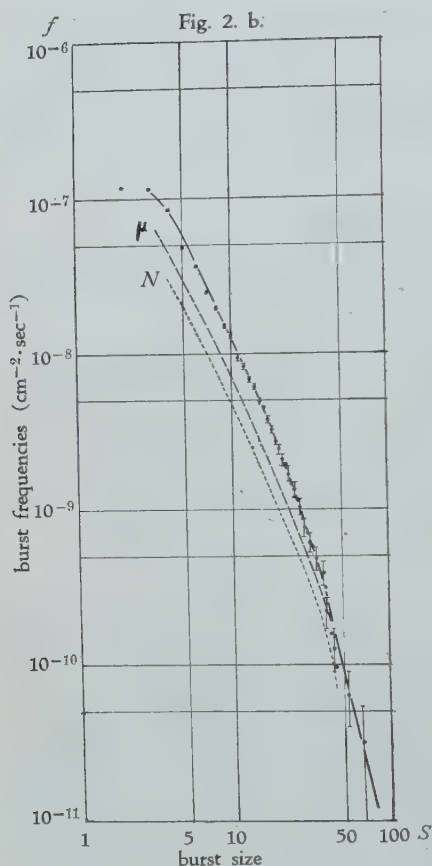
The bursts reported here are picked up from the recording films obtained in the continuous observation of cosmic ray intensities. The details of measurements are summarized in Table I.

Table I.						
Station	Altitude	Atm. depth	Apparatus	Gas press. (at 0°C)	Duration	Total time
Mt. Norikura	2830 m	750 g/cm ²	SIR ion chamber No. 4	38.5 kg/cm ²	Sept. 1950	792 hr
Mamiana, Tokyo	40 m	1030 g/cm ²	" No. 1	39.5 kg/cm ²	May 1942– Sept. 1943	7779 hr
Komagome, Tokyo	Basement	1030 g/cm ² + 170 g/cm ² concrete	" No. 1	39.5 kg/cm ²	Mar. 1939– Apr. 1941	15733 hr



The size of the bursts, tabulated in the tables left on record, was again checked by one of the authors (W), and the films recorded at Mt. Norikura were read by him also. Corrections for the data at mountain due to the usage of different apparatus were strictly carried out following the report (5), in order to compare it directly with those at sea level and at the basement.

The integral size-frequency distributions of bursts thus obtained are shown in Fig. 2a, b, and c. In the figures, burst size is represented in terms of the deflection of electrometer needle on the film in 1/10 mm at normal condition, and size 1 corresponds to 3.34×10^6 ion pairs. The frequencies of the bursts are indicated in the number of bursts divided by the time of total duration and by the horizontal cross-sectional area of the chamber. The materials above the apparatus are thin wooden roof at Mt. Norikura and at sea level, and those at the basement are four-storied concrete



ceiling, whose thickness is 60 cm of concrete (density 2.4) and 24 cm mortar (density 1.5 \sim 1.6), being estimated as 170 g/cm² in total thickness in vertical direction.

§ 3. Discussion of the results

a) Intensities of total ionization

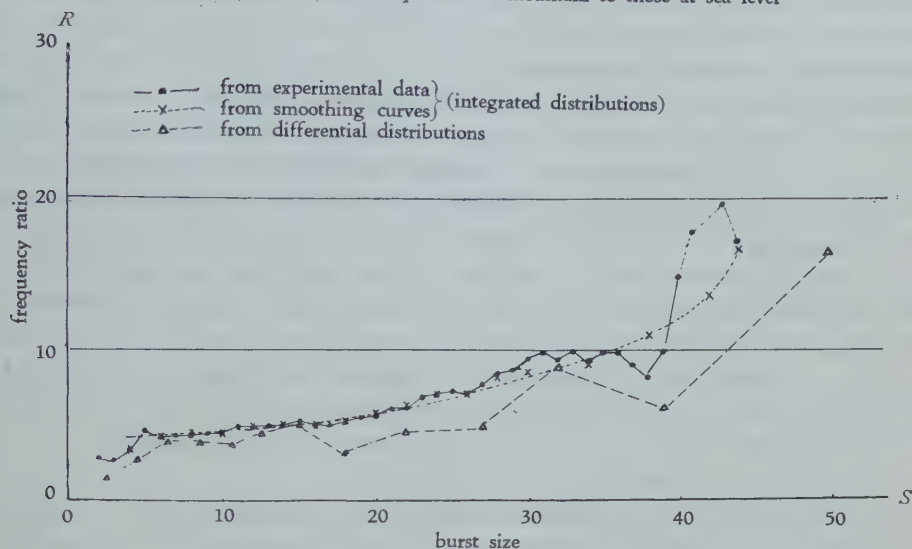
The intensities of total ionizations measured by the above mentioned ion chambers are tabulated in Table II.

Table II.

Station	Atm. depth	Duration	Intensities
Mt. Norikura	750 g/cm ²	Sept. 1950	1.79*
Mamiana	1030 g/cm ²	Nov. 1942	1.00
Komagome	1030 g/cm ² + 170 g/cm ²	Jan.-Dec. 1941	0.74

* corrected for the apparatus.

Fig. 3. Ratio of burst frequencies at mountain to those at sea level

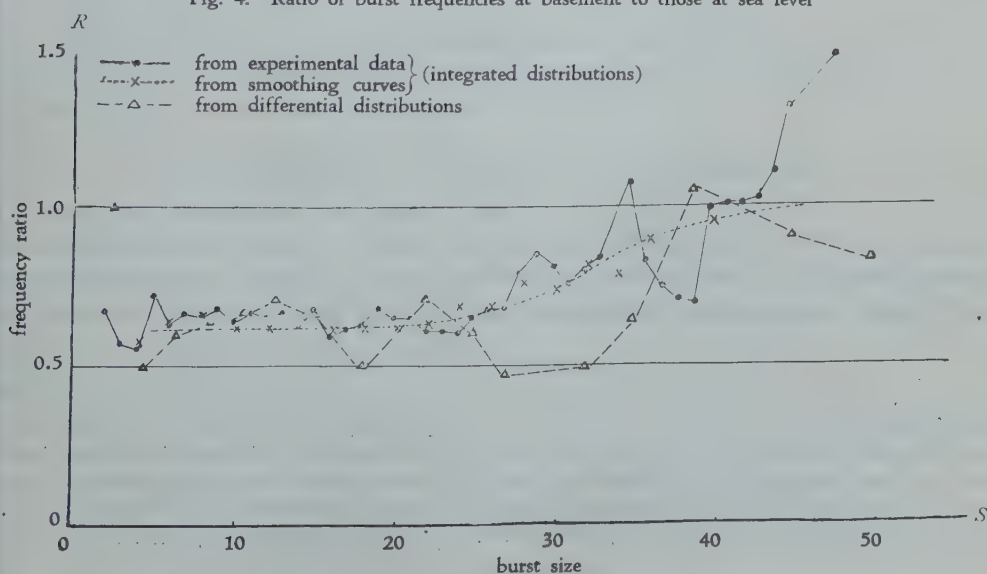


b) Frequency ratio of the bursts

Frequency ratio of the bursts at mountain to those at sea level against burst size, is shown in Fig. 3. in which the solid line was obtained from Fig. 2. directly, the dotted line from the smoothing curve in Fig. 2. and the broken line from the differential size-frequency distributions (averaged over several sizes to reduce fluctuation). Similarly, Fig. 4. shows the frequency ratio of the bursts at basement and at sea level.

From Fig. 4. it will be easily noticed that the decrease of burst frequencies at smaller

Fig. 4. Ratio of burst frequencies at basement to those at sea level



size is more remarkable than the total ionizations and that the frequency ratio gradually increases from 0.65 to roughly 1 with increasing burst size.

The former fact can not be explained if we assume that all of the bursts at sea level are μ -bursts, because the mesons capable of producing the bursts have fairly high energies and are less absorbable than the total components. However, if we assume that the burst-producing rays at sea level are mixture of mesons and N -rays,* the decrease of the burst frequencies at basement will be easily understood. The latter fact that the ratio increases with increasing size and at larger size it approaches to roughly 1, represents that the portion of μ -bursts at sea level gradually increases with increasing size, and that μ -mesons are predominant components which produce very large bursts at sea level and at basement.

By comparing the second fact with Fig. 3. showing that the ratio of burst frequencies at mountain to those at sea level increases with increasing burst size, it may be concluded that the burst-producing N -rays have not the same absorption MFP in air for bursts of all size. That is, N -rays producing small bursts have longer absorption MFP, and those of large bursts have much shorter MFP.

c) Tentative deduction of the absorption MFP of burst-producing rays

According to the studies on penetrating showers, the absorption MFP of the particles producing penetrating showers are $120 \sim 125 \text{ g/cm}^2$ in air (6), that is, twice of the collision MFP deduced from the geometrical cross section of air nucleus. On the other hand, from the results obtained by nuclear emulsion (7), it is known that the absorption MFP of star-producing rays in aluminium is similarly about twice of the collision MFP in it. Referring to these results, we tentatively assumed that the absorption MFP of N -rays producing bursts of size 4** is 120 g/cm^2 in air and 150 g/cm^2 in concrete. Next, designating the fraction of μ -bursts and N -bursts at sea level as μ and N respectively, we assume the following equations,

$$\begin{aligned} \text{at sea level} \quad & 1 = \mu + N, \\ \text{at mountain altitude} \quad & R = a\mu + N \exp\left(-\frac{280 \cdot \epsilon}{120}\right), \\ \text{at basement} \quad & R' = a'\mu + N \exp\left(-\frac{170 \cdot \epsilon}{150}\right) \end{aligned} \quad (1)$$

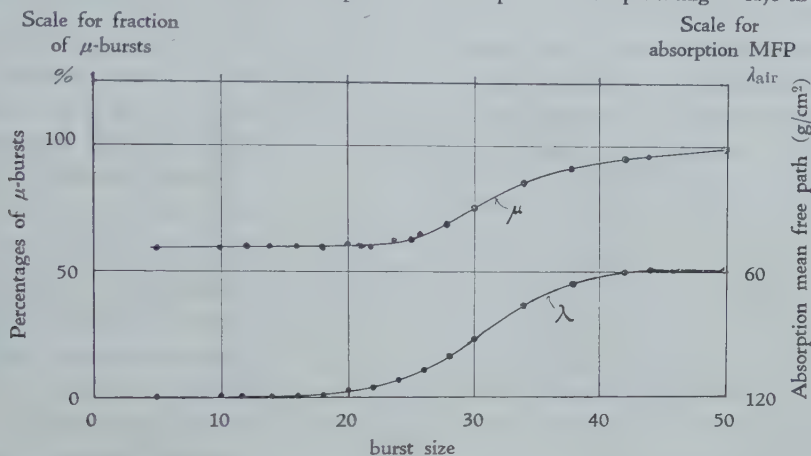
where R and R' are ratio of burst frequencies at mountain and at basement to those at sea level, and they are determined from the dotted curves of Fig. 3 and 4 respectively. a and a' are the factors relating to ionization loss and decay loss of μ -mesons, and are func-

* Of course, though we must not exclude the possible contribution of high energy electronic components which impinge upon the apparatus, the electrons capable of producing such large bursts as containing several hundred ionizing rays under more than 20 radiation units will be very rare, and if they exist, we may include them in N -rays because they are considered to be produced by nuclear events near the apparatus or contained in air showers, which we regard as the same as N -rays for the burst production.

** Bursts smaller than size 4 are excluded for their less reliability, because of possible introduction of errors when distinguishing the true bursts from the fluctuation of the total ionizations.

tions of their energies or burst size. The value of α lies between 1.07 (for size 4)* and about 1.0 (for size 40 or more), and that of α' is estimated to be between 0.92 (for size 4) and about 1.0 (for size 40 or more)**, taking into account the structure of building. ϵ is a constant indicating the rate of decrease of adsorption MFP, and depends on the burst size. Assuming that ϵ takes the same value in concrete as in air at a given size***, these equations are solved with respect to μ , N , and ϵ .

Fig. 5. Fraction of μ -bursts, and absorption mean free path of burst-producing N -rays in air



The results thus obtained are shown in Fig. 5, and each portion of μ -bursts and N -bursts at sea level is shown in Fig. 2 b.

From the Figs., the followings are found ;

- I) At sea level, $40 \pm 3\%$ of bursts of size 4 are N -bursts, and with increasing burst size they diminish gradually.
- II) Absorption MFP of the burst-producing N -rays is about 120 g/cm^2 for bursts of size 4, decrease with increasing size, and at the greatest burst (of size 40) approach to the collision MFP.

§ 4. Comparison of the results obtained with others

a) Burst frequencies under the concrete roof of 75 g/cm^2 thickness

According to the burst data obtained by SRI ion chamber No. 5 which is now in operation at Nishina Laboratory of Scientific Research Institute in Tokyo, the burst frequencies under the concrete roof of 75 g/cm^2 thickness are 0.82 of those at sea level for the bursts of size 4. This value is in good agreement with that estimated from the third equation of (1), 0.83, where $\alpha' = 0.98$, $\epsilon = 1$.

* As we are comparing the integral size-frequency distributions each other, bursts of size 4 mean all bursts of size greater than 4. This terminology will be used throughout this report.

** Estimation of minimum energies corresponding to each size will be shown later.

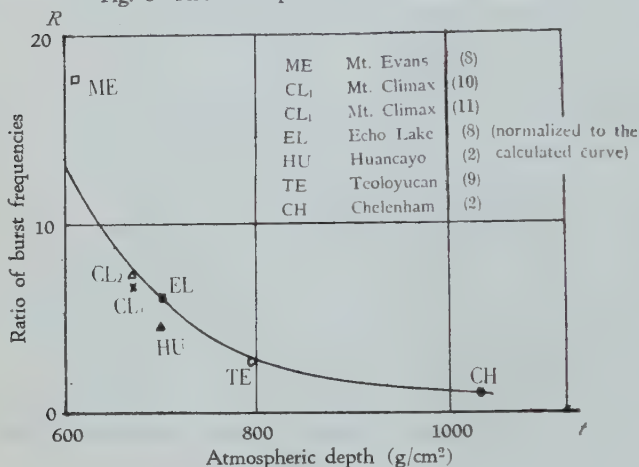
*** Taking account of the small difference of atomic number between air and concrete, this assumption may not be so far from reality.

b) Bursts by Model C Meter

Concerning the burst production by Model C Meter, many valuable results have hitherto been reported. However, some discrepancies are found between them in altitude dependence, perhaps, resulted from statistical poorness and neglect of the materials above the apparatus*. Therefore, in so far as the conditions in each experiment are not known precisely, comparison of each result is difficult and dangerous. Nevertheless, we may obtain some informations about the altitude dependence and the nucleonic origin of bursts by comparing their results each other.

As the burst frequencies at sea level for Model C Meter, we cited those at Cheltenham reported by Lapp²⁾, and compared them with the frequencies at several altitudes, referring to the papers reported by Benett et al⁸⁾, Schein and Gill⁹⁾, Lapp²⁾, Fahy and Schein¹⁰⁾ and Stinchcomb¹¹⁾.

Fig. 6 Altitude dependence for Model C Meter



The result is illustrated in Fig. 6. On the other hand, we roughly estimated the ratio of N -bursts to μ -bursts at sea level to be 0.61 ± 0.06 or 38 : 62 for Model C Meter, taking into account the thickness of the lead shield**** and assuming for the absorption MFP of burst-producing N -rays to be 350 g/cm^2 in lead*****. The full line in Fig. 6 is the estimated altitude dependence thus obtained. From the Fig., the followings are shown ;

* Unfortunately, we could not know the materials above the apparatus at each observatory or station cited here. As the materials above the apparatus affect seriously the frequencies of burst, some descriptions reported here may be somewhat altered.

** Here we do not cite the data at Chicago and Denver because of the statistical poorness.

*** The data at Teoloyucan are cited.

**** Mean thickness of the lead shield was computed from the following expression (Fig. 7):

$$t = \frac{2}{3r^2} \cdot (l^3 - (l^2 - r^2)^{\frac{3}{2}}) - \frac{2}{3}r \quad (\text{A})$$

In our chambers, the mean thickness of the lead shield was estimated to be 11.5 cm, taking accounts of the shape of the shield and the zenith angle distribution of N -rays deduced from absorption MFP.

***** Absorption MFP of 350 g/cm^2 is deduced from the experimental data of Schein and Fahy¹⁰⁾, taking the contribution of μ -bursts into consideration.

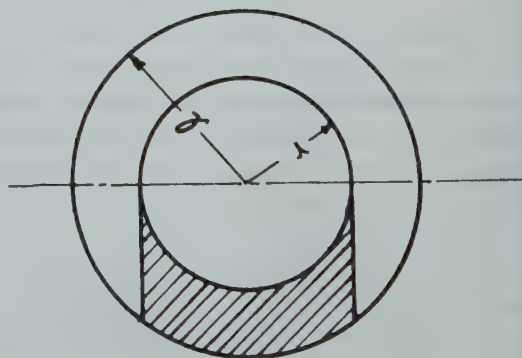


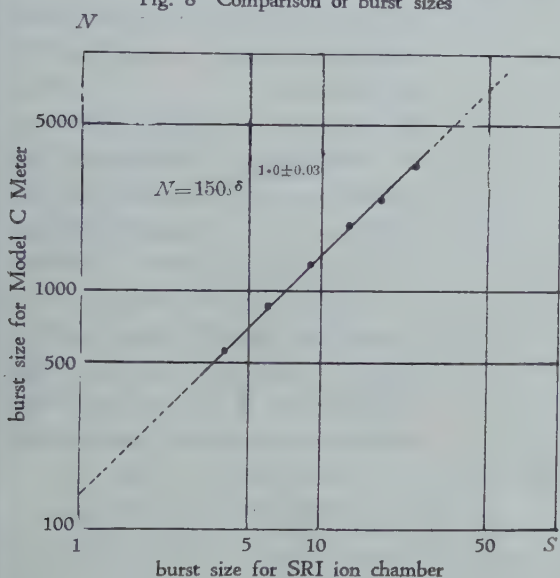
Fig. 7.

- I) Estimation of N -bursts as 38% at sea level are not in so far from reality.
- II) The frequencies at Huancayo reported by Lapp are appreciably deviated from the estimated one. Considering the lack of latitude effect of such high energy N -rays as capable of producing bursts, we can not understand why such discrepancy exists.*
- III) At higher elevation, the observed ratio is appreciably higher than the estimated one. Concerning this point, we shall discuss in § 5.

c) Recombination

To study the correspondence between the bursts measured by Model C Meter and those by our apparatus, the frequencies of N -bursts at sea level in Fig. 2b were reduced a factor corresponding to an absorption in 13.7 cm lead referring to Fig. 5, that is, the

Fig. 8 Comparison of burst sizes



ratio of N -bursts to μ -bursts was changed to be same as that by Model C Meter. Thus the relation between burst sizes of both chambers was compared at same frequencies**. The result is shown in Fig. 8. Denoting the burst sizes of our chamber and Model C Meter at the same frequencies as S and N respectively, next expression is obtained from Fig. 8.

$$N = 150 S^{\delta},$$

$$\delta = 1.0 \pm 0.03. \quad (2)$$

This means that the size-frequency distribution obtained by our apparatus is completely same as that obtained by Model C Meter, if size

1 in the former is assumed to correspond to 150 particles in the latter. This figure shows that some impurities in gas, or recombination, and difference of the shape of chamber does not affect the size-frequency distribution.

In connection with the above mentioned, we compared the size frequency distribution for Model C Meter with that for Neher's ion chamber which was obtained on the Pacific Lines during 1937 to 1939 and shown in Fig. 9***¹²⁾. Thereby, the frequencies of N -burst for Model C Meter are reduced by a factor corresponding to the difference of shield thickness (referring to Figs. 5 and 7). As illustrated in Fig. 10 the relation between both sizes is expressed as follows;

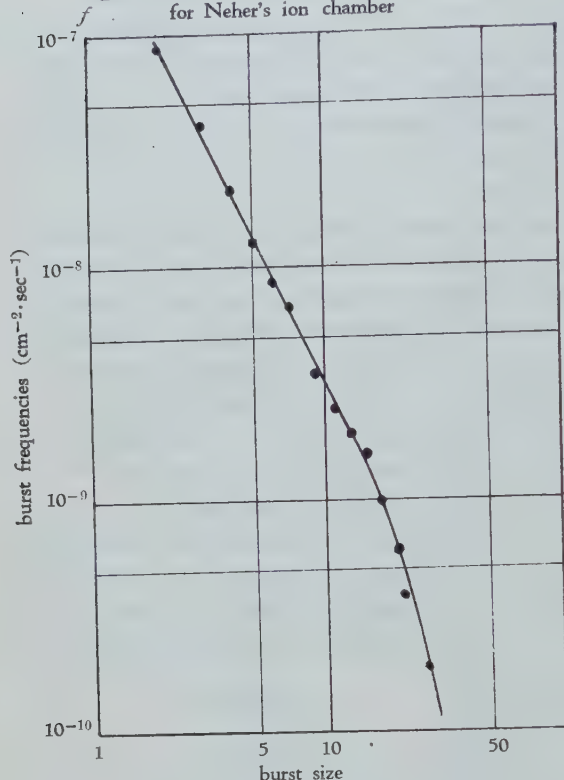
$$N = 60 S_N^{1.17 \pm 0.05} \quad (3)$$

* Here we doubt if the material above the apparatus might affect the frequencies.

** Strictly speaking, same frequencies have no definite meaning, because the cross-sectional area of our chamber depends on zenith angle and azimuth for its complicated shape.

*** Dr. Sekido kindly lent us the unpublished data.

Fig. 9 Size-frequency distribution of bursts for Neher's ion chamber



ing the current notion concerning to μ -meson and its energy spectrum at sea level deduced from the intensity-depth relation underground¹³⁾.

However, in comparing the theoretical results with the experimental data, the followings must be noted;

I) The experimental data at sea level contain some portions of N -bursts. By Model C Meter, μ -bursts are about 62% of the total bursts at small size, and at larger size their percentages increase.

II) It is necessary to recalculate the value of critical energy in lead-iron (1.2 cm thickness) transition, taking into account the existence of N -bursts.

where S_N designates the burst size for Neher's ion chamber.

Thus, if the burst frequencies f is related to the burst size for Model C Meters as

$$f = a_1 N^{-\tau} \quad (4)$$

the relation between f and S_N is described by

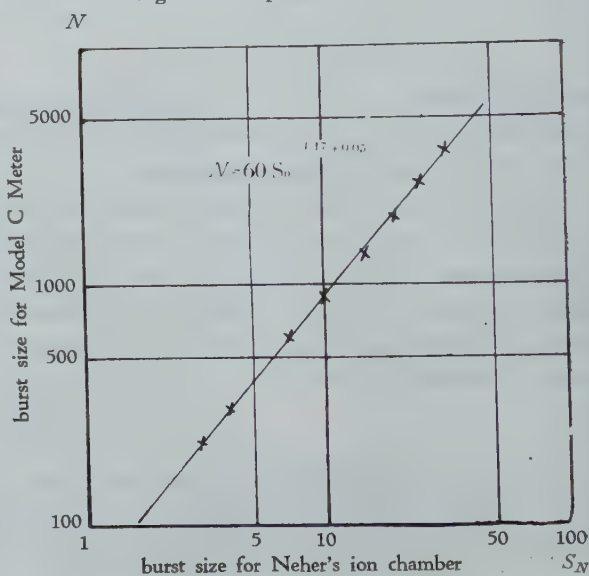
$$f = a_2 S^{-1.17\tau} \quad (4')$$

The difference of power factor may be attributed to small size of the chamber.

d) Note on μ -bursts at sea level

Size-frequency distribution of the bursts was theoretically deduced by Christy and Kusaka¹⁾, assuming that they are all induced by μ -mesons, and the result was compared with experimental data by them, and by Lapp²⁾. Thereafter, Fujimoto and Hayakawa³⁾ computed the size-frequency distribution of bursts, follow-

Fig. 10 Comparison of burst sizes



In evaluating $\beta_{\text{Pb-Fe}}$ (denoted by β_c for abbreviation hereafter) from Lasp's measurement carried out with 10.7 cm Pb and 35 cm Fe shield, we assume as follows for the sake of simplicity;

- 1) In equal number of atoms, the production ratio R of N -bursts and μ -bursts is proportional to $A^{2/3}/Z^2$, where A and Z are atomic weight and atomic number respectively.
- 2) Absorption MFP of burst-producing N -rays are 350 g/cm² and 220 g/cm² ¹⁴⁾* in lead and iron respectively.

Then, the ratio R_{Fe} of N -bursts to μ -bursts under 35 cm Fe shield is represented by

$$R_{\text{Fe}} = R_{\text{Pb}} \exp \left(-\frac{t_{\text{Fe}}}{220} + \frac{t_{\text{Pb}}}{350} \right) \left(\frac{A_{\text{Fe}}}{A_{\text{Pb}}} \right)^{2/3} \left(\frac{Z_{\text{Pb}}}{Z_{\text{Fe}}} \right)^2. \quad (5)$$

By substituting 307 g/cm², 150 g/cm², and 38/62 for t_{Fe} , t_{Pb} , and R_{Pb} respectively, we obtain 0.89 for R_{Fe} , that is, 44% of the total bursts under 35 cm Fe are initiated by N -rays.

Thus, taking the critical energy β_{Fe} of iron as 25 Mev ¹⁵⁾, β_c is determined to be 16.1 Mev, which is in good agreement with the value cited by Lapp, though he did not take into considerations the contribution of N -bursts.

Considering the above note, discrepancy between the experimental result and the theoretical one deduced by Fujimoto and Hayakawa, amounts to a factor 6.0 or more. Meanwhile, according to Nishimura and Ida ¹⁶⁾, the theoretical result should be lowered by a factor 3.5 or more by taking into account Coulomb scattering of the cascade electrons in the shield. Thus, though the discrepancy between them becomes remarkably small, the theoretical result is still greater than the experimental one by a factor about 2.

§ 5. Some difficulties in an interpretation of our results

a) Altitude dependence and barometric effect at mountain altitude

According to the result obtained by Benett et al ⁸⁾, the number of bursts larger than 2×10^7 ion pairs** increased by factor 2.9 between Echo Lake (3240 m) and Mt. Evans (4300 m). If μ -bursts are subtracted, the absorption MFP of burst-producing N -rays between both altitudes is found to be about 80 g/cm² in air. Similar result was obtained by Stinchcomb ¹¹⁾ from the barometric effect of the burst frequencies at Mt. Climax (3510 m). Therefore, we must conclude that the absorption MFP of the burst-producing N -rays in air above and near 3500 m is approximately equal to the collision MFP of air. These results immediately contradict with our result that the burst-producing N -rays are absorbed by air with an absorption MFP of 120 g/cm² at bursts of smaller size.

b) Flux of primary cosmic ray particles

An exact computation of the mean length traversed by a particle in our chamber is

* For iron, we cited the value slightly longer than that reported by Tinlot and Gregory, ¹⁴⁾ so as to be consistent with the absorption MFP in lead.

** Although the correspondence of the size of these bursts with our burst is not known, they seem to correspond to our relatively small bursts, taking into consideration their frequencies.

very difficult, because the mean length is a function of zenith angle and azimuth, and the zenith angle distribution of burst-producing rays must be taken into consideration. Therefore, we took the following procedure in order to obtain a particle number-frequency distribution at mountain;

I) From Fig. 8, the number of particles corresponding to each size of bursts for our chamber is estimated.

II) The size-frequency curve of N -bursts at 2830 m is depicted, taking number of particles as abscissa.*

Next, we assume as follows;

I) N -bursts are produced by cascade electrons initiated by π_0 -mesons which are produced by N -rays.

II) The number of particles corresponds to that at shower maximum.

III) Total energy of a nucleon is consumed by nucleonic collision, and in average $1/3$ of it is transferred to π_0 -meson.

Following these assumptions, energy E of a nucleon which produces a burst containing N particles, and the integrated intensity $J_2(E, 750)$ of N -rays at mountain altitude, of energy more than E , were determined, taking into account the production probability** of burst in the shield. On the other hand, directional intensity $I(E, 0)$ of primary nucleons of energy greater than E was obtained by referring to the paper reported by Winckler et al¹⁷⁾. From both intensities thus obtained, the absorption MFP of burst-producing N -rays in air was calculated***, and the followings were revealed;

I) Absorption MFP of N -rays capable of producing the bursts of size 4, is $100 \sim 105$ g/cm².

II) That of N -rays capable of producing the bursts of size 40 is $95 \sim 100$ g/cm².

For simple exponential absorption****, the above figures become $85 \sim 90$ g/cm² and $80 \sim 85$ g/cm², respectively.

Of course, owing to some ambiguities in the burst production probability, above figures may not be accurate enough, nevertheless, we may conclude that the absorption MFP of burst-producing N -rays above mountain altitude is shorter than those near sea level for the bursts of smaller size.

* This procedure is equivalent to replace our chamber by Model C Meter with thinner lead shield at 2830 m.

** Burst production probability $p(E, S)$ means the probability that a N -rays impinging upon the shield with energy greater than E produce a burst greater than size S in the shield.

*** Absorption MFP was calculated by the following expression,

$$J_2(E, 750) = 2\pi H(t) I(E, 0) \quad (B)$$

where

$$H(t) = \exp(-t) + t E_4(-t),$$

$t = 750/\lambda$, and λ is absorption MFP in air.

**** λ is determined by the following relation;

$$J_2(E, 750) = 2\pi \exp(-t) I(E, 0) / (t+2) \quad (C)$$

and $t = 750/\lambda$.

On the other hand, for the bursts of the greatest size the absorption MFP of burst-producing N -rays derived by the above procedure is not same as that obtained in § 3, but is appreciably larger than the latter. Concerning this inagreement, we are now in study.

Acknowledgement

We should like to express our sincere gratitude to the late Dr. Y. Nishina, and Mr. Y. Miyazaki for their great interest in this work. We also express our appreciation to Mr. J. Nishimura for his many valuable suggestions.

Note added in proof

Two parers, recently published by Fahy* and Stinchcomb**, reported the detailed studies on large cosmic ray bursts and revealed much of the nature of the burst-producing rays. Meanwhile, most figures concerning Model C Meter described in this report seem not to be altered appreciably by their papers.

However, the followings must be noted;

- a) Throughout this report, we did not take into account the role of air-showers in the burst-production, but regarded it as the same as that of N -rays. Therefore, our conclusions described in § 3 c) and § 5 b) must be reexamined, standing on the facts found by them that the cointident rate of bursts with air showers increases with increasing burst size, and at burst size of, say, 4000 particles, most of them coincide with air showers. Thus, some informations will be deduced on the burst-production by air showers.***
- b) The result concerning the fraction of μ -bursts at sea level, deduced by Fahy, is in good agreement with our result notwithstanding that he derived it from the revised calculation of CK. While, the similar revised calculation of CK gave a result greater than the experimental one by a factor 2 or more as described in § 4 d). However, if we, as Fahy, quote the usual value of specific ionization of pure argon as cited by Schein and Gill,**** the above discrepancy may nearly disappear. Considering some ambiguities in cascade function and in fluctuation, it seems to be dangerous to decide the fraction of μ -bursts at sea level following CK's theory.

* E. F. Fahy, Phys. Rev. **83** (1951), 413.

** T. G. Stinchcomb, Phys. Rev. **83** (1951), 422.

*** Concerning these points, possible explanation will be presented elsewhere in connection with the variation of absorption MFP above mountain altitude and near sea level.

**** Fujimoto and Hayakawa cited the value higher than that of Schein and Gill by a factor 1.4 for specific ionization of argon.

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Theory of Some Magnetic Properties of Cobalt Tutton Salts*

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The g -values in the paramagnetic resonance experiment and the static magnetic susceptibilities of the cobalt Tutton salts are calculated under the assumption that the crystalline field is nearly cubic with a small tetragonal component. The values of these quantities depend on the value of a parameter $\eta = 4\delta/3\lambda$, where δ is the energy separation between the lowest energy levels when the spin-orbit coupling is disregarded, and λ is the spin-orbit coupling constant. Comparing the theoretical results with the experimental data, we have estimated $\eta = 3.5 \sim 4.5$ from g -values, $\eta = 4 \sim 5$ from χ_{11} , but the measured values of χ_{11} do not fall between the theoretical curves of χ_{11} versus T with $\eta = 4$ and 5. This discrepancy is discussed.

§ 1. Introduction

The behaviour of the paramagnetic ion in the magnetically dilute crystal can be many experimental methods. One of the most direct method is the paramagnetic resonance experiment. The measurement of the static magnetic susceptibility offers also a powerful but somewhat indirect means for the purpose. Making some reasonable assumptions on the symmetry and magnitude of the electrostatic field acting on the paramagnetic ion in the crystal, we can calculate theoretically the energy levels of the ion, the g -values and its static magnetic susceptibility.

If the experimental values of those quantities can be explained consistently by the adjustment of some parameters appearing in the theoretical calculation, the theory will be warranted. We performed this type of calculation for the Cobalt Tutton salt, since the experimental data are available in this case.

§ 2. Crystallography and crystalline field

All the Tutton salts have a monoclinic crystal structure. The ratio of the axes $a:b:c$ is nearly $3:4:2$. The b axis normal to ac -plane, and the angle β between a and c axes is about 105° . As for Co-NH_4 Tutton salt, $a = 9.28\text{Å}$, $b = 12.57\text{Å}$, $c = 6.22\text{Å}$ and $\beta = 106^\circ 56'$.

We shall assume that the atomic arrangement of Co-NH_4 Tutton salt is analogous to that of Mg-NH_4 Tutton salt, of which the X-ray analysis has been performed by Hofmann.¹⁾ This is based on the fact that the similar assumption made in Polder's treatment of the susceptibility of Cu Tutton salt²⁾ has been confirmed by Bleaney *et al.*³⁾

* Read on November 4, 1950 and May 8, 1951 before the meeting of the Physical Society of Japan held at Osaka University and Kyoto University respectively.

through the paramagnetic resonance experiment. According to this assumption, the Co^{++} ion is surrounded by six water molecules which make a slightly distorted octahedron. On the basis of the dimension of the unit cell, it has been estimated that the two H_2O molecules are at a somewhat larger distance 2.2 Å from the central ion, while the other four are located at the distance 1.9 Å. The system composed of the Co^{++} ion and the six H_2O molecules has a tetragonal symmetry, the axis being the line which passes through the two, more distant H_2O molecules. The unit cell contains two Co^{++} ions, one at (0,0,0), the other at (1/2,1/2,0). The tetragonal system with the central ion at (1/2,1/2,0) can be obtained from that at (0,0,0) by a simple symmetry operation, namely, a translation ($a \rightarrow a + 1/2$, $b \rightarrow b + 1/2$, $c \rightarrow c$) followed by a reflection in ab -plane. The tetragonal axis of the complex makes an angle u with the ac -plane. The estimation described in Section 4 shows that u is about 33° for Co-NH_4 Tutton salt.

The internal electrostatic field acting on a Co^{++} ion is determined mainly by the six H_2O molecules surrounding it. It can, therefore, be safely assumed that this crystalline field consists of a large cubic component and a small tetragonal component. Then, as far as it concerns d -electrons the crystalline field potential can be written in the following form :

$$A(x^2 + y^2 - 2z^2) + D(x^4 + y^4 + z^4) + Q(z^4 + 6x^2y^2) + f(r)$$

where A , D , Q and $f(r)$ are functions of r . Here the tetragonal axis has been taken as z -direction. The second term represents the cubic field component and the first and third terms, the tetragonal component. As will be seen later on, we are concerned with the states originating in a single electronic configuration of the corresponding free ion, and hence we can omit the spherically symmetrical field $f(r)$.

Treating the functions A , D and Q as constant parameters, Polder²⁾ has calculated their values on the basis of a dipole model. He has further used an appropriate Slater function for the radial part of $3d$ atomic orbital and showed that the values of the above parameters are reasonable in the order of magnitude for the cupric salt. In this paper, however, we shall regard $\langle Ar^2 \rangle_{AV}$, $\langle Dr^4 \rangle_{AV}$ and $\langle Qr^4 \rangle_{AV}$ as parameters to be determined through comparison with experiments. Here the average is to be taken over the density of the $3d$ -atomic orbital.

§ 3. Energy levels and g -values

The energy levels of Co^{++} ion in a crystalline field was at one time investigated by Schlapp and Penney.⁴⁾ The field they treated was of rhombic symmetry and the secular equation was not solved except the two extreme cases: the case where the spin-orbit interaction overrules the rhombic field and the oppositely extreme case. In contrast to this our crystalline field has a tetragonal symmetry and besides in our case the spin-orbit coupling are comparable in magnitude with the tetragonal component of the field.

The lowest energy state of a free Co^{++} ion is $(3d)^7 {}^4F(L=3, S=3/2)$. The first excited state is ${}^4P(L=1, S=3/2)$ and lies about 1500 cm^{-1} above the ground state. These two are all the quartet states that are originating in the $(3d)^7$ configuration.

When the crystalline field is not so strong that the Russell-Saunders coupling is not destroyed, as is the case, we can perform the perturbation calculation starting from the free ion states. The relevant perturbation is composed of two parts: spin-orbit interaction and crystalline Stark energy. Its dominant contribution is due to the Stark energy arising from the cubic field component. The only state that this energy combines with the ground state is 4P , but Van Vleck⁽⁵⁾ estimated that the contamination of $^4F\Gamma_4$ by 4P is of the order of 5 percent and is too small to alter materially Schlapp and Penney's calculation made on the assumption that the basic state is pure 4F . We shall, therefore, take the 4F free ion state as the unperturbed state. The perturbation hamiltonian is

$$H' = V_c + \lambda(\mathbf{L} \cdot \mathbf{S}) + \beta(\mathbf{H} \cdot \mathbf{L} + 2\mathbf{S}) \quad (1)$$

with

$$V_c = e \sum_i [A(x_i^2 + y_i^2 - 2z_i^2) + D(x_i^4 + y_i^4 + z_i^4) + Q(z_i^4 + 6x_i^2 y_i^2) + f(r_i)] \quad (2)$$

where \mathbf{L} and \mathbf{S} are orbital and spin angular momentum operators respectively, $\lambda = -180 \text{ cm}^{-1}$ is the spin-orbit coupling constant, β is the Bohr magneton, \mathbf{H} is the external static magnetic field, e is the electronic charge, (x_i, y_i, z_i) is the coordinate of the i -th $3d$ electron.

When we take into account only the cubic component of the crystalline field, the unperturbed level which has sevenfold orbital degeneracy splits into a singlet Γ_2 and two triplets, Γ_4 and Γ_5 , Γ_4 being the lowest level and Γ_2 the highest. The energy separation

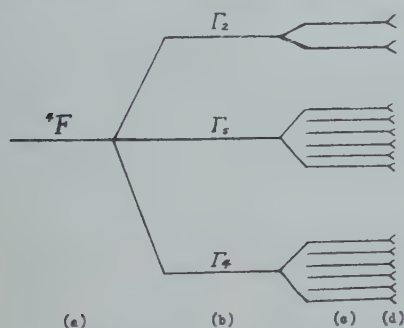


Fig. 1. Scheme of the level splitting for Co^{++} ion under the influence of crystalline field and spin-orbit coupling.

- free ion state 4F .
- splitting in cubic field only.
- splitting in tetragonal field with spin-orbit coupling.
- under the external magnetic field where all degeneracies are removed.

between Γ_4 and Γ_5 is given by $(104/21)e \langle D r^4 \rangle_{4F}$, and of the order of 10^4 cm^{-1} . We can therefore neglect Γ_5 and Γ_2 states for our purpose. The lowest orbital state Γ_4 has a further fourfold spin degeneracy, so we have a twelvefold degenerate state. This state splits into six doublets under the combined action of the tetragonal field and the spin-orbit coupling. Each doublet splits further under the influence of external magnetic field. The scheme of these level splittings is given in Fig. 1.

The three orbital states belonging to the Γ_4 level will be called Γ_4' , Γ_4'' , Γ_4''' and the corresponding wave functions are as follows:

$$\left. \begin{aligned} \varphi(\Gamma_4') &= \frac{1}{4}(\sqrt{10}\varphi_3 + \sqrt{6}\varphi_{-1}), \\ \varphi(\Gamma_4'') &= \frac{1}{4}(\sqrt{6}\varphi_1 + \sqrt{10}\varphi_{-3}), \\ \varphi(\Gamma_4''') &= \varphi_0, \end{aligned} \right\} \quad (3)$$

where $\varphi_{\pm 3}, \varphi_{\pm 2}, \varphi_{\pm 1}, \varphi_0$ are the orbital wave functions with the magnetic quantum number $M_L = \pm 3, \pm 2, \pm 1, 0$ ($L=3$) respectively.

The crystalline field energy V_c gives the matrix elements only between the levels Γ_4' and Γ_5 , and the states $\Gamma_4', \Gamma_4'', \Gamma_4'''$ are chosen in such a way that the energy V_c is diagonal in Γ_4 . But when we supplement the orbital wave functions (3) with the spin functions and consider the spin orbit coupling, we obtain a secular matrix of order twelve, which is given in table I.

Table I. Secular Matrix
(The common term is omitted from the diagonal elements)

M_S	Γ_4'				Γ_4''				Γ_4'''			
	3/2	1/2	-1/2	-3/2	3/2	1/2	-1/2	-3/2	3/2	1/2	-1/2	-3/2
Γ_4'	$\frac{3}{2}$	$\frac{9}{4}\lambda + \delta$								$\frac{3\sqrt{6}}{4}\lambda$		
	$\frac{1}{2}$		$\frac{3}{4}\lambda + \delta$								$\frac{3\sqrt{2}}{2}\lambda$	
	$-\frac{1}{2}$			$-\frac{3}{4}\lambda + \delta$								$\frac{3\sqrt{6}}{4}\lambda$
	$-\frac{3}{2}$				$-\frac{9}{4}\lambda + \delta$							
Γ_4''	$\frac{3}{2}$				$-\frac{9}{4}\lambda + \delta$							
	$\frac{1}{2}$					$-\frac{3}{4}\lambda + \delta$			$\frac{3\sqrt{6}}{4}\lambda$			
	$-\frac{1}{2}$						$\frac{3}{4}\lambda + \delta$			$\frac{3\sqrt{2}}{2}\lambda$		
	$-\frac{3}{2}$							$\frac{9}{4}\lambda + \delta$			$\frac{3\sqrt{6}}{4}\lambda$	
Γ_4'''	$\frac{3}{2}$					$\frac{3\sqrt{6}}{4}\lambda$						
	$\frac{1}{2}$	$\frac{3\sqrt{6}}{4}\lambda$					$\frac{3\sqrt{2}}{2}\lambda$					
	$-\frac{1}{2}$		$\frac{3\sqrt{2}}{2}\lambda$					$\frac{3\sqrt{6}}{4}\lambda$				
	$-\frac{3}{2}$			$\frac{3\sqrt{6}}{4}\lambda$								

In the secular matrix given by Table I, we have omitted from the diagonal elements the common term, and δ is given by the following expression :

$$\delta = \frac{10}{35}e\langle Qr^4 \rangle_{AV} + \frac{12}{35}e\langle Ar^2 \rangle_{AV}. \quad (4)$$

If we neglect the spin-orbit coupling, the level Γ_4 splits into a doublet (Γ_4', Γ_4'') and a singlet (Γ_4'''), the energy separation of which is given by δ . Since, however, the magnitudes of δ and λ are comparable, we must treat them on the same footing. The secular equation of twelfth order are split into the factors of 3rd, 2nd and 1st order, each appearing twice. If we measure the level energy W in the unit of $(3/4)\lambda (= -135 \text{ cm}^{-1})$,

and write $W/(\frac{3}{4}\lambda) = \eta$, x must satisfy one of the following equations with the parameter η :

$$x^2 - (4 + 2\eta)x^2 - (11 - 4\eta - \eta^2)x + (30 + 14\eta) = 0, \quad (5, a)$$

$$x^2 + (1 - \eta)x - 6 = 0, \quad (5, b)$$

$$x + 3 - \eta = 0. \quad (5, c)$$

The roots of these equations are denoted by x_1 , x_2 and x_3 for Eq. (5, a), x_4 and x_5 for Eq. (5, b), x_6 for Eq. (5, c). We have calculated the variation of these roots with the parameter η , and the results are shown in Fig. 2. The lowest energy level corresponds

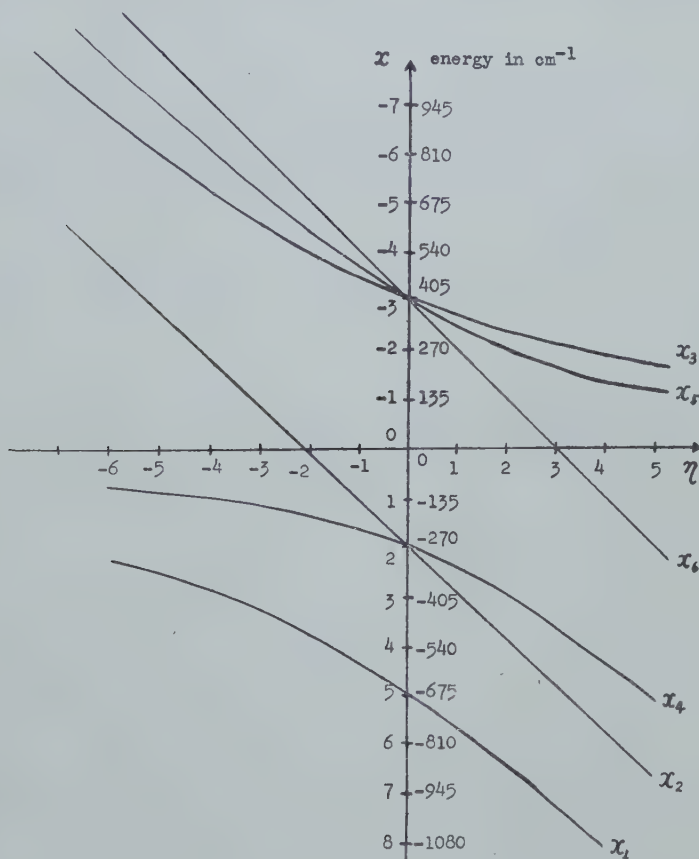


Fig. 2. Variation of the roots of Eqs. (5, a, b, c) with η .

to the highest root x_1 , since the spin-orbit coupling constant λ has negative sign in our case. From Fig. 2, the energy level scheme can be determined if we find the value of η corresponding to the actual case. To determine the value of η for Co Tutton salt, we shall make use of the two quantities, namely the g -value and static magnetic susceptibility.

The g -value of the lowest level may be calculated from the first order Zeeman splitting ΔW with the formula

$$\Delta W = g\beta H \quad (6)$$

where H is the magnitude of the external magnetic field.

The wave functions of the lowest level are

$$\Psi_1 = c_1 \varphi(\Gamma_4') u(3/2) + c_2 \varphi(\Gamma_4'') u(-1/2) + c_3 \varphi(\Gamma_4''') u(1/2), \quad (6, a)$$

$$\Psi_2 = c_2 \varphi(\Gamma_4') u(1/2) + c_1 \varphi(\Gamma_4'') u(-3/2) + c_3 \varphi(\Gamma_4''') u(-1/2) \quad (6, b)$$

where $u(\pm 3/2)$ and $u(\pm 1/2)$ are the spin eigenfunction with $M_s = \pm 3/2$ and $\pm 1/2$ respectively, and c 's are given by

$$c_1 = \sqrt{6} c_3 / (x_1 - 3 - \eta), \quad c_2 = \sqrt{8} c_3 / (x_1 - 1 - \eta),$$

$$c_3 = [1 + 6 / (x_1 - 3 - \eta)^2 + 8 / (x_1 - 1 - \eta)^2]^{-1/2}. \quad (7)$$

The first order Zeeman energy can be obtained from the secular equation with the matrix elements $(\Psi_i | \mathcal{H}_m | \Psi_j)$ ($i, j = 1, 2$), where $\mathcal{H}_m = \beta(\mathbf{H} \cdot \mathbf{L} + 2\mathbf{S})$ is the magnetic energy. If the external field is parallel to the tetragonal axis (z -axis), we have $\mathcal{H}_m = \beta H_z (L_z + 2S_z)$, the matrix elements of which are

$$(\Psi_1 | \mathcal{H}_m | \Psi_1) = -(\Psi_2 | \mathcal{H}_m | \Psi_2) = \left(\frac{9}{2} c_1^2 - \frac{5}{2} c_2^2 + c_3^2 \right) \beta H_z,$$

$$(\Psi_1 | \mathcal{H}_m | \Psi_2) = (\Psi_2 | \mathcal{H}_m | \Psi_1) = 0.$$

The first order Zeeman splitting is therefore

$$\Delta W = (\Psi_1 | \mathcal{H}_m | \Psi_1) - (\Psi_2 | \mathcal{H}_m | \Psi_2) = (9c_1^2 - 5c_2^2 + 2c_3^2) \beta H_z.$$

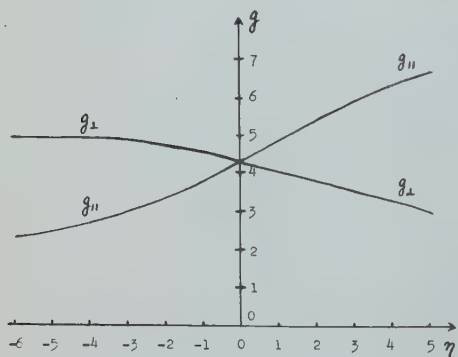


Fig. 3. Variation of g_{\parallel} and g_{\perp} with η

For the Co-NH_4 Tutton salt which is highly diluted magnetically with Zn-NH_4 salt, Bleaney and Ingram⁽⁶⁾ has observed the paramagnetic resonance line with the hyperfine structure. The g -values determined by them are

$$g_{\parallel} = 6.2 \quad \text{and} \quad g_{\perp} = 3.0.$$

Inspecting in Fig. 3, we find that the experimental g -values correspond to $\eta = 3.5 \sim 4.5$. Then, $\delta = 2.6\lambda \sim 3.4\lambda = -470 \sim 670 \text{ cm}^{-1}$.

Hence, the g -value for the parallel field are

$$g_{\parallel} = 9c_1^2 - 5c_2^2 + 2c_3^2. \quad (8)$$

In the case of the field perpendicular to the tetragonal axis, we obtain similarly the g -value as follows:

$$g_{\perp} = 3\sqrt{2} c_2 c_3 + 4\sqrt{3} c_1 c_2 + 4c_3^2. \quad (9)$$

Calculating the c 's with the formulae given by (7), we obtained the behaviour of g_{\parallel} and g_{\perp} when η varies. The results are presented graphically in Fig. 3.

§ 4. Magnetic susceptibility

In order to test the result of the last section, we have calculated the static magnetic susceptibility. We have treated the problem from the stand-point stated at the beginning of the last section. The matrix elements of the \mathcal{H}_m in $(I \cdot M_s)$ representation are given in Table II. This matrix must be transformed by the unitary matrix U which diagonalizes the secular matrix given in Table I. From the resulting secular matrix we can easily obtain the perturbation energy terms for the crystalline levels when the external magnetic field is present. If we expand the energy in powers of the field strength H , we get

$$W_i = W_i^0 + W_i^{(1)}H + W_i^{(2)}H^2 + \dots \quad (10)$$

Table II. The Matrix Elements of the Magnetic Energy
in $(I \cdot M_s)$ Representation. (unit β)

	M_s	3/2	1/2	-1/2	-3/2	3/2	1/2	-1/2	-3/2	3/2	1/2	-1/2	-3/2
Γ_4'	$\frac{3}{2}$	$\frac{9}{2}H_z$	$\sqrt{3}H_-$							$\frac{3}{\sqrt{8}}H_+$			
	$\frac{1}{2}$	$\sqrt{3}H_+$	$\frac{5}{2}H_z$	$2H_-$						$\frac{3}{\sqrt{8}}H_+$			
	$-\frac{1}{2}$		$2H_+$	$\frac{1}{2}H_z$	$\sqrt{3}H_z$						$\frac{3}{\sqrt{8}}H_+$		
	$-\frac{3}{2}$			$\sqrt{3}H_+$	$-\frac{3}{2}H_z$							$\frac{3}{\sqrt{8}}H_+$	
Γ_4''	$\frac{3}{2}$					$\frac{3}{2}H_z$	$\sqrt{3}H_-$			$\frac{3}{\sqrt{8}}H_-$			
	$\frac{1}{2}$					$\sqrt{3}H_+$	$-\frac{1}{2}H_z$	$2H_-$		$\frac{3}{\sqrt{8}}H_-$			
	$-\frac{1}{2}$						$2H_+$	$-\frac{5}{2}H_z$	$\sqrt{3}H_-$		$\frac{3}{\sqrt{8}}H_-$		
	$-\frac{3}{2}$						$\sqrt{3}H_+$	$-\frac{9}{2}H_z$				$\frac{3}{\sqrt{8}}H_-$	
Γ_4'''	$\frac{3}{2}$	$\frac{3}{\sqrt{8}}H_-$				$\frac{3}{\sqrt{8}}H_+$				$3H_z$	$\sqrt{3}H_-$		
	$\frac{1}{2}$		$\frac{3}{\sqrt{8}}H_-$				$\frac{3}{\sqrt{8}}H_+$			$\sqrt{3}H_+H_z$	$2H_-$		
	$-\frac{1}{2}$			$\frac{3}{\sqrt{8}}H_-$				$\frac{3}{\sqrt{8}}H_+$			$2H_+$	$-H_z$	$\sqrt{3}H_-$
	$-\frac{3}{2}$				$\frac{3}{\sqrt{8}}H_-$				$\frac{3}{\sqrt{8}}H_+$		$\sqrt{3}H_+$	$-3H_z$	

where the first term is the unperturbed energy which equals to one of the $(3/4)\lambda x_i$, the second and third terms are the first order and second order Zeeman energy terms respectively. Then magnetic susceptibility is given by the well known formula:⁷⁾

$$\chi = N \frac{\sum_i [(W_i^{(1)2}/kT) - 2W_i^{(2)}] \exp(-W_i^0/kT)}{\sum_i \exp(-W_i^0/kT)} \quad (11)$$

Table III. Static Susceptibilities for the Parallel and Perpendicular Fields. (theoretical)

	T	-10	-6	-2	2	3	4	5
χ_{\perp}	100°	0.039	0.036	0.033	0.021	0.019	0.017	0.016
	200°	0.017	0.019	0.019	0.013	0.012	0.0114	0.0106
	300°	0.011	0.013	0.013	0.0097	0.0092	0.0088	0.0082
χ_{\parallel}	100°	0.0080	0.0082	0.0167	0.036	0.042	0.046	0.050
	200°	0.0062	0.0064	0.0110	0.0204	0.0226	0.024	0.026
	300°	0.0046	0.0054	0.0081	0.0137	0.0147	0.0156	0.0163

We have calculated the susceptibilities for the parallel and perpendicular fields, which are presented in Table III and Fig. 4.

The unit cell of the Co Tutton salt contains two Co^{++} ions and the tetragonal axes referred to them make an angle 2α . The principal susceptibilities of the whole crystal, therefore, are given by the following expressions:

$$\chi_1 = \chi_{\parallel} \cos^2 \alpha + \chi_{\perp} \sin^2 \alpha, \quad (11, a)$$

$$\chi_2 = \chi_{\perp}, \quad (11, b)$$

$$\chi_3 = \chi_{\parallel} \sin^2 \alpha + \chi_{\perp} \cos^2 \alpha. \quad (11, c)$$

The corresponding magnetic axes K_1 , K_2 and K_3 are illustrated in Fig. 5. The three principal susceptibilities of some Co Tutton salts have been measured by Jackson, Rabi, Bartlett, and recently by Bose.

From their experimental data we can calculate the values of χ_{\parallel} and χ_{\perp} from equations (11, a)—(11, b). There is, however, an ambiguity concerning the assignment of the measured principal susceptibilities to the expressions in Eqs. (11, a, b, c). If we denote the measured principal molar susceptibilities χ_{m1} , χ_{m2} and χ_{m3} , which satisfy the relation $\chi_{m1} > \chi_{m3} > \chi_{m2}$ at all temperatures used in the experiments, we have two possibilities:

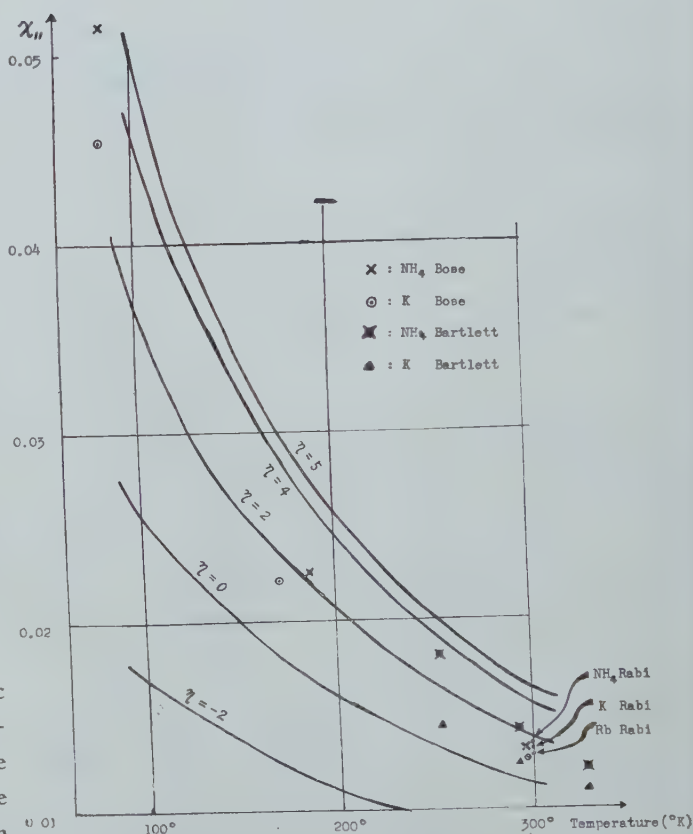


Fig. 4. (a). Static susceptibility for the parallel field.

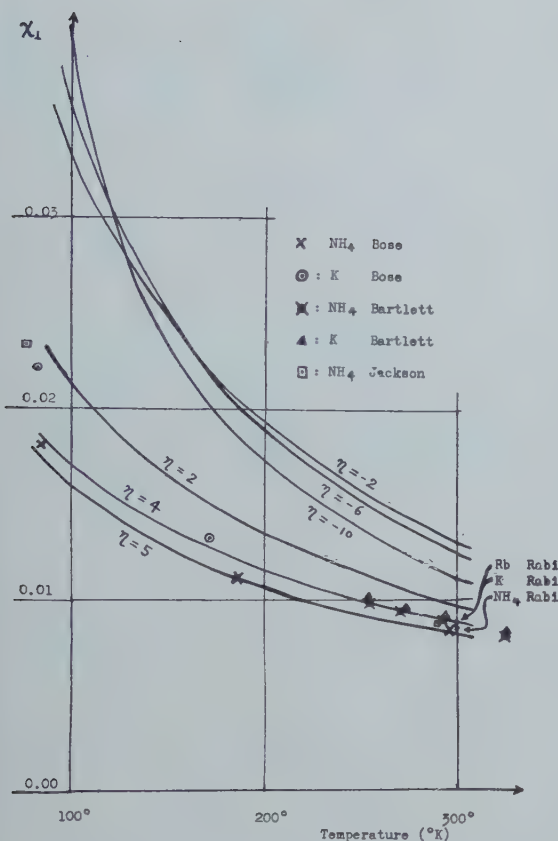


Fig. 4 (b). Static susceptibility for the perpendicular field

Table IV. Static Susceptibilities for the Parallel and Perpendicular Fields. (Calculated from the experimental data for $\text{Co}(\text{NH}_4)_2(\text{SO}_4)_2 \cdot 6\text{HO}^{(1)}$)

T	χ_{\parallel}	χ_{\perp}	$\cos^2 \alpha$
84.7°	0.05163	0.01913	0.7194
185.2°	0.022505	0.01124	0.6989
296.0°	0.013109	0.00837	0.6740

the measured values of χ_{\parallel} do not fall between the corresponding theoretical curves of χ_{\perp} . For other salts, too, we have similar situations. We cannot, therefore, determine consistently the value of the parameter η for these salts. Possible causes of this discrepancy are suggested in the next section.

§ 5. Discussion

Owing to the lack of the precise knowledge of the crystalline field and the orbital

(A) $\chi_{\parallel} > \chi_{\perp}$ and $\chi_{m1} = \chi_1$, $\chi_{m2} = \chi_2$, $\chi_{m3} = \chi_3$. (B) $\chi_{\parallel} < \chi_{\perp}$ and $\chi_{m1} = \chi_2$, $\chi_{m2} = \chi_3$, $\chi_{m3} = \chi_1$. The case (B), however, seems not to hold, because from the experimental data for Co-NH_4 salt given by Bose, for example, will follow the result that χ_{\parallel} decreases with decreasing temperature, which is a highly improbable behaviour from the theoretical point of view. We assume the case (A) holds for all available experiments. Then we can calculate the molar susceptibilities χ_{\parallel} and χ_{\perp} and the angle α from the following formulae.

$$\chi_{\parallel} = \chi_{m1} + \chi_{m3} - \chi_{m2}, \quad (12, a)$$

$$\chi_{\perp} = \chi_{m2}, \quad (12, b)$$

$$\cos^2 \alpha = (\chi_{m1} - \chi_{m2}) / (\chi_{m1} + \chi_{m3} - 2\chi_{m2}). \quad (12, c)$$

As an example, we give in Table IV the values of χ_{\parallel} , χ_{\perp} and $\cos^2 \alpha$ calculated by Eqs. (12, a, b, c) with the experimental value for $\text{Co}(\text{NH}_4)_2(\text{SO}_4)_2 \cdot 6\text{H}_2\text{O}$ determined by Bose.

The average value of $\cos^2 \alpha$ is 0.6974

which corresponds to $\alpha = 33^\circ 23'$ in agreement with the value obtained from the paramagnetic resonance measurement.³⁾ The values of χ_{\parallel} and χ_{\perp} calculated by combining the measured values of the principal susceptibilities for various salts are plotted in Fig. 4.

For Co-NH_4 Tutton salt, all measured values of χ_{\perp} fall between the theoretical curve of χ_{\perp} for $\eta=4$ and that for $\eta=5$, but

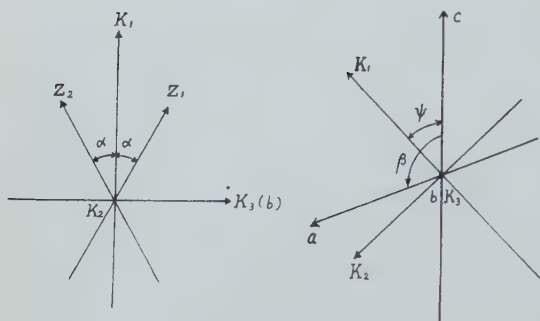


Fig. 5. Principal magnetic axes K_1 , K_2 and K_3 , K_1 and K_2 being in the ac -plane, K_3 identical with b -axis. The tetragonal axes Z_1 , Z_2 are in the K_1K_3 -plane, K_1 being the internal bisector of the angle made by the axes Z_1 and Z_2 . $a=33^\circ$, $\beta=106.9^\circ$ and $\psi=43.2^\circ$ for $\text{Co}(\text{NH}_4)_2(\text{SO}_4)_2 \cdot 6\text{H}_2\text{O}$.

parameters. When moreover the admixing of the excited state 4P is included in the treatment, number of adjustable parameters becomes larger. But in our paper we have adopted the different standpoint. We investigated whether we can explain the many experimental results consistently with the introduction of only small number of adjustable parameters or not. Although the theoretical treatment becomes cruder because of the necessary approximations to avoid the introduction of more parameters, the values of parameters adjusted to fit the theory with the experimental results may be considered as confirmed. Following this idea, we have neglected all the excited levels and retained only the Γ_4 state in § 3 and § 4. Then we have only one parameter η to be adjusted in comparison with experimental data. The experimental values of χ_{\parallel} , however, can not be explained with the value of η estimated from the g -values and χ_{\perp} . The discrepancy comes probably from the inaccuracies of both theoretical and experimental results. The theoretical treatment may be improved by the inclusion of the excited states neglected by us. Abragam and Pryce⁽¹²⁾ has calculated the g -values of cobalt Tutton salts in this approximation. Since they have not, however, calculated the static magnetic susceptibilities of these salts, it is uncertain whether their values of parameters estimated from the g -values can also explain consistently the experimental data for χ_{\parallel} and χ_{\perp} or not. The curves of g -values in Fig. 3 of our paper correspond to the curve D in Fig. 3 of their paper. Our treatment is, therefore, not a bad approximation compared with their more accurate treatment as far as the g -values are concerned.

In conclusion we should like to express our sincere gratitude to Professor Masao Kotani for his encouragement.

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On the Antiferromagnetism of $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$ Single Crystal

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A theory of the antiferromagnetism of $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$ single crystal is developed, basing on the Van Vleck model modified by the introduction of the anisotropic spin-spin coupling besides the usual isotropic exchange coupling. Since the spin quantum number S of the magnetic Cu^{2+} ion in this substance is $1/2$, the effect of the crystalline electric field can only be taken into account by adopting an adequate tensor form for the Landé g -factor. Starting from this point of view, formulas for the anisotropy of the susceptibility, both above and below the Curie point, and for the temperature variation of the anisotropy constant are derived. The critical field discovered by C. J. Gorter *et al.* and the antiferromagnetic resonance frequency are also discussed.

§ 1. Introduction

C. J. Gorter *et al.*¹⁾ have measured the susceptibility of $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$ single crystal in the range of temperature of liquid helium. According to their measurements, this crystal is an antiferromagnetic substance with a Curie temperature just above the boiling point of liquid helium, i.e., at about 5°K . Below this Curie temperature, the susceptibility along the a -direction decreases with decreasing temperature, while that along the b -direction remains constant, independent of temperature. These behaviors indicate that the former corresponds to $\chi_{||}$ and the latter to χ_{\perp} of the Van Vleck theory²⁾. Furthermore, a remarkable field dependency was observed for the susceptibility along the a -axis that, when the applied magnetic field exceeds a certain value, this susceptibility jumps abruptly up to a constant value corresponding to the susceptibility along the b -axis and almost independent of the field strength; however this critical field strength ranges from a value of 7000 oersteds at a low temperature to a value of 8500 oersteds at a higher temperature.

In a preceding paper³⁾, the writer has developed a theory of antiferromagnetism for a single crystal of MnF_2 and derived the critical field strength. This theory cannot, however, be applied without alteration to the case of $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$ where the magnetic Cu^{2+} ion has a spin quantum number S of $1/2$ and hence the anisotropy energy cannot be expressed as a function of one spin variable alone, as it was done in the case of MnF_2 . It is therefore necessary to extend the preceding theory to the case of $S=1/2$.

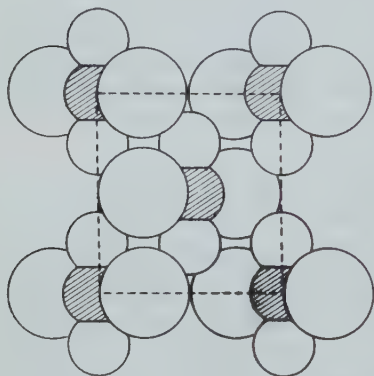
The anisotropy energy of the antiferromagnetic substance may be considered to arise from the anisotropy of the crystalline electric field at the position of the magnetic ion and from such an anisotropic coupling between two spins as dipole-dipole interaction or anisotropic exchange interaction. The former part is expressible as a function of one spin variable, while the latter contains spin variables of two magnetic ions. In the case of MnF_2 , the anisotropy energy could effectively be represented by the anisotropy of the first type alone, as shown in the preceding paper, whereas in the case of $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$ the anisotropy of the

second type must be the sole origin of the anisotropy energy, because the anisotropy of the first type gives rise merely to a constant value on account of S being equal to $1/2$.

Another difference between MnF_2 and $\text{CuCl}_2 \cdot 2\text{H}_2\text{O}$ is that Mn^{2+} ion has no orbital angular momentum, whereas Cu^{2+} ion has a finite value of it. The Lande g -factor of the magnetic ion in the latter case must therefore be considered as a tensor, its principal axes coinciding with those of the crystalline electric field at the position of that ion.

The purpose of the present paper is therefore to extend the previous calculations for MnF_2 to the case of $\text{CuCl}_2 \cdot 2\text{H}_2\text{O}$ by taking into account the two circumstances as mentioned above.

§ 2. The crystal structure and the g -tensors



Projection on $(0, 0, 1)$

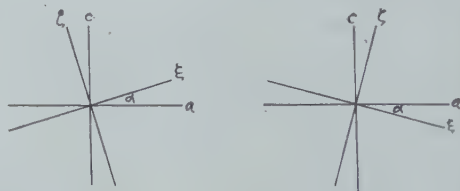
Fig. 1. The crystal structure of $\text{CuCl}_2 \cdot 2\text{H}_2\text{O}$. Large open circles represent chlorine ions, small open circles represent oxygen ions, and shaded regions represent copper ions.

applying for the Cu ion on $(0, 0, 0)$ and $-\alpha$ for that on $(1/2, 1/2, 0)$. Thus these two magnetic ions are crystallographically not equivalent to each other. This circumstance is illustrated schematically in Fig. 2.

If we take three crystal axes a , b and c as the coordinate system x , y and z and denote the directions of the three principal axes of the crystalline electric field by ξ , η and ζ , then the η -axis coincides with the y -axis and ξ and x , and ζ and z make an angle of α to each other, respectively. Since the principal axes of the g -tensor should coincide with those of the crystalline field, the components of the g -tensors referred to x , y - and z -axes for the two magnetic ions can be expressed in terms of the three principal values of the g -tensor, g_ξ , g_η and g_ζ , according to the tensor transformation rule as follows :

The crystal of $\text{CuCl}_2 \cdot 2\text{H}_2\text{O}$ is orthorhombic as shown by Fig. 1.⁴⁾ The lengths of the three vectors a , b and c , which map up a unit cell, are 7.38Å, 8.04Å and 3.72Å, respectively. A unit cell contains two copper ions at the positions of $(0, 0, 0)$ and $(1/2, 1/2, 0)$, four water molecules at $(0, \pm u_0, 0)$ and $(1/2, 1/2 \pm u_0, 0)$, with $u_0 = 0.25 \pm 0.02$, and four chlorine ions at $(\pm u, 0, \pm v)$ and $(1/2 \pm u, 1/2, \pm v)$, with $u = 0.25 \pm 0.05$ and $v = 0.37 \pm 0.01$.

Each copper ion in a unit cell is thus situated at the center of a crystalline electric field of orthorhombic symmetry, one of whose principal axes coincides with the b -axis and the remaining two are rotated from the a - and b -axes through an angle of $+\alpha$ or $-\alpha$ about the b -axis, $+\alpha$



1. Cu^{2+} at $(0, 0, 0)$ 2. Cu^{2+} at $(1/2, 1/2, 0)$

Fig. 2. The principal axes of the crystalline fields at the positions of two copper ions in a unit cell.

$$\begin{aligned}
 g_{xx}^{(1)} &= g_{xx}^{(2)} = g_{\xi} \cos^2 a + g_{\zeta} \sin^2 a, & g_{zz}^{(1)} &= g_{zz}^{(2)} = g_{\xi} \sin^2 a + g_{\zeta} \cos^2 a, \\
 g_{yy}^{(1)} &= g_{yy}^{(2)} = g_{\eta}, & g_{xz}^{(1)} &= -g_{xz}^{(2)} = (g_{\xi} - g_{\zeta}) \sin a \cos a, \\
 g_{xy}^{(1)} &= g_{xy}^{(2)} = g_{yz}^{(1)} = g_{yz}^{(2)} = 0,
 \end{aligned} \tag{1}$$

where indices (1) and (2) are used to distinguish between two different magnetic ions. These equations indicate that all the components of $g^{(1)}$ referred to x -, y - and z -axes are respectively equal to those of $g^{(2)}$ except for the xz -component which takes opposite signs for the two Cu^{2+} ions.

The spins of Cu^{2+} ions in $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$ take an antiferromagnetic orientation below its Curie temperature, and so the spin lattice constituted from the copper ions can be divided into two sublattices, each having $+$ or $-$ spins alone. Since the observed susceptibility along the x -axis corresponds to χ_{\parallel} of the Van Vleck theory, the preferred axis of the spins must be considered to lie in the direction of the x -axis. The magnetic superstructure can not be decided until for instance the neutron diffraction experiment is carried out for this substance, but three structures shown in Fig. 3 may be considered as probable ones. In case I, a magnetic unit cell has the same size as that of the chemical unit cell,

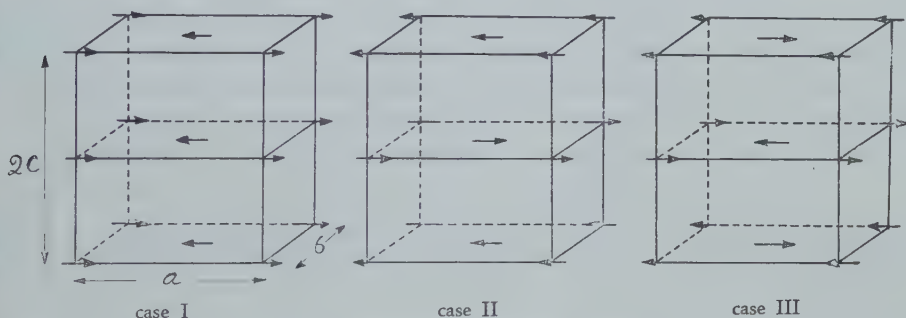


Fig. 3. Possible superstructures in $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$.

and in cases II and III it is twice as large as the latter. In case I a spontaneous magnetization appears along the z -axis, since the sign of g_{xz} is opposite for the two antiparallel spins in the unit cell. Accordingly this case is to be excluded. In the following calculations we shall assume the third structure which seems to be the most probable one. In it the spin lattice is divided into two penetrating body-centered sublattices.

§ 3. The interaction energy between two spins

Let us consider the interaction between two spins of copper ions. According to Van Vleck⁵⁾ the most general form of the interaction V between the l -th and the m -th spins in the case of spin quantum number S equal to $1/2$ is given by

$$\begin{aligned}
 V = & V_0 - 2J_{lm}(S_{lx}S_{mx} + S_{ly}S_{my} + S_{lz}S_{mz}) + C_{lm}(2S_{lz}S_{mz} - S_{lx}S_{mx} - S_{ly}S_{my}) + D_{lm}(S_{lx}S_{mx} \\
 & - S_{ly}S_{my}) + E_{lm}(S_{lx}S_{my} + S_{mx}S_{ly}) + F_{lm}(S_{lx}S_{mz} + S_{mz}S_{lx}) + G_{lm}(S_{lx}S_{mz} + S_{my}S_{ly}), \quad (2)
 \end{aligned}$$

where J , C , D , E , F and G are the coupling constants dependent on the distance between the two spins.

If we take the direction of the line joining the two spins as z' -axis and assume V to be invariant against a rotation about this z' -axis, then D , E , F and G must vanish. In the case of $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$ this condition would not strictly be satisfied, but we shall assume it to hold approximately. Then Eq. (2) becomes, referring to the new primed coordinate system,

$$V = V_0 - 2J_{lm}(S_{lx'}S_{mx'} + S_{ly'}S_{my'} + S_{lz'}S_{mz'}) + C_{lm}(2S_{lx'}S_{mz'} - S_{lx'}S_{mx'} - S_{ly'}S_{my'}) \quad (3)$$

Transforming this coordinate system into a general one, V can be written as follows:

$$V = V_0 - 2J_{lm}(\mathbf{S}_l \cdot \mathbf{S}_m) - C_{lm}\{(\mathbf{S}_l \cdot \mathbf{S}_m) - 3(\mathbf{S}_l \cdot \mathbf{e}_{lm})(\mathbf{S}_m \cdot \mathbf{e}_{lm})\}, \quad (4)$$

where \mathbf{e}_{lm} is the unit vector in the direction joining the l -th and the m -th spins.

Thus a general form of the spin-spin interaction in the case of $S=1/2$ consists of an isotropic coupling and an interaction of the dipole-dipole type, as it was actually shown by Van Vleck⁵⁾.

§ 4. The anisotropy energy below the Curie point

For simplicity we shall assume that the interaction expressed by Eq. (4) exists only between nearest dissimilar and similar spins and adopt an approximation of the *molecular field*. In this section our considerations will be confined to the case of no external magnetic field. In such a case the anisotropy of the g -factor does not play any role.

By use of Eq. (4), the average potential energy for the spin l is given by

$$V_l = V_0 - \sum_m \{2J_{lm} + C_{lm}(1 - 3\alpha_{lm}^2)\} \bar{S}_{mx} \cdot S_{lx} - \sum_m \{2J_{lm} + C_{lm}(1 - 3\beta_{lm}^2)\} \bar{S}_{my} \cdot S_{ly} - \sum_m \{2J_{lm} + C_{lm}(1 - 3\gamma_{lm}^2)\} \bar{S}_{mz} \cdot S_{lz} + 3 \sum_m C_{lm} \beta_{lm} \gamma_{lm} (\bar{S}_{my} \cdot S_{lx} + \bar{S}_{mz} \cdot S_{ly}) + 3 \sum_m C_{lm} \gamma_{lm} \alpha_{lm} (\bar{S}_{mz} \cdot S_{lx} + \bar{S}_{mx} \cdot S_{lz}) + 3 \sum_m C_{lm} \alpha_{lm} \beta_{lm} (\bar{S}_{mx} \cdot S_{ly} + \bar{S}_{my} \cdot S_{lx}), \quad (5)$$

where α , β and γ are the direction cosines of the vector \mathbf{e}_{lm} and the summation with respect to m is carried over nearest spins which are situated on the sublattice different from that of the l -th spin, and \bar{S}_m is the value of \mathbf{S} averaged according to the molecular field approximation. We shall assume that this average is the same for two kinds of the spins situated on the same sublattice in the absence of external field. Then the summation over m can explicitly be carried out. If we denote the spin vectors on two sublattices by \mathbf{S}_1 and \mathbf{S}_2 and choose the crystal axes as the coordinate system, Eq. (5) can generally be written in the following form:

$$V_1 = V_0 - (2J_z - C_x) \bar{S}_{2x} \cdot S_{1x} - (2J_z - C_y) \bar{S}_{2y} \cdot S_{1y} - (2J_z - C_z) \bar{S}_{2z} \cdot S_{1z}. \quad (6)$$

Here z means the effective number of nearest neighboring spins, and for our superstructure we obtain the following relations:

$$\begin{aligned} 2J_z &= 2J_1 z_1 + 2J_2 z_2, \\ -C_x &= C_1(1 - 3\alpha_1^2) z_1 + C_2(1 - 3\alpha_2^2) z_2, \\ -C_y &= C_1(1 - 3\beta_1^2) z_1 + C_2(1 - 3\beta_2^2) z_2, \\ -C_z &= C_1(1 - 3\gamma_1^2) z_1 + C_2(1 - 3\gamma_2^2) z_2, \end{aligned} \quad (7)$$

where suffix 1 is related to similar ions (the ions l and m have the same g -tensor) and 2 to dissimilar ions (they have different g -tensors). Thus $z_1=2$, and $z_2=4$. Since J has a negative sign in antiferromagnetic case, Eq. (7) becomes

$$V_1 = V_0 + (2 |J| z + C_x) \bar{S}_{2x} \cdot S_{1x} + (2 |J| z + C_y) \bar{S}_{2y} \cdot S_{1y} \\ + (2 |J| z + C_z) \bar{S}_{2z} \cdot S_{1z} . \quad (8)$$

An analogous expression is also obtained for V_2 .

If we could put $C_x = C_y = C_z = 0$, we would obtain the following results which were already derived by Van Vleck⁹⁾:

$$\bar{S}_1 = -\bar{S}_2 = S, \quad S = \frac{1}{2} \tanh \frac{1}{2kT} 2 |J| z S . \quad (9)$$

In this case the directions of S_1 and S_2 remain undetermined. From Eq. (9) the Curie temperature θ is given by

$$\theta = |J| z / 2k , \quad (10)$$

From Eq. (8), the eigenvalues of V_1 and V_2 can easily be obtained, apart from a constant term, as follows:

$$E_1 = \pm \frac{1}{2} [(2 |J| z + C_x)^2 \bar{S}_{2x}^2 + (2 |J| z + C_y)^2 \bar{S}_{2y}^2 + (2 |J| z + C_z)^2 \bar{S}_{2z}^2]^{\frac{1}{2}}, \\ E_2 = \pm \frac{1}{2} [(2 |J| z + C_x)^2 \bar{S}_{1x}^2 + (2 |J| z + C_y)^2 \bar{S}_{1y}^2 + (2 |J| z + C_z)^2 \bar{S}_{1z}^2]^{\frac{1}{2}} . \quad (11)$$

Using these eigenvalues, the partition function for each spin situated on each sublattice becomes

$$\zeta_1 = 2 \cosh \frac{1}{2kT} [(2 |J| z + C_x)^2 \bar{S}_{2x}^2 + (2 |J| z + C_y)^2 \bar{S}_{2y}^2 + (2 |J| z + C_z)^2 \bar{S}_{2z}^2]^{\frac{1}{2}}, \\ \zeta_2 = 2 \cosh \frac{1}{2kT} [(2 |J| z + C_x)^2 \bar{S}_{1x}^2 + (2 |J| z + C_y)^2 \bar{S}_{1y}^2 + (2 |J| z + C_z)^2 \bar{S}_{1z}^2]^{\frac{1}{2}} . \quad (12)$$

From the relations $F = -kT \log Z$ and $Z = \zeta_1^{\frac{N}{2}} \zeta_2^{\frac{N}{2}}$, where N denotes the number of total spins, the free energy F can be obtained as follows:

$$F = -NkT \left\{ \log 2 + \frac{1}{2} \log \cosh \frac{1}{2kT} [(2 |J| z + C_x)^2 \bar{S}_{2x}^2 + (2 |J| z + C_y)^2 \bar{S}_{2y}^2 \\ + (2 |J| z + C_z)^2 \bar{S}_{2z}^2]^{\frac{1}{2}} + \frac{1}{2} \log \cosh \frac{1}{2kT} [(2 |J| z + C_x)^2 \bar{S}_{1x}^2 \\ + (2 |J| z + C_y)^2 \bar{S}_{1y}^2 + (2 |J| z + C_z)^2 \bar{S}_{1z}^2]^{\frac{1}{2}} \right\} \quad (13)$$

$$- \frac{N}{2} [(2 |J| z + C_x) \bar{S}_{1x} \cdot \bar{S}_{2x} + (2 |J| z + C_y) \bar{S}_{1y} \cdot \bar{S}_{2y} + (2 |J| z + C_z) \bar{S}_{1z} \cdot \bar{S}_{2z}] .$$

Here the last term which corresponds to the negative value of the average interaction

energy was introduced into this expression in order to care of the fact that the interaction energy is taken into account twice in Z through the molecular field.

Now we shall assume that the absolute values of C_x , C_y , and C_z are small compared with J and denote the deviations of S_1 and S_2 from their values in absence of the anisotropy terms C 's by δS_1 and δS_2 , which would be also small quantities, and further denote the arbitrarily assumed direction cosines of S_1 and S_2 in absence of the anisotropy terms by (u, β, γ) and $(-u, -\beta, -\gamma)$, respectively. Then we obtain the following relations :

$$\begin{aligned}\bar{S}_{1x} &= uS + \delta S_{1x}, & \bar{S}_{1y} &= \beta S + \delta S_{1y}, & \bar{S}_{1z} &= \gamma S + \delta S_{1z}, \\ \bar{S}_{2x} &= -uS + \delta S_{2x}, & \bar{S}_{2y} &= -\beta S + \delta S_{2y}, & \bar{S}_{2z} &= -\gamma S + \delta S_{2z}.\end{aligned}\quad (14)$$

Expanding the free energy per spin, i.e., $f = F/N$ in powers of C_x , C_y , C_z , δS_1 and δS_2 and adopting only the terms of the first order, we obtain the following result after simple calculations :

$$f = -kT \log \cosh \left(\frac{1}{2kT} 2 |J| S \right) + |J| S^2 - \frac{S^2}{2} (u^2 C_x + \beta^2 C_y + \gamma^2 C_z). \quad (15)$$

Taking out the terms dependent upon u, β, γ from this expression, the macroscopic anisotropy energy $f_{\text{an.}}$ can be given by

$$f_{\text{an.}} = -\frac{S^2}{2} (u^2 C_x + \beta^2 C_y + \gamma^2 C_z). \quad (16)$$

Thus, in our approximation, the anisotropy constant shows the same temperature variation as S^2 and therefore vanishes linearly with temperature at the Curie point.

Since the field dependency of the susceptibility mentioned in section 1 indicates that the preferred direction of the spins coincides with the x -axis, C_x must be larger than C_y and C_z . Therefore, putting $\varepsilon_1 = 2$, $u_1 = \beta_1 = 0$ and $\varepsilon_2 = 4$, $\gamma_2 = 0$ in Eq. (7), we obtain the following relations :

$$\begin{aligned}2C_1 + 4C_2(1 - 3u_2^2) &< 2C_1 + 4C_2(1 - 3\beta_2^2), \\ 2C_1 + 4C_2(1 - 3u_2^2) &< -4C_1 + 4C_2.\end{aligned}\quad (17)$$

Since $u_2^2 < \beta_2^2$ from the dimensions of the unit cell, these inequalities are reduced to

$$C_2 < 0 \quad \text{and} \quad C_1 < 2C_2u_2^2. \quad (18)$$

Moreover, the following relation can be derived :

$$\begin{aligned}\text{either } C_y &> C_z \quad \text{if } C_1 < 2C_2\beta_2^2, \\ \text{or } C_y &< C_z \quad \text{if } 2C_2u_2^2 > C_1 > 2C_2\beta_2^2.\end{aligned}\quad (19)$$

These relations can be satisfied with an appropriate choice of the values of the parameters.

§ 5. The anisotropy of the susceptibility below the Curie point

In calculating the susceptibility, we shall omit the anisotropy terms which would be small compared with J . Since the Zeeman energy includes the g -factor, we have to distinguish in the following calculations between two kinds of spins which are on the same

sublattice and which have different g -values. Then the potential energies for four kinds of spins on two sublattices are given by

$$\begin{aligned} V_1^{(1)} &= (2 |J_1| z_1 \bar{S}_2^{(1)} + 2 |J_2| z_2 S_2^{(2)}) \cdot \bar{S}_1^{(1)} + \mu_B (g^{(1)} S_1^{(1)} H), \\ V_1^{(2)} &= (2 |J_1| z_1 \bar{S}_2^{(2)} + 2 |J_2| z_2 S_2^{(1)}) \cdot \bar{S}_1^{(2)} + \mu_B (g^{(2)} S_1^{(2)} H), \\ V_2^{(1)} &= (2 |J_1| z_1 \bar{S}_1^{(1)} + 2 |J_2| z_2 S_1^{(2)}) \cdot \bar{S}_2^{(1)} + \mu_B (g^{(1)} S_2^{(1)} H), \\ V_2^{(2)} &= (2 |J_1| z_1 \bar{S}_1^{(2)} + 2 |J_2| z_2 S_1^{(1)}) \cdot \bar{S}_2^{(2)} + \mu_B (g^{(2)} S_2^{(2)} H), \end{aligned} \quad (20)$$

where J_1 and J_2 are respectively the exchange integral between nearest similar ions and that between nearest dissimilar ions, z_1 and z_2 are, as before, respectively the number of nearest similar spins and that of nearest dissimilar spins, $S_1^{(1)}$, $S_1^{(2)}$ etc. are respectively the spin of the copper ion (1) situated on the sublattice 1 and the spin of the copper ion (2) on the sublattice 1, etc., and μ_B and H denote the Bohr magneton and the external field.

From (20) the eigenvalues of $V_1^{(1)}$ etc. are obtained as

$$E_1^{(1)} = \pm \frac{1}{2} | 2 |J_1| z_1 \bar{S}_2^{(1)} + 2 |J_2| z_2 \bar{S}_2^{(2)} + \mu_B g^{(1)} H |, \quad \text{etc.}, \quad (21)$$

and $S_1^{(1)}$ etc. are given by

$$\bar{S}_1^{(1)} = \frac{1}{2} \tanh \frac{1}{2kT} | 2 |J_1| z_1 \bar{S}_2^{(1)} + 2 |J_2| z_2 \bar{S}_2^{(2)} + \mu_B g^{(1)} H |, \quad \text{etc.} \quad (22)$$

Putting $H=0$, these equations are reduced to

$$\bar{S}_1^{(1)} = \bar{S}_1^{(2)} = -\bar{S}_2^{(1)} = -\bar{S}_2^{(2)} = S,$$

where S satisfies Eq. (9).

Again we shall treat the variations $\partial S_1^{(1)}$, $\partial S_1^{(2)}$, etc. of $S_1^{(1)}$, $S_1^{(2)}$ etc. due to the external field as small quantities.

(a) The calculations of the parallel susceptibility

We shall calculate the parallel susceptibility along the x -axis in the same way as was done by Van Vleck²⁾. In this case, $\alpha=1$, $\beta=0$, $\gamma=0$ and $H_x=H$, $H_y=0$, $H_z=0$, and therefore, using the relation (9), the first terms in the series expansion of (22) with respect to ∂S and H obey the following equations:

$$\begin{aligned} \partial S_{1x}^{(1)} &= -\frac{1}{2} (1-4S^2) \frac{1}{2kT} (2 |J_1| z_1 \partial S_{2x}^{(1)} + 2 |J_2| z_2 \partial S_{2x}^{(2)} + \mu_B g_{xx}^{(1)} H), \\ \partial S_{1x}^{(2)} &= -\frac{1}{2} (1-4S^2) \frac{1}{2kT} (2 |J_1| z_1 \partial S_{2x}^{(2)} + 2 |J_2| z_2 \partial S_{2x}^{(1)} + \mu_B g_{xx}^{(2)} H), \\ \partial S_{2x}^{(1)} &= -\frac{1}{2} (1-4S^2) \frac{1}{2kT} (2 |J_1| z_1 \partial S_{1x}^{(1)} + 2 |J_2| z_2 \partial S_{1x}^{(2)} + \mu_B g_{xx}^{(1)} H), \\ \partial S_{2x}^{(2)} &= -\frac{1}{2} (1-4S^2) \frac{1}{2kT} (2 |J_1| z_1 \partial S_{1x}^{(2)} + 2 |J_2| z_2 \partial S_{1x}^{(1)} + \mu_B g_{xx}^{(2)} H). \end{aligned} \quad (23)$$

Since $g_{xx}^{(1)} = g_{xx}^{(2)}$, these equations are reduced to

$$\begin{aligned} \partial S_{1x}^{(1)} &= \partial S_{1x}^{(2)} = \partial S_{2x}^{(1)} = \partial S_{2x}^{(2)}, \\ \partial S_x^{(1)} &= -\frac{\frac{1}{2}(1-4S^2) \frac{2|J|z}{2kT} \mu_B g_{xx}^{(1)} H}{1 + \frac{1}{2}(1-4S^2) \frac{2|J|z}{2kT}}. \end{aligned} \quad (24)$$

Considering further that the total magnetic moment is equal to

$$-\frac{N\mu_B}{4} \{ g_{xx}^{(1)} (\partial S_{1x}^{(1)} + \partial S_{2x}^{(1)}) + g_{xx}^{(2)} (\partial S_{1x}^{(2)} + \partial S_{2x}^{(2)}) \},$$

the parallel susceptibility $\chi_{x\parallel}$ becomes

$$\chi_{x\parallel} = \frac{N\mu_B^2 g_{xx}^{(1)2}}{1 + \frac{2kT}{|J|z} (1-4S^2)^{-1}}. \quad (25)$$

By use of the relations (1), the effective g -value for the parallel susceptibility along the x -direction is given by

$$g_x = g_{\xi} \cos^2 \alpha + g_{\eta} \sin^2 \alpha, \quad (26)$$

and similarly those for the y - and z -directions are given respectively by

$$g_y = g_{\eta}, \quad (27)$$

$$g_z = g_{\xi} \sin^2 \alpha + g_{\eta} \cos^2 \alpha. \quad (28)$$

(b) The calculation of the perpendicular susceptibility

The perpendicular susceptibility can easily be obtained by a simple geometrical consideration that the direction of the spins coincides with that of the internal field, including the applied field.

We shall first consider the case that \mathbf{S} is parallel to the x -axis and the external field H is applied to the direction of the y -axis.

Denoting the variations of $\mathbf{S}_1^{(1)}$, $\mathbf{S}_1^{(2)}$, $\mathbf{S}_2^{(1)}$ and $\mathbf{S}_2^{(2)}$ by $\partial \mathbf{S}_1^{(1)}$, $\partial \mathbf{S}_1^{(2)}$, $\partial \mathbf{S}_2^{(1)}$ and $\partial \mathbf{S}_2^{(2)}$, respectively, we have the following relations:

$$\begin{aligned} \mathbf{S} + \partial \mathbf{S}_1^{(1)} &= x \{ 2 |J_1| z_1 (-\mathbf{S} + \partial \mathbf{S}_2^{(1)}) + 2 |J_2| z_2 (-\mathbf{S} + \partial \mathbf{S}_2^{(2)}) + \mu_B \mathbf{g}^{(1)} \mathbf{H} \}, \\ \mathbf{S} + \partial \mathbf{S}_1^{(2)} &= x' \{ 2 |J_1| z_1 (-\mathbf{S} + \partial \mathbf{S}_2^{(2)}) + 2 |J_2| z_2 (-\mathbf{S} + \partial \mathbf{S}_2^{(1)}) + \mu_B \mathbf{g}^{(2)} \mathbf{H} \}, \\ -\mathbf{S} + \partial \mathbf{S}_2^{(1)} &= x \{ 2 |J_1| z_1 (\mathbf{S} + \partial \mathbf{S}_1^{(1)}) + 2 |J_2| z_2 (\mathbf{S} + \partial \mathbf{S}_1^{(2)}) + \mu_B \mathbf{g}^{(1)} \mathbf{H} \}, \\ -\mathbf{S} + \partial \mathbf{S}_2^{(2)} &= x' \{ 2 |J_1| z_1 (\mathbf{S} + \partial \mathbf{S}_1^{(2)}) + 2 |J_2| z_2 (\mathbf{S} + \partial \mathbf{S}_1^{(1)}) + \mu_B \mathbf{g}^{(2)} \mathbf{H} \}. \end{aligned} \quad (29)$$

From the x -components of these vector equations, we obtain

$$x = x' = -\frac{1}{2 |J| z}, \quad (30)$$

$$\partial S_{1x}^{(1)} = x \{ 2 |J_1| z_1 \partial S_{2x}^{(1)} + 2 |J_2| z_2 \partial S_{2x}^{(2)} \}; \text{ etc.}, \quad (31)$$

and from the y - and z -components we obtain

$$\delta S_{1y}^{(1)} = \kappa \{ 2 |J_1| z_1 \delta S_{2y}^{(1)} + 2 |J_2| z_2 \delta S_{2y}^{(2)} + \mu_R g_{yy}^{(1)} H \}, \quad \text{etc.}, \quad (32)$$

$$\delta S_{1z}^{(1)} = \kappa \{ 2 |J_1| z_1 \delta S_{2z}^{(1)} + 2 |J_2| z_2 \delta S_{2z}^{(2)} \}, \quad \text{etc.} \quad (33)$$

From the considerations of symmetry, the solutions of these equations become

$$\begin{aligned} \delta S_{1y}^{(1)} &= \delta S_{1y}^{(2)} = \delta S_{2y}^{(1)} = \delta S_{2y}^{(2)} \\ &= \frac{1}{2} \kappa \mu_R g_{yy}^{(1)} H = - \frac{\mu_R g_{yy}^{(1)} H}{4 |J| z}. \end{aligned} \quad (34)$$

Therefore the perpendicular susceptibility for this case is given by

$$\chi_{x,y,z} = \frac{N \mu_R^2 g_{yy}^{(1)2}}{2 |J| z} = \frac{N \mu_R^2 g_{yy}^2}{4 |J| z}. \quad (35)$$

The other perpendicular susceptibility can easily be calculated in a similar way. In case that \mathbf{S} is parallel to the y -axis and H is along the x -axis, the z -component of the magnetic moment appears on account of the xz -component of the g -tensor, but the effect on the susceptibility along the x -axis would become so small that it can be neglected.

§ 6. The anisotropy of the susceptibility above the Curie point

For the susceptibility above the Curie point we can apply the diagonal-sum-method used by W. Opechowski⁽⁶⁾ in his calculation of the susceptibility of $\text{CuK}_2(\text{SO}_4)_2 \cdot 6\text{H}_2\text{O}$.

The Hamiltonian for the total spin system is given by

$$H = \sum_{(lm)} \sum_{\lambda, \mu} C_{\lambda\mu}^{(lm)} S_{l\lambda} S_{m\mu} + \sum_l \sum_{\lambda} \mu_B (\mathbf{g}_\lambda^{(l)} \mathbf{H}) S_{l\lambda}, \quad (36)$$

where

$$\begin{aligned} C_{xx}^{(lm)} &= -2J_{lm} - C_{lm}(1 - 3\alpha_{lm}^2), \\ C_{xy}^{(lm)} &= 3C_{lm}\alpha_{lm}\beta_{lm}, \quad \text{etc.}, \end{aligned} \quad (37)$$

and (l, m) denotes the pair of the l -th and the m -th spins and λ and μ represent x, y and z .

According to Opechowski, the free energy F can be expressed as the following power series of $1/kT$:

$$F = -NkT \log 2 - \frac{g_1 + h_1}{kT} + \frac{h_2}{(kT)^2} + \dots, \quad (38)$$

$$g_1 = \frac{1}{32} \sum_{(lm)} \sum_{\lambda\mu} (C_{\lambda\mu}^{(lm)})^2, \quad (39)$$

$$h_1 = \frac{1}{8} \sum_l \sum_{\mu} P_{lm}^2, \quad (40)$$

$$h_2 = \frac{1}{16} \sum_{(lm)} \sum_{\lambda\mu} C_{\lambda\mu}^{(lm)} P_{l\lambda} P_{m\mu}, \quad (41)$$

$$P_{l\lambda} = \mu_B (\mathbf{g}_\lambda^{(l)} \mathbf{H}). \quad (42)$$

Three principal susceptibilities can be calculated from

$$\chi_x = -\frac{\partial^2 F}{\partial H_x^2}, \quad \chi_y = -\frac{\partial^2 F}{\partial H_y^2}, \quad \chi_z = -\frac{\partial^2 F}{\partial H_z^2}. \quad (43)$$

Using (38) to (43), χ_x becomes

$$\chi_x = \frac{\mu_B^2}{4kT} \sum_l \sum_\mu (g_{\mu l}^l)^2 - \frac{\mu_B^2}{8(kT)^2} \sum_{(lm)} \sum_{\lambda\mu} C_{\lambda\mu}^{(lm)} g_{\lambda\mu}^{(l)} g_{\mu\lambda}^{(m)}, \quad (44)$$

$$\begin{aligned} \sum_l \sum_\mu (g_{\mu l}^l)^2 &= \frac{N}{2} \{g_{xx}^{(1)2} + g_{xx}^{(2)2} + g_{yx}^{(1)2} + g_{yx}^{(2)2} + g_{zx}^{(1)2} + g_{zx}^{(2)2}\} \\ &= N(g_\xi^2 \cos^2 u + g_\zeta^2 \sin^2 u). \end{aligned} \quad (45)$$

Here we shall adopt only the interactions between nearest spins. Then we have

$$\begin{aligned} &\sum_{(lm)} \sum_{\lambda\mu} C_{\lambda\mu}^{(lm)} g_{\lambda\mu}^{(l)} g_{\mu\lambda}^{(m)} \\ &= \frac{N}{2} \left[\frac{1}{2} \{2 |J_1| z_1 - C_1(1-3\alpha_1^2)z_1\} (g_{xx}^{(1)2} + g_{xx}^{(2)2}) + \frac{1}{2} \{2 |J_1| z_1 - C_1(1-3\beta_1^2)z_1\} \right. \\ &\quad \times (g_{yx}^{(1)2} + g_{yx}^{(2)2}) + \frac{1}{2} \{2 |J_1| z_1 - C_1(1-3\gamma_1^2)z_1\} (g_{zx}^{(1)2} + g_{zx}^{(2)2}) + \{2 |J_2| z_2 \\ &\quad - C_2(1-3\alpha_2^2)z_2\} g_{xx}^{(1)} g_{xx}^{(2)} + \{2 |J_2| z_2 - C_2(1-3\beta_2^2)z_2\} g_{yx}^{(1)} g_{yx}^{(2)} + \{2 |J_2| z_2 \\ &\quad - C_2(1-3\gamma_2^2)z_2\} g_{zx}^{(1)} g_{zx}^{(2)} \Big] \\ &= \frac{N}{2} \left[(2 |J| z + C_x) (g_\xi^2 \cos^2 u + g_\zeta^2 \sin^2 u)^2 + \{(2 |J_1| z_1 - 2 |J_2| z_2) \right. \\ &\quad \left. - C_1(1-3\gamma_1^2)z_1 + C_2(1-3\gamma_2^2)z_2\} (g_\xi - g_\zeta)^2 \sin^2 u \cos^2 u \right]. \end{aligned} \quad (46)$$

Substituting (45) and (46) into (44), we obtain

$$\begin{aligned} \chi_x &= \frac{N\mu_B^2}{4kT} (g_\xi^2 \cos^2 u + g_\zeta^2 \sin^2 u) \\ &\quad - \frac{N\mu_B^2}{16(kT)^2} \left[(2 |J| z + C_x) (g_\xi^2 \cos^2 u + g_\zeta^2 \sin^2 u)^2 \right. \\ &\quad \left. + \{2 |J_1| z_1 - 2 |J_2| z_2 - C_1(1-3\gamma_1^2)z_1 + C_2(1-3\gamma_2^2)z_2\} \right. \\ &\quad \left. \times (g_\xi - g_\zeta)^2 \sin^2 u \cos^2 u \right]. \end{aligned} \quad (47)$$

Since the Curie temperature of $\text{CuCl}_2 \cdot 2\text{H}_2\text{O}$ lies at about 5°K , the first term of (47) may be considered as giving the susceptibility at room temperature to a considerably good approximation. Then the effective g -values become

$$g_e = (g_\xi^2 \cos^2 u + g_\zeta^2 \sin^2 u)^{\frac{1}{2}}, \quad (48)$$

and similarly we obtain

$$g_y = g_\eta, \quad (49)$$

$$g_z = (g_\xi^2 \sin^2 u + g_\zeta^2 \cos^2 u)^{\frac{1}{2}}. \quad (50)$$

Here it is to be noted that these g -values are different from those below the Curie point given by Eqs. (26), (27) and (28).

The spectroscopic splitting factors as measured by experiments on magnetic resonance absorption correspond to the g -values given by (48), (49) and (50). Such experiments at room temperature have been made by J. Itoh and his collaborators⁷⁾, who obtained the following results:

$$\begin{aligned} (g_{\xi}^2 \cos^2 \alpha + g_{\eta}^2 \sin^2 \alpha)^{\frac{1}{2}} &= 2.195, \\ g_{\eta} &= 2.075, \\ (g_{\xi}^2 \sin^2 \alpha + g_{\eta}^2 \cos^2 \alpha)^{\frac{1}{2}} &= 2.260. \end{aligned} \quad (51)$$

According to these results, the values of g do not deviate from 2 so much.

§ 7. The rotation of spins under a comparatively strong field

We shall calculate the critical field strength at which the direction of the spins rotates from their preferred axis to the direction perpendicular to it. In this calculation, we shall assume g as a scalar quantity having a value of 2, and that the next preferred direction of the spins lies in the direction of the y -axis, that is,

$$C_x > C_y > C_z. \quad (52)$$

Thus the rotation of the spins occurs in the xy -plane, and therefore we can put $\gamma=0$. From Eq. (16), the anisotropy energy in this plane can be expressed as

$$f_{an.} = -\frac{S^2}{2} (C_x - C_y) \sin^2 \theta, \quad (53)$$

where θ means the angle between the direction of the spins and the x -axis. Now putting $D' = (S^2/2)(C_x - C_y)$, the equation (27) in the previous paper³⁾, giving the critical field, can also be applied to this case. We have

$$H_c^2 = \frac{2ND'}{\chi_{\perp} - \chi_{\parallel}}. \quad (54)$$

At sufficiently low temperatures we can put

$$\chi_{\parallel} = 0, \quad D' = \frac{1}{8} (C_x - C_y), \quad (55)$$

and using (55) and (10), the critical field strength at the absolute zero of temperature is given by

$$H_c(0) = \left[\frac{k\theta(C_x - C_y)}{2\mu_B^2} \right]^{\frac{1}{2}}. \quad (56)$$

Putting $\theta = 5^\circ\text{K}$ and $H_c = 7000$ oersteds in this expression, we obtain for $C_x - C_y$

$$C_x - C_y \cong 0.06 \text{ cm}^{-1}. \quad (57)$$

This value shows that the absolute value of C is, in its order, a hundredth of that of

$2J/z$ and so the assumption $C \ll 2|J|z$ is reasonable. From (55), D' is estimated as $D' \sim 0.008 \text{ cm}^{-1}$, which is considerably small when compared with the corresponding value 0.6 cm^{-1} of MnF_2 .

In consequence of the temperature dependency of the anisotropy constant through Eq. (10), the critical field increases with rising temperature, as in the case of MnF_2 .

§ 8. The antiferromagnetic resonance frequency

The theory of antiferromagnetic resonance absorption has been developed by T. Nagamiya⁸⁾ who treated the case of the uniaxial or cubic symmetry. However his theory can easily be extended to the orthorhombic crystal. In this section, we shall adopt the notations used in Nagamiya's paper.

According to Eq. (16), the anisotropy energy of the total crystal can be written as

$$F_{\text{an.}} = \frac{1}{2} (K_1 \beta^{+2} + K_2 \gamma^{+2}) + \frac{1}{2} (K_1 \beta^{-2} + K_2 \gamma^{-2}), \quad (58)$$

where $K_1 = (NS^2/2)(C_x - C_y)$; and $K_2 = (NS^2/2)(C_x - C_z)$ and β^\pm and γ^\pm are the direction cosines of $+$ spins and $-$ spins, respectively. From (52), we obtain the following relation:

$$K_2 > K_1 > 0. \quad (59)$$

The x , y , z components of the torque derived from (58) acting on the magnetic moments \mathbf{M}^+ and \mathbf{M}^- of the two $+$ and $-$ sublattices are given by

$$\begin{aligned} & \left(\frac{\partial F_{\text{an.}}}{\partial \beta^\pm} \gamma^\pm - \frac{\partial F_{\text{an.}}}{\partial \gamma^\pm} \beta^\pm, \quad \frac{\partial F_{\text{an.}}}{\partial \gamma^\pm} u^\pm - \frac{\partial F_{\text{an.}}}{\partial u^\pm} \gamma^\pm, \quad \frac{\partial F_{\text{an.}}}{\partial u^\pm} \beta^\pm - \frac{\partial F_{\text{an.}}}{\partial \beta^\pm} u^\pm \right) \\ & = (- (K_2 - K_1) \gamma^\pm \beta^\pm, \quad K_2 u^\pm \gamma^\pm, \quad -K_1 u^\pm \beta^\pm). \end{aligned} \quad (60)$$

(1) The case that the direction of the spins lies in the preferred axis

Since the magnetic moments \mathbf{M}^\pm in this case take an orientation almost parallel to the x -axis, (60) can be written as

$$\begin{aligned} & \left(-\frac{(K_2 - K_1)}{M_0} \frac{M_z^\pm M_y^\pm}{M_0}, \quad \pm \frac{K_2}{M_0} M_z^\pm, \quad \mp \frac{K_1}{M_0} M_y^\pm \right) \\ & \simeq \left(0, \quad \pm \frac{K_2}{M_0} M_z^\pm, \quad \mp \frac{K_1}{M_0} M_y^\pm \right). \end{aligned} \quad (61)$$

Adopting this form as the torque arising from the anisotropy energy, the equations (8) in Nagamiya's paper giving the oscillation part of the equation of motion become

$$\begin{aligned} \frac{1}{\gamma} \frac{d\delta\mathbf{M}}{dt} &= \delta\mathbf{M} \times \mathbf{H} + \mathbf{M} \times \delta\mathbf{H} + \left(0, \quad \frac{K_2}{M_0} \delta M_z', \quad -\frac{K_1}{M_0} \delta M_y' \right), \\ \frac{1}{\gamma} \frac{d\delta\mathbf{M}'}{dt} &= \delta\mathbf{M}' \times (\mathbf{H} - A\mathbf{M}) + \mathbf{M}' (\delta\mathbf{H} - A\delta\mathbf{M}) \\ &+ \left(0, \quad \frac{K_2}{M_0} \delta M_z, \quad -\frac{K_1}{M_0} \delta M_y \right), \end{aligned} \quad (62)$$

where $\mathbf{H}-\mathbf{AM}=(aH_x, 0, 0)$, and $a=1-\chi_{\parallel}/\chi_{\perp}$.

When the static magnetic field is applied to the x -axis, we can put $H_x=H$, $H_y=H_z=0$, and $\mathbf{M}'=(2M_0, 0, 0)$. In this case (62) leads to the following resonance condition :

$$\frac{\omega^4}{\gamma^4} - \frac{\omega^2}{\gamma^2} \{ (a^2 + 1)H^2 + 2A(K_1 + K_2) \} + a^2 H^4 - 2A(K_1 + K_2)aH^2 + 2AK_1 \cdot 2AK_2 = 0. \quad (63)$$

Putting $H_y=H$, $H_x=H_z=0$, and $\mathbf{M}'=(2M_0, 0, 0)$ in the case of the applied static field parallel to the y -axis, we obtain the following resonance frequencies :

$$\frac{\omega^2}{\gamma^2} = 2AK_2, \quad \frac{\omega^2}{\gamma^2} = H^2 + AK_1. \quad (64)$$

If we put $K_1=K_2$ in (63) and (64), we obtain the results derived by Nagamiya.

(2) Next, we shall consider the case that the magnetic moments have the direction parallel to the y -axis in consequence of the stronger static magnetic field applied to the x -axis than the critical field strength.

In this case the torque due to the anisotropy field can be written as

$$\begin{aligned} & \left(\mp \frac{(K_2 - K_1)}{M_0} M_x^{\pm}, \quad \frac{K_2}{M_0} \frac{M_x^{\pm} M_z^{\pm}}{M_0}, \quad \mp \frac{K_1}{M_0} M_x^{\pm} \right) \\ & \sim \left(\mp \frac{K_2 - K_1}{M_0} M_x^{\pm}, \quad 0, \quad \mp \frac{K_1}{M_0} M_x^{\pm} \right). \end{aligned} \quad (65)$$

Then the equations corresponding to (62) are

$$\begin{aligned} \frac{1}{\gamma} \frac{d\delta\mathbf{M}}{dt} &= \delta\mathbf{M} \times \mathbf{H} + \mathbf{M} \times \delta\mathbf{H} + \left(-\frac{K_2 - K_1}{M_0} \delta M_z', \quad 0, \quad -\frac{K_1}{M_0} \delta M_x' \right), \\ \frac{1}{\gamma} \frac{d\delta\mathbf{M}'}{dt} &= \delta\mathbf{M}' \times (\mathbf{H} - \mathbf{AM}) + \mathbf{M}' \times (\delta\mathbf{H} - A\delta\mathbf{M}) \\ & \quad + \left(-\frac{K_2 - K_1}{M_0} \delta M_z, \quad 0, \quad -\frac{K_1}{M_0} \delta M_x \right); \end{aligned} \quad (66)$$

where $\mathbf{H}-\mathbf{AM}=(0, aH_y, 0)$, and $\mathbf{M}'=(0, 2M_0, 0)$.

Putting $H_x=H$ and $H_y=H_z=0$, the resonance frequencies are given by

$$\frac{\omega^2}{\gamma^2} = H^2 - 2AK_1, \quad \frac{\omega^2}{\gamma^2} = 2A(K_2 - K_1). \quad (67)$$

In this case it is to be noted that the static field H is larger than the critical field H_c which is given by $(2AK_1/a)^{1/2}$.

§ 9. Conclusion

The treatment above may be summarized as follows. $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$ manifests an anti-ferromagnetic behavior below its Curie point which is at about 5°K. Three possible

superstructures arising from the arrangement of the spin orientations are considered. One of them, the case I in Fig. 3, would produce a spontaneous magnetization along the z -axis, since the xz -components of the g -tensors of the spins situated on plus and minus sublattices have opposite signs. In the remaining two superstructures, the case II and the case III, the spontaneous magnetizations on the two sublattices compensate with each other. Although it is obscure which of these two superstructures corresponds to the actuality at the present stage, and we must await experiments by neutron diffraction or by nuclear magnetic resonance⁹⁾ for the elucidation of this point, we arbitrarily assumed the case III to be the most stable structure, and on this basis calculated the anisotropy of the susceptibility and the temperature dependency of the anisotropy constant. In these calculations, we adopted the general form of the anisotropic interaction energy between nearest similar (with the same g -tensor) and dissimilar (with different g -tensors) spins. It was found possible through the introduction of such interactions between only nearest neighbors and through a suitable choice of the parameter values that the easy direction coincides with the x -axis. In addition, we calculated the absolute value of the anisotropy constant, referring to the data of the critical field strength measured by C. J. Gorter *et al.* The antiferromagnetic resonance frequency of this substance was also calculated on the basis of Nagamiya's theory.

In conclusion, the writer wishes to express his cordial thanks to Professor T. Nagamiya for his kind advices on this problem.

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The Unitary Transformation and the Quantization

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Particle mechanics classical as well as quantum being observed to be representations of the unitary transformation of finite dimensions, field mechanics is formulated as the representation of the unitary transformation of infinite dimensions.

§ 1. The contact transformation

The contact transformation $(q^r, p^r \rightarrow q'^r, p'^r)$, q^r, p^r being canonical variables, is defined as the transformation such that

$$\sum p^r dq^r - \sum p'^r dq'^r = \text{a perfect differential.}$$

Since the *dérivée extérieure* of a perfect differential vanishes,

$$\sum (dq^r \delta p^r - dp^r \delta q^r) = \sum (dq'^r \delta p'^r - dp'^r \delta q'^r).$$

Therefore the contact transformation may be defined as the transformation that leaves unaltered the bilinear form, which is called the skew product of the two contravariant vectors $(dq^1, \dots, dq^n, dp^1, \dots, dp^n)$ and $(\delta q^1, \dots, \delta q^n, \delta p^1, \dots, \delta p^n)$. In other words, the contact transformation is the symplectic transformation discussed in detail in H. Weyl's "Classical Groups". Hence, some theorems in classical mechanics take clearer aspects.

When q^r, p^r are regarded as functions of two parameters u, v , $(\partial q^r / \partial u, \partial p^r / \partial u)$ and $(\partial q^r / \partial v, \partial p^r / \partial v)$ are two contravariant vectors, whose skew product

$$\sum \left(\frac{\partial q^r}{\partial u} \frac{\partial p^r}{\partial v} - \frac{\partial p^r}{\partial u} \frac{\partial q^r}{\partial v} \right) = (u, v)$$

is an invariant of the contact transformation, which is called a Lagrange bracket.

Since the differential

$$dF = \sum \left(\frac{\partial F}{\partial q^r} dq^r + \frac{\partial F}{\partial p^r} dp^r \right)$$

of a scalar function $F(q, p)$ is an invariant, $(\partial F / \partial q^r, \partial F / \partial p^r)$ is a covariant vector.

With regard to the symplectic transformation a covariant vector $(x_r, x_{r'})$ can be changed into a contravariant vector $(x_r, -x_{r'})$, a contravariant vector $(y^r, y^{r'})$ into a covariant vector $(y^{r'}, -y^r)$.

Hence $(\partial F / \partial p^r, -\partial F / \partial q^r)$ is a contravariant vector. Therefore we have tensor equations of the contact transformation

$$\frac{dq^r}{ds} = \frac{\partial F}{\partial p^r}, \quad \frac{dp^r}{ds} = -\frac{\partial F}{\partial q^r}$$

which are called Hamilton's canonical equations, provided that F represents the energy of a mechanical system, s the time.

The skew product

$$\Sigma \left(\frac{\partial F}{\partial q^r} \frac{\partial G}{\partial p^r} - \frac{\partial F}{\partial p^r} \frac{\partial G}{\partial q^r} \right) = [F, G]$$

of the two contravariant vectors $(\partial F / \partial p^r, -\partial F / \partial q^r)$ and $(\partial G / \partial p^r, -\partial G / \partial q^r)$, the latter of which is derived from another scalar function $G(q, p)$, is also an invariant of the contact transformation, which is called a Poisson bracket.

If we take a set of $2n$ functions $(F^r, F^{r'})$, the differentials of which $(dF^r, dF^{r'})$ constitute the components of a contravariant vector, the Poisson brackets $[F^\rho, F^\sigma]$ with ρ, σ ranging over $1, 2, \dots, n, 1', \dots, n'$, constitute the components of an antisymmetric tensor of the second rank.

If we put $F^r = q^r$, $F^{r'} = p^r$, the tensor of the second rank turns out to be

$$\left. \begin{aligned} [q^r, q^s] &= 0, & [p^r, p^s] &= 0 \\ [q^r, p^s] &= \delta^{rs}. \end{aligned} \right\} \quad (1)$$

In other words the relations (1) are tensor relations of the contact transformation.

§ 2. The unitary transformation in particle mechanics

The contact transformation group has a subgroup that is linear and unitary. Changing the variables q^r, p^r into x^r, x_r defined by

$$x^r = q^r + i\lambda p^r, \quad x_r = q^r - i\lambda p^r$$

λ being a real constant, we can represent a transformation S of the subgroup as such that

$$x \rightarrow Sx, \quad x^\dagger \rightarrow x^\dagger S^\dagger = x^\dagger S^{-1},$$

$$S^\dagger S = 1$$

where $x = (x^r)$ is a contravariant vector of the unitary transformation, (x_r) a covariant vector.

With respect to the unitary transformation, the relations (1) may be written

$$[x^r, x^s] = 0, \quad [x_r, x_s] = 0, \quad [x^r, x_s] = -2i\lambda \delta_s^r.$$

The transition from classical mechanics to quantum mechanics may be carried out by establishing the commutation relations on a set of canonical variables q^r, p^r or x^r, x_r .

The commutators

$$F^{rs} = x^r x^s - x^s x^r, \quad F_{rs} = x_r x_s - x_s x_r, \quad F_s^r = x^r x_s - x_s x^r,$$

constructed with canonical variables are tensors of rank 2 of the unitary transformation.

If they are required to be invariant under unitary transformations, one sees that

$$F^{rs}=0, \quad F_{rs}=0, \quad F_s^r=a\delta_s^r,$$

a being a real coefficient.

Returning to the old variables one gets

$$q^r q^s - q^s q^r = 0, \quad p^r p^s - p^s p^r = 0,$$

$$q^r p^s - p^s q^r = ia/2\lambda \cdot \delta_s^r.$$

In other words, commutation relations in quantum particle mechanics are observed to be tensor equations of the unitary transformation. The observation will help us in quantum field mechanics.

§ 3. Transition to field mechanics

While particle mechanics deals with the mechanical systems of finite degrees of freedom, field mechanics treats the mechanical systems of infinite degrees of freedom.

To treat the systems of infinite degrees of freedom, the representation coordinate will be introduced.

With respect to the unitary transformation of finite dimensions, the components of a vector are labeled with the indices ranging over $1, 2, \dots, n$. When the dimension of the transformation becomes infinite, the eigenvalues of an hermitian operator or a set of commuting hermitian operators $\hat{\xi}$, ranging from $-\infty$ to ∞ will label the components of a tensor.

So the components of tensors will be denoted as follows

$$(\hat{\xi}|U) \quad : \quad \text{contravariant vector,}$$

$$(V|\hat{\xi}) \quad : \quad \text{covariant vector,}$$

$$(\hat{\xi}|U|\hat{\xi}') \quad : \quad \text{mixed tensor of rank 2 etc..}$$

The unitary transformation S induces linear transformations on the tensors

$$(\hat{\xi}|U) \quad \rightarrow \quad (\hat{\xi}|S|\hat{\xi}')(\hat{\xi}'|U),$$

$$(V|\hat{\xi}) \quad \rightarrow \quad (V|\hat{\xi}')(\hat{\xi}'|S^{-1}|\hat{\xi}),$$

$$(\hat{\xi}|U|\hat{\xi}') \quad \rightarrow \quad (\hat{\xi}|S|\hat{\xi}'')(\hat{\xi}''|U|\hat{\xi}''')(\hat{\xi}'''|S^{-1}|\hat{\xi}')$$

where the signs of integration with respect to dummy suffices are omitted conforming to the rule of tensor calculus.

The unitarity of the transformation S is represented by the condition

$$S^\dagger S = 1$$

or

$$(\hat{\xi}|S^\dagger|\hat{\xi}'')(\hat{\xi}''|S|\hat{\xi}') = (\hat{\xi}|\hat{\xi}').$$

So far the $\hat{\xi}$ representation is used. When the representation coordinate is changed from the $\hat{\xi}$ representation to the η representation, η being another hermitian operator with

the eigenvalues ranging from $-\infty$ to ∞ , the tensors undergo linear substitutions

$$\begin{aligned}(\xi|U) &\rightarrow (\eta|U) = (\eta|\xi) (\xi|U), \\(V|\xi) &\rightarrow (V|\eta) = (V|\xi) (\xi|\eta), \\(\xi|U|\xi') &\rightarrow (\eta|U|\eta') = (\eta|\xi) (\xi|U|\xi') (\xi'|\eta')\end{aligned}$$

$(\xi|\eta)$, $(\eta|\xi)$ being the transformation functions complex conjugate to each other.

§ 4. The commutation relations

In § 2, the commutation relations in quantum particle mechanics have been observed to be invariant under unitary transformations of finite dimensions. If the commutation relations in quantum field mechanics are required also to be invariant under unitary transformations of infinite dimensions, they will be almost uniquely determined except for numerical coefficients.

When the two operators z , z^\dagger hermite conjugate to each other satisfy the commutation relation $zz^\dagger - \rho z^\dagger z = 1$, the eigenvalues of the operator $z^\dagger z$ are 0, 1, $1+\rho$, $1+\rho+\rho^2$, \dots . A permissible value of ρ will be then either 1 or -1 , provided that an eigenvalue of $z^\dagger z$ can be interpreted as a number of particles in a state. The case $\rho=1$ corresponds to the Boson field, the case $\rho=-1$ to the Fermion field.

To set up the commutation relations, we impose two postulates

- 1) field variables referring to different degrees of freedom commute,
- 2) the commutation relations have invariant forms under unitary transformations of infinite dimensions.

We have then

$$\left. \begin{aligned}1) \quad &(\xi|U)(\xi'|U) - \rho(\xi'|U)(\xi|U) = 0 \\ &(U^\dagger|\xi)(U^\dagger|\xi') - \rho(U^\dagger|\xi')(U^\dagger|\xi) = 0, \\ &(\xi|U)(U^\dagger|\xi') - \rho(U^\dagger|\xi')(\xi|U) = a(\xi|\xi'),\end{aligned} \right\} \rho^2=1 \quad (2)$$

$$\left. \begin{aligned}2) \quad &(\xi|U|\xi'')(\xi'|U|\xi''') - (\xi'|U|\xi''')(\xi|U|\xi'') = 0, \\ &(\xi|U^\dagger|\xi'')(\xi'|U^\dagger|\xi''') - (\xi'|U^\dagger|\xi''')(\xi|U^\dagger|\xi'') = 0, \\ &(\xi|U|\xi'')(\xi'|U^\dagger|\xi''') - (\xi'|U^\dagger|\xi''')(\xi|U|\xi'') = a(\xi|\xi'')(\xi'|\xi''')\end{aligned} \right\} \quad (3)$$

etc.,

a being a real constant.

When an operator is hermitian, $A^\dagger = A$, the commutators thereof all vanish. In the case, we decompose A into the sum of two hermite-conjugate operators as $A = a + a^\dagger$ and we get the rule (3) for the components $(\xi|a|\xi')$, $(\xi|a^\dagger|\xi')$.

§ 5. The field equations

Non-quantized field equations have been derived from the invariant Lagrangian with the help of the variation principle. The variation principle however has been replaced by

the Hamiltonian formalism in the quantization procedure.

If the relativity of representation coordinates is admitted as in the preceding paper (this journal, 6, 5), the signs of integration and differentiation all disappear from the expressions of the field theories, since the differentiation operator $\partial/\partial \xi$ is to be replaced by the operator π , satisfying the commutation relation $\pi \xi - \xi \pi = 1$, and the signs of integration is to be omitted following the rule of tensor calculus. Therefore the field equations provided by the variation principle are equivalent to the vanishing of the partial derivatives of the Lagrangian with respect to each of the independent field variables.

As an example, we take the case of a scalar field, the Lagrangian thereof is given in usual notations

$$L = \int_{-\infty}^{\infty} (dx)^4 \left\{ g^{ij} \frac{\partial U^\dagger}{\partial x^i} \frac{\partial U}{\partial x^j} + m^2 U^\dagger U \right\}$$

which is written in tensor notations

$$L = (U^\dagger | x) (x | Q | x') (x' | U), \quad Q = g^{ij} p_i p_j + m^2$$

where $U(x) = (x | U)$, $U^\dagger(x) = (U^\dagger | x)$ are counted for contravariant and covariant vectors respectively, and p_j stand for the momentum operators satisfying the commutation relations

$$x^i p_j - p_j x^i = i \delta_j^i.$$

L may be written in the ξ representation

$$L = (U^\dagger | \xi) (\xi | Q | \xi') (\xi' | U)$$

that provides the field equations

$$L_{(U^\dagger | \xi)} = (\xi | Q | \xi') (\xi' | U) = 0 \quad \text{or} \quad Q U = 0,$$

$$L_{(\xi' | U)} = (U^\dagger | \xi) (\xi | Q | \xi') = 0 \quad \text{or} \quad U^\dagger Q = 0,$$

with the aid of the variation principle, for the variation principle postulates the Lagrangian to be extremum for realizable fields.

If the commutation relations (2) are imposed on $(\xi | U)$, $(U^\dagger | \xi)$, the commutators of L with $(\xi | U)$, $(U^\dagger | \xi)$ turn out to be

$$\left. \begin{aligned} [L, (\xi'' | U)] &= [(U^\dagger | \xi) (\xi' | U), (\xi'' | U)] (\xi | Q | \xi') \\ &= -[(\xi'' | U) (U^\dagger | \xi) - \rho (U^\dagger | \xi) (\xi'' | U)] (\xi' | U) (\xi | Q | \xi') \\ &= -a (\xi'' | \xi) (\xi' | U) (\xi | Q | \xi') \\ &= -a (\xi'' | Q | \xi') (\xi' | U), \\ [L, (U^\dagger | \xi'')] &= (U^\dagger | \xi) [(\xi' | U) (U^\dagger | \xi'') - \rho (U^\dagger | \xi'') (\xi' | U)] (\xi | Q | \xi') \\ &= (U^\dagger | \xi) a (\xi' | \xi'') (\xi | Q | \xi') \\ &= a (U^\dagger | \xi) (\xi | Q | \xi'') \end{aligned} \right\} \quad (4)$$

where the equality

$$[AB, C] = A(BC - \rho CB) - (CA - \rho AC)B$$

is used.

The relations (4) suggest us a general formula

$$[L, (\xi|U)] = -aL_{(U^\dagger|\xi)}, \quad [L, (U^\dagger|\xi)] = aL_{(\xi|U)}$$

§ 6. The variation principle

If the Lagrangian of a system of the fields satisfy the conditions

- 1) hermiticity,
 - 2) Lorentz invariance,
 - 3) invariance under the unitary transformation as well as the representation transformation,
 - 4) to be bilinear in pairs of the Fermion field variables hermite conjugate to each other,
- there exists a general formula

$$[L, F] = -aL_{F^\dagger}, \quad [L, F^\dagger] = aL_F$$

F, F^\dagger being a pair of field variables hermite conjugate to each other.

Let Ψ be the state vector on which field variables operate, Φ its adjoint vector, then field equations will be

$$\Phi[L, F]\Psi = 0, \quad \Phi[L, F^\dagger]\Psi = 0 \quad (5)$$

for each of the independent field variables. In other words, the expectation values of the partial derivatives of the Lagrangian L with respect to each of the independent field variables will vanish.

The solution of the simultaneous equations (5) can be reduced to the solution of the eigenvalue problem of the operator L as was mentioned in the preceding paper.

There is another way to get the result. Normalizing $\Phi\Psi$ to be unity, we require $\Phi L\Psi$ to be extremum for realizable states. We have then

$$\delta(\Phi L\Psi) = 0$$

with an additional condition

$$\delta(\Phi\Psi) = 0.$$

Hence, we have, introducing an indeterminate multiplier l ,

$$\begin{aligned} & \delta(\Phi L\Psi) - l\delta(\Phi\Psi) \\ &= \delta\Phi(L\Psi - l\Psi) + (\Phi L - \Phi l)\delta\Psi = 0. \end{aligned}$$

Therefore we get

$$L\Psi - l\Psi = 0, \quad \Phi L - \Phi l = 0.$$

This process is a revival of the variation principle. Thus the postulate of the relativity of representation coordinates leads us to the standpoint of view from which we are forced to gaze upon physical phenomena in their entireties (sub specie unitatis; $\epsilon\lambda\zeta \mu\acute{\iota}\alpha\nu \iota\delta\acute{\epsilon}\alpha\nu$).

The Meson Theory of Nuclear Forces, I*

— The Deuteron Ground State and Low Energy Neutron-Proton Scattering —

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Adopting the new method of the theory of nuclear forces proposed by one of us (M.T) and others, the deuteron ground state and low energy neutron-proton scattering have been investigated. The symmetrical pseudoscalar π -meson potentials including second- and fourth-order terms are assumed in the outside region, while in the inside region where the static meson potential becomes meaningless, phenomenological potentials represented by square well are adopted. It is shown that we can then account for the experimental data, if we take the value 0.09~0.10 as the magnitude of the coupling constant between π -meson and nucleon. Saturation does not result from these potentials in the static approximation.

§ 1. Introduction and summary

Although it is well known that the problem of nuclear forces is one of the most important in nuclear theory, little is known with certainty of their detailed natures. This problem has so far been attacked essentially from two different quarters. On the one hand simple assumptions about the forces are made which account for as many facts as possible, such as binding energies, stability rules and various data from low and high energy scattering experiments. This is the so called "phenomenological" theory of nuclear forces. On the other hand it has been attempted to derive the nuclear forces from field theory and this is the meson theory of nuclear forces firstly suggested by Yukawa.

Since π -meson were produced artificially, many important natures of mesons have been clarified. Especially, it has been made clear that π -mesons interact strongly with nucleons and most part of the nuclear forces are due to π -mesons. In spite of the fact that each phenomenological nuclear potentials proposed so far can explain the experimental results in each energy region within which they are assumed, one know little about the relations between these phenomenological natures of nuclear forces and the properties of π -meson which would be essential to these forces. Therefore, in the present stage, the analysis of nuclear forces from the point of view of meson theory is much preferable to the phenomenological method and it is necessary to make clear how far we can explain experimental results about nuclear forces assuming the present meson theory.

* Preliminary reports have been published in this journal.¹⁾⁻³⁾

In this paper we shall treat the deuteron ground state and low energy neutron-proton scattering. Many investigations have been made on the same problem assuming various forms of interactions between neutron and proton.

A new method proposed by one of us (M.T) and others¹⁾ for attacking the problem of nuclear forces is employed in the present paper. Briefly speaking, the static π -meson potential is assumed in the outside region, and in the inside region where the static meson potential becomes meaningless, phenomenological potential represented, for simplicity, by the square well is adopted. The latter stands for various effects which become important at small distances of the nucleon, namely the effects of non-static forces, higher-order terms, strong coupling and heavier mesons, etc., and is so adjusted to fit the experimental results according as each process and according as each energy region.*

As the potential assumed outside we have adopted the symmetrical p_s^{**} meson potential with both p_s and p_v couplings in the static approximation.*** In fact, recent experiments^{6), 7)} on the process $\pi^+ + d \rightleftharpoons p + p$ and π^- -capture by deuterium definitely show that charged π -mesons are pseudoscalar. Though the type of neutral π -meson is not yet well known beyond the fact that they have the spin zero, it is much plausible that both charged and neutral π -mesons have the same type, viz., pseudoscalar. Taking these facts into consideration, besides the charge independent nature of nuclear forces established at least in the low energy region,^{8) - 10)} symmetrical theory was assumed. We have, therefore, calculated neutron-proton scattering only.

Methods of calculation used throughout this paper are developed in § 2. They are direct numerical integrations of the wave equations using the Störmer-Levy method, essentially same as that used by Bethe.¹¹⁾

Using these methods, we shall at first investigate the deuteron ground state and low energy neutron-proton scattering under the assumption of second-order meson potential outside in § 3. From the results which are obtained there, we suggest strongly spin dependent nature of the inside potentials.

The largest correction to these second order meson potential assumed in § 2 is probably the effect due to fourth-order potential. Higher-order terms will be negligible in the outside region, because the range of $2n$ -th order meson potential is $1/n$ times of that of the second-order.^{4), 12)}

The effects of the fourth-order potentials will be discussed in § 4. Deriving the fourth-order p_s meson potential in Part (A) of that section by the S -matrix method firstly developed by Nambu¹³⁾, some qualitative but important features of the p_s meson potential including second- and fourth-order terms are given in Part (B), and we have confirmed the inference about the inside potential as has been expected in § 2.

Results of calculations assuming these p_s meson potential are given in Part (C). We also consider the saturation property of nuclear forces in that section.

* Estimated value of the critical distance where static approximation breaks down, is about half or one third of force range.⁵⁾

** Hereafter we shall use the following abbreviations: p_s pseudoscalar, p_v pseudovector.

*** Concerning the validity of weak coupling theory, see references 4), 5) and 12).

The analysis of high energy nucleon-nucleon scattering will be developed in Part II of our paper using the same method as that of the present Part I.

§ 2. Computational methods

Various experimental results of the low energy region (including some derived quantities) employed in this paper are as follows:—

- a) binding energy of the deuteron: $|e| = 2.227(1 \pm 0.0015) \text{ Mev.}^{14)}$
- b) electric quadrupole moment of the deuteron: $Q = (2.738 \pm 0.016) \times 10^{-27} \text{ cm}^2.^{15)*}$
- c) singlet effective range for neutron-proton scattering: $^1r = (2.7 \pm 0.5) \times 10^{-13} \text{ cm.}^{14)**}$
- d) triplet effective range for neutron-proton scattering:
 $^3r = (1.704 \pm 0.030) \times 10^{-13} \text{ cm.}^{14)}$
- e) singlet scattering length for neutron-proton scattering:
 $^1a = -23.68(1 \pm 0.0025) \times 10^{-13} \text{ cm.}^{14)}$
- f) triplet scattering length for neutron-proton scattering:
 $^3a = 5.388(1 \pm 0.0045) \times 10^{-13} \text{ cm.}^{14)}$
- g) radius of the deuteron: $r_d = 4.314 \cdot (1 \pm 0.0008) \times 10^{-13} \text{ cm.}^{14)}$
- h) mass of charged π -meson: $m_\pi = (276 \pm 6) m_e.^{17)}$

The value chosen here gives for the π -meson Compton wave length $\lambda^{-1} = \hbar/m_\pi c = (1.40 \pm 0.03) \times 10^{-13} \text{ cm}$ which is also the range of the π -meson potential.

The general form of the static interaction potential adopted in this paper is

$$V(x) = V_o(x) + V_i(x) \cdot S_{12}, \quad (1)$$

where x is the relative position vector for the neutron-proton system measured in units of λ^{-1} , while S_{12} is the usual tensor operator

$$S_{12} = [3(\sigma_1 \cdot x)(\sigma_2 \cdot x)/x^2] - (\sigma_1 \cdot \sigma_2). \quad (2)$$

(A) Low energy singlet S scattering

The radial wave function u of the singlet S state for zero energy obeys the wave equation

$$d^2u(x)/dx^2 - W_o(x)u(x) = 0, \quad (3)$$

where the notation $W_o(x) = (M/x^2) \cdot V_o(x)$ is introduced and M is the nucleon mass.***

As has been stated in § 1, we consider the two alternative ways of cutting-off:

- a) "zero cut-off"¹¹⁾

$$W_o(x) = 0 \quad \text{for } x < x_0, \quad (4a)$$

- b) infinite repulsive well inside (hard core model)^{2), 13)}

$$W_o(x) = +\infty \quad \text{for } x < x_0. \quad (4b)$$

* Newell has obtained a different value for the quadrupole moment of deuteron, see ref. 16).

** Singlet effective range for proton-proton scattering coincides with that for neutron-proton scattering within the experimental error.

*** We adopt the natural units $\hbar=c=1$ throughout this paper.

x_0 is the cut-off radius where the meson potential assumed in the outside region breaks down.⁴⁾

The above wave equation is to be solved subject to the boundary conditions

$$u(x=0)=0. \quad (5)$$

$$u(x \rightarrow \infty) \sim 1 - ({}^1a)^{-1}r = 1 - ({}^1a)^{-1}x^{-1} \cdot x. \quad (6)$$

The integration for $x < x_0$ being elementary, we have for zero cut-off

$$u = cx \quad (7a)$$

with an arbitrary constant c and for hard core inside

$$u = 0. \quad (7b)$$

For $x > x_0$, direct numerical integrations (Stormer-Levy method) have been carried out from the outside in, starting from the boundary condition (6) at large x . Since the potential $V(x)$ increases rapidly with decreasing x , the transformations

$$y = \log x, \quad U = u \cdot x^{-1/2} \quad (8)$$

are very convenient for such calculations.¹¹⁾

Then the wave equation (3) reduces to the form

$$d^2U/dy^2 = (x^2 W_c + 1/4) \cdot U. \quad (3')$$

The starting point for numerical integration was in general $y=1.65$, corresponding to $x=5.207$, while the interval was chosen equal to 0.15 units of the natural logarithm.

Cut-off radius x_0 is determined by the continuity condition, a smooth join of wave function and its derivative to the interior solution (7a) or (7b). This condition is for zero cut-off

$$(1/U) \cdot (dU/dy) = 1/2 \text{ at } x = x_0 \quad (9a)$$

and for hard core inside

$$U = 0 \text{ at } x = x_0. \quad (9b)$$

Singlet effective range 1r is expressed in terms of the wave function $u(x)$ by the expression¹⁰⁾

$${}^1r = 2 \int_0^\infty \{ [1 - ({}^1ax)^{-1}x]^2 - u^2(x) \} dx \quad (10)$$

where $u(x)$ is normalized so that it approaches the asymptotic form (6) at large distances.

(B) The deuteron ground state and low energy triplet S scattering

The wave function for the deuteron ground state may be written as a linear combination of S and D states

$$\psi = (1/x) [u(x) + (1/8^{1/2}) \cdot S_{12} \cdot w(x)] \chi_m \quad (11)$$

where χ_m is the spin function with magnetic quantum number m . Then, if we make the following abbreviations,

$$\begin{aligned} \gamma^2 &= (M/x^2) \cdot |\epsilon|, \quad A = \gamma^2 + (M/x^2) \cdot V_c(x), \\ B &= -2(M/x^2) V_t(x), \end{aligned} \quad (12)$$

$u(x)$ and $w(x)$ satisfy the two simultaneous differential equations

$$d^2u(x)/dx^2 = Au(x) - \sqrt{2}Bw(x) \quad (13a)$$

$$d^2w(x)/dx^2 = (A+B+6/x_2) \cdot w(x) - \sqrt{2}Bu(x). \quad (13b)$$

We consider again the two alternatives for cutting-off:

a) zero cut-off

$$A=B=0 \quad \text{for } x < x_0, \quad (14a)$$

b) infinite repulsive well inside

$$A=B=+\infty \quad \text{for } x < x_0. \quad (14b)$$

Numerical integrations in the region from $x=5.207$ to $x=x_0$ were the same as that of (A). The wave functions $u(x)$ and $w(x)$ thus obtained were normalized according to

$$\int_0^\infty \{u^2(x) + w^2(x)\} \cdot dx = 1. \quad (15)$$

Then the quadrupole moment

$$Q = (\sqrt{2}/10x^2) \int_0^\infty x^2 [uw - (1/2\sqrt{2})w^2] dx \quad (16)$$

and D -state probability p_D

$$p_D = \int_0^\infty w^2(x) \cdot dx \quad (17)$$

are readily calculated using the above wave functions. We can also calculate the triplet effective range 3r using the expression²⁰⁾

$$^3r = 2 \int_0^\infty [\exp(-2\gamma x) - (u^2 + w^2)] dx. \quad (18)$$

Here the normalization of $(u^2 + w^2)$ must be chosen so that it approaches $\exp(-2\gamma x)$ as $x \rightarrow \infty$.

§ 3. Results of the calculations using the second-order ρs meson potential

The second-order ρs meson potential with ρs coupling does not appear in the static approximation. So we have the ρv coupling term only,

$$V_c^{(2)} = -x \cdot g^2/4\pi \cdot \exp(-x)/x, \quad (19)$$

$$V_t^{(2)} = -x \cdot g^2/4\pi \cdot (1 + 3/x + 3/x^2) \cdot \exp(-x)/x, \quad (20)$$

where the prime "2" indicates that $V^{(2)}$ is the second-order potential.

For $x < x_0$, we adopted for convenience' sake, the hard core*

$$V_c = V_t = +\infty \quad \text{for } x < x_0. \quad (21)$$

* Ferretti has also treated the same problem assuming repulsive potential inside.²¹⁾ We are indebted to Prof. Ferretti for sending us a reprint of his paper.

The results are given in Table 1.

Table 1. Second order π -meson potential outside and hard core inside

$g^2/4\pi$	x_0	Q (in 10^{-27} cm ²)	f_D (%)
0.025	0.122	1.07	4.2
0.050	0.228	1.877	6.3
0.075	0.387	2.542	7.4
0.084*	0.458	2.766	7.54

* These values were obtained by extrapolation.

of the π -meson potential is somewhat shorter than that of phenomenological potentials with short tail.⁴⁾ The quadrupole moment decreases rapidly with decreasing force range. Therefore, to get the correct magnitude of quadrupole moment, we must assume a large value for coupling constant and, as a necessary consequence, the D -state probability increases together.²⁰⁾ Concerning the relation between the magnetic moment of deuteron and the D -state probability, we shall discuss further in next section.

The repulsive potential assumed in the inside region plays an important role. That is, the wave function is pushed out, so the quadrupole moment and triplet effective range increase in comparison with the other cutting-off method.^{1), 22)}

From these results, we can infer that the potential at small distances where the second-order meson potential breaks down may be fairly repulsive for triplet S state.

On the other hand, it is impossible to account for the experimental value of singlet scattering length provided that the value of the coupling constant is so chosen as to fit the correct value for the deuteron quadrupole moment. From our standpoint, we must adopt the same potential with the same value of coupling constant in the outside region for all states. Therefore, the inside potential for singlet even state must necessarily attractive in contrast to the strong repulsion required in the triplet even state. This inference is confirmed by the calculation of fourth-order meson potential in next section.

§ 4. Fourth-order meson potential

(A). Derivation of fourth-order meson potential*

Fourth-order terms of the static ps meson theory have been calculated by the method of S -matrix.¹³⁾ From the Feynman diagrams (Fig. 1), we get the results up to first order of the expansion in terms of (m_π/M) as follows:²⁾

$$\begin{aligned}
 V_{ps}^{(4)} &= x \cdot \left(\frac{f^2}{4\pi} \right)^2 \frac{1}{8\pi} \left(\frac{m_\pi}{2M} \right)^2 \cdot \sum_{k,l=1}^3 [\rho_2 \tau_k, \rho_2 \tau_l]_+^{(1)} [\rho_2 \tau_k, \rho_2 \tau_l]_+^{(2)} \frac{K_1(2x)}{x^2} \\
 &= x \cdot \left(\frac{f^2}{4\pi} \right)^2 \frac{3}{8\pi} \left(\frac{m_\pi}{M} \right)^2 \frac{K_1(2x)}{x^2},
 \end{aligned} \tag{22}$$

* K. Nishijima has obtained the same results for pv coupling, using a different method of calculation (canonical transformations). See, reference¹²⁾.

$$\begin{aligned}
 V_{pv}^{(4)} &= \chi \cdot \left(\frac{g^2}{4\pi} \right)^2 \cdot \sum_{k,l=1}^3 [(\sigma \nabla) \tau_k, (\sigma \nabla) \tau_l]^{(1)} [(\sigma \nabla) \tau_k, (\sigma \nabla) \tau_l]^{(2)} \frac{K_0(2x)}{x} \\
 &= \chi \cdot \left(\frac{g^2}{4\pi} \right)^2 [(\tau^{(1)} \tau^{(2)}) U_\tau(x) + (\sigma^{(1)} \sigma^{(2)}) U_\sigma(x) + S_{12} U_T(x)]. \quad (23)
 \end{aligned}$$

If one attempts to carry out this calculation relativistically, unrenormalizable divergences appear in the pv coupling case. In the static approximation, however, these divergences disappear fortunately. Furthermore, we can reduce the divergences resulting from the diagrams shown in Fig. 2 to renormalizable terms and to contact interaction terms, provided that the static approximation is allowed.¹²⁾

Under these inevitable circumstances arising from the incompleteness of the present field theory, we could not help confining ourselves to the static approximation. Important terms such as spin-orbit coupling will probably result from non-static part of the fourth-order potentials, but unambiguous calculation is very difficult in the present stage of the field theory.

As can be seen from the commutator in the expression of $V^{(4)}$, fourth-order meson potentials are purely quantum mechanical effects.⁵⁾

(B). Qualitative natures of the meson potential including fourth-order terms

Assuming that the value of the ps coupling constant $f^2/4\pi$ is not much larger than that of the pv coupling, $g^2/4\pi$, we confine ourselves to the pv coupling only.* Then the fourth-order meson potentials (23) reduces to the following form in each state:

a) triplet even state

$$V_c^{(4)} = \chi (g^2/4\pi)^2 \cdot (-3U_\tau + U_\sigma), \quad (24a)$$

$$V_t^{(4)} = \chi (g^2/4\pi)^2 \cdot U_T, \quad (24b)$$

b) singlet even state

$$V_c^{(4)} = \chi (g^2/4\pi)^2 \cdot (U_\tau - 3U_\sigma), \quad (25)$$

c) triplet odd state

$$V_c^{(4)} = \chi (g^2/4\pi)^2 \cdot (U_\tau + U_\sigma), \quad (26a)$$

$$V_t^{(4)} = \chi (g^2/4\pi)^2 \cdot U_T, \quad (26b)$$

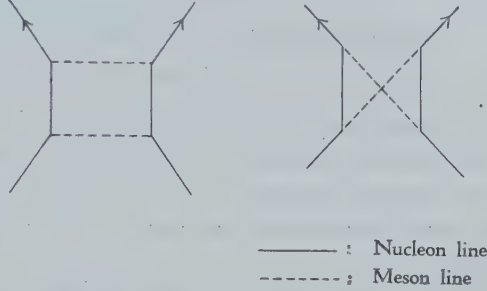


Fig. 1.



Fig. 2.

* Since $V_{ps}^{(4)}$ is an ordinary (Wigner) type, neglect of this term does not alter the following results essentially.

d) singlet odd state

$$V_o^{(4)} = -x(g^2/4\pi)(3U_\tau + 3U_o). \quad (27)$$

Here U_o , U_τ and U_T are defined using the modified Bessel functions K_0 and K_1^* ,

$$U_o = 8/\pi \{ (3/x^2)K_0(2x) + (2/x^2 + 3/x^4)K_1(2x) \}, \quad (28)$$

$$U_\tau = -8/\pi \{ (1/x + 23/4x^3)K_0(2x) + (2/x^2 + 23/4x^4)K_1(2x) \}, \quad (29)$$

$$U_T = -8/\pi \{ (3/x^3)K_0(2x) + (1/x^2 + 15/4x^4)K_1(2x) \}. \quad (30)$$

These potentials are shown in Figs. 3, 4, 5 and 6 respectively.

In the same figures, the second-order potentials $V^{(2)}$ and the resultant potentials $V = V^{(2)} + V^{(4)}$ are illustrated together.

From these figures, one can see some qualitative but very important features of the ps meson potential.

1). In the triplet even state, central force is strongly repulsive and tensor force is strongly attractive, being similar to that of Bethe's neutral vector meson theory.¹¹⁾ This repulsive potential was represented in last section by the infinite repulsive well assumed in the inside region.

2). Strong attractive force in the singlet even state may account for the experimental value for the singlet scattering length. We have suggested this attractive force in last

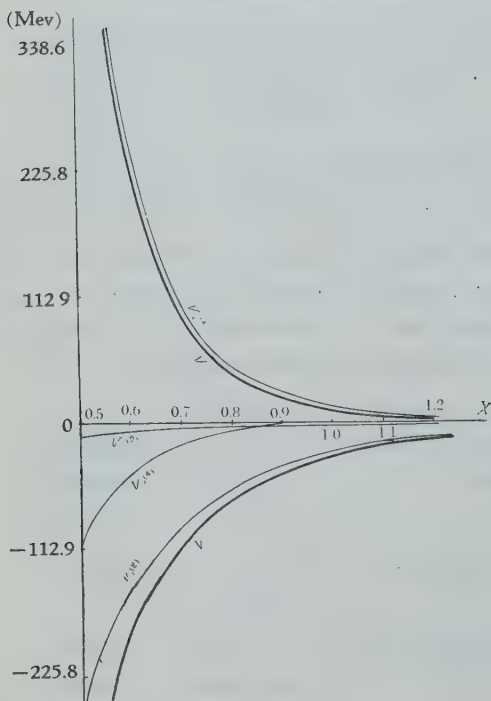


Fig. 3 Triplet even states $g^2/4\pi=0.08$

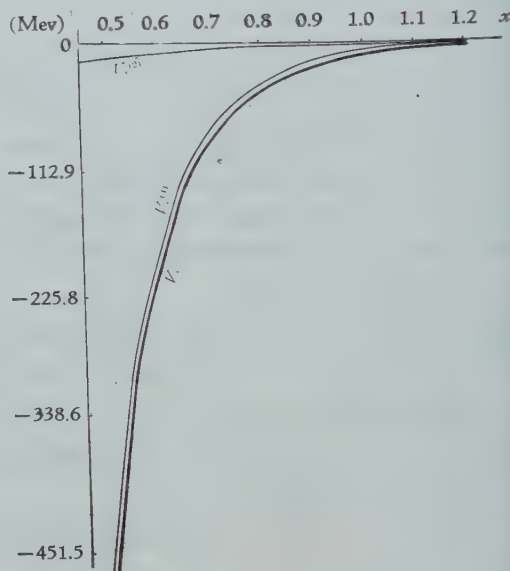


Fig. 4 Singlet even state $g^2/4\pi=0.08$

* U_τ and U_o are negative while U_T is positive. Roughly speaking, $|U_o| \sim U_T \sim 1/2|U_\tau|$.

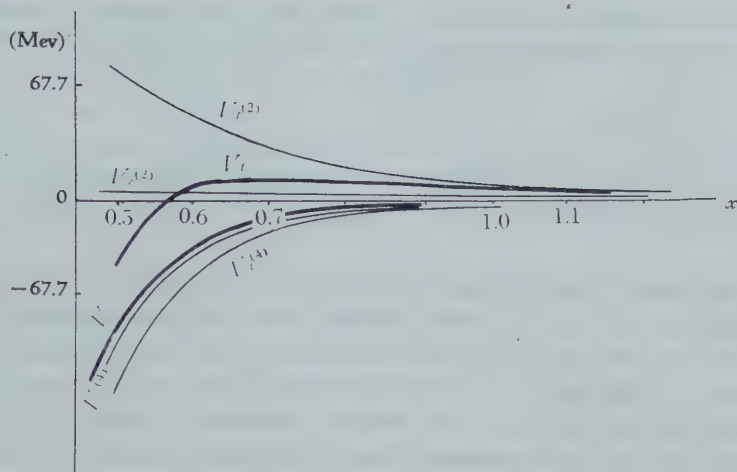


Fig. 5 Triplet odd state $g^2/4\pi=0.08$

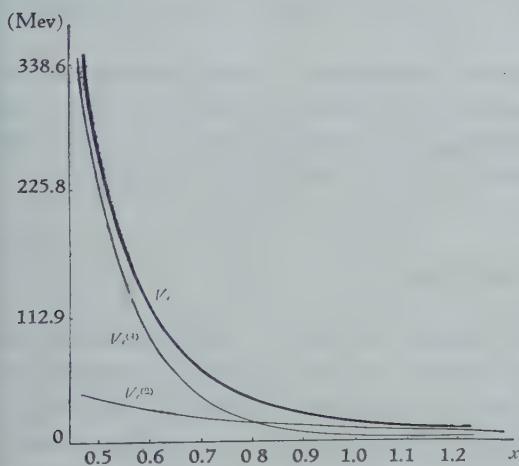


Fig. 6 Singlet odd state $g^2/4\pi=0.08$

section.

3). Low and high energy proton-proton scattering may be explained fairly well if a hard core is assumed inside, since these potentials are similar to those adopted by Jastrow.¹⁸⁾

4). The fact that odd state interactions are much weaker than those of the even state is just what is required for accounting for the experimental angular distribution of high energy neutron-proton scattering.

We shall discuss the high energy neutron-proton scattering in the subsequent paper (Part II).

(C). Results of calculations and discussions

Tables 2 and 3 show the results which have been obtained using the above potentials $V=V^{(2)}+V^{(4)}$.

Table 2a. Singlet neutron-proton scattering;
zero cut-off

$g^2/4\pi$	r_1 (in 10^{-12} cm)	x_0
0.075	1.925	0.5886
0.090	2.457	0.665

Table 2b. Singlet neutron-proton scattering;
infinite repulsive well inside

$g^2/4\pi$	r_1	x_0
0.075	2.104	0.3285
0.090	2.585	0.384

Table 3. The deuteron ground state and triplet neutron-proton scattering; zero cut-off *

$g^2/4\pi$	x_0	Q (in 10^{-27}cm^2)	3γ (in 10^{-13}cm)	p_D (%)
0.05	0.16	1.85	1.325	7.42
0.08	0.232	2.37	1.582	8.65
0.10	0.303	3.25	1.75	11.3

* Infinite repulsive potential assumed inside does not give a correct binding energy for deuteron.

As is well known, the percentage of D -state wave function in the ground state of deuteron is closely related to the magnetic moment, and the value 4% obtained from this relation has been adopted so far. This relation forms, however, an unreliable restriction, not only because of the uncertain relativistic corrections²³, but also of a dependence of the magnetic moment of one nucleon on the proximity of another. Recently, the latter effect has been confirmed by the experimental ratio of hyperfine structure in deuterium and hydrogen.²⁴ Taking these facts into consideration, it seems meaningless to adhere to the value 4%, although the correct value is not known.** Therefore, the D -state probability obtained here, viz., 9~10%, is not necessarily incompatible with the present experimental data.

Whether this potential satisfies the saturation requirements or not has also been investigated. Rough estimation shows that conditions for saturation²⁵ break down in several respects. For example, spin saturated neutron cluster may be stable. Even if we take into account the term of ps coupling, not all saturation requirements cannot be satisfied.

§ 5. Conclusions

From the above results, it is possible to conclude that the ps meson potentials with pv coupling including second- and fourth-order terms may well account for the experimental data for the deuteron ground state and low energy neutron-proton scattering, provided that suitable potentials are assumed in the inside region. The best value of the coupling constant between π -meson and nucleon is 0.09~0.10.

Saturation does not result on the assumption of the ps meson potential only in the static approximation.

** Note added in proof: Recently, H. Miyazawa has estimated the magnitude of D -state admixture. The value obtained here is not in contradictory to his results. See H. Miyazawa, Prog. Theor. Phys. 7 (1952), No. 2.

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Remarks on the Adiabatic Nuclear Potential

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We discuss on the validity and approximation method of the adiabatic nuclear potentials, especially of the fourth order nuclear potential derived from pseudoscalar meson theory⁵⁾. In § 2, it is shown that effects of time component of pseudovector coupling on the pseudoscalar meson potential are not large when distance between two nucleons is larger than about half of the force range. In § 3, time variation of nucleon spin and τ -spin are evaluated for phenomenological and meson potentials, and critical radii for their applicability are derived. In § 4, implications of the velocity-dependent forces in the interior region are discussed, and largeness of the purely quantum mechanical effects in the meson theory is pointed out. Hamiltonian for a two nucleon system in the intermediate coupling approximation is given. In § 5, features of the meson-nucleon interaction are compared with that of the photon-electron interaction. It is pointed out that the transition matrix of the nucleon-nucleon collision is not analytic with respect to nucleon energy at the threshold for meson production. An ambiguity occurs when one intend to calculate the sixth order nuclear potential.

§ 1. Introduction

Recently experiments on nucleon-nucleon scattering at various energies have been performed and the nature of nuclear forces especially at high energies has become clearer. So it is a quite important problem to study to what extent the meson theory can give an unified description of the nature of both the artificially produced mesons (i.e. real mesons) and the nuclear forces (i.e. virtual mesons). At present the qualitative properties of mesons are analyzed on the group theoretical basis rather than on the questionable field theory.

General principles which hold regardless of the approximations are the following:

- (1) conservation laws
(of energy-momentum, angular momentum, parity etc.),
- (2) gauge invariance
(the direct consequence of the charge conservation law),
- (3) Lorentz invariance.

These laws only are enough to determine the transformation properties of the pi-mesons, which we shall state simply in the following:

- (a) the charged pi-meson

The spin of the charged pi-meson was determined as being equal to zero by the detailed balancing consideration of the following processes¹⁾;

$$\pi^+ + d \rightleftharpoons p + p.$$

Next it was shown with the aid of Pauli exclusion principle that the charged pi-meson

cannot be scalar in explanation of the neutron absorption of the negative pi-meson in deuterium²⁾, i.e.

$$\pi^- + d \rightarrow n + n.$$

Thus we can conclude that the charged pi-meson is pseudoscalar.

(b) The neutral pi-meson

The neutral pi-meson cannot have spin 1 since it decays into two photons³⁾; i.e.

$$\pi^0 \rightarrow 2\gamma,$$

and the same parity with the charged pi-meson is favoured from the mesic absorption of the negative pi-meson in hydrogen⁴⁾, i.e.

$$\pi^- + p \rightarrow \pi^0 + n.$$

Thus the neutral pi-meson seems to be also pseudoscalar.

The correctness of the choice of the pseudoscalar meson theory in our former calculations⁵⁾ is now guaranteed.

For the convenience of the phenomenological descriptions of mesonic phenomena, we usually employ

(4) S matrix formalism.

Since the energies of artificially produced mesons are comparatively low at present, we can employ as a suitable approximation

(5) the expansion in energies

(i.e. shape independent approximation, partial wave analysis).

Moreover, if we confine ourselves only to the non-radiative mesonic phenomena we may assume as the most probable coupling

(6) the symmetrical theory.

Utilizing these general laws only, Watson and Brueckner⁶⁾ were able to discuss the meson production in nucleon-nucleon-collisions in a quite beautiful manner.

But, when we once intend to study the properties of pi-mesons one step further from these qualitative investigations, we inevitably face the incomplete meson theory, so that it is necessary to explore the difficulties and the limit of applicability of the present meson theory. From such a point of view, we shall investigate the nuclear forces on symmetrical pseudoscalar meson theory by the perturbation method. In this case, because of the singular natures of the pseudoscalar meson (simply denote P s meson hereafter) that the lowest order terms differ from higher order ones even in qualitatively, we cannot draw any definite conclusion from the lowest order results. Besides, in higher order terms divergences arise which cannot be renormalized⁷⁾ and this difficulty prevents us to draw even the qualitative conclusions. Since these divergent terms become larger as the energy increases, we shall be obliged to do with only low energy phenomena not to be suffered seriously from these defects. Along these lines we shall search for restrictions on the validity of the adiabatic nuclear potentials in the following sections.

§ 2. The role of the matrix γ_5 in nuclear forces

The most severe limitation upon the static approximation employed before⁵⁾ results from the omission of the time component of the pseudovector ($p\nu$) coupling, and so we will estimate its contribution to the nuclear potential. For the sake of simplicity, we shall be content with a rough qualitative discussion for this term brings divergent contributions beyond the renormalization technique, that is, clearly outside the applicability of the present field theory.

Now the spatial and temporal parts in the $p\nu$ -coupling are given respectively by

$$\frac{\mathcal{E}}{\kappa} \phi^* \sigma \phi \cdot \nabla \phi, \quad \frac{\mathcal{E}}{\kappa} \phi^* \gamma_5 \phi \frac{\partial \phi}{\partial t}. \quad (1)$$

The temporal part is quite small compared with the spatial part for virtual mesons of low enough energies, because the temporal part of the $p\nu$ -coupling is almost equal to the ps coupling at low energies, i.e.

$$\frac{\mathcal{E}}{\kappa} \phi^* \gamma_5 \phi \frac{\partial \phi}{\partial t} \sim \mathcal{E} \phi^* \gamma_5 \phi \cdot \phi \quad (2)$$

and as is well known from the equivalence theorem, the contribution of the ps coupling to the second order nuclear forces is only $(\kappa/2M)^2$ of the $p\nu$ coupling.

In other words, the contribution of the temporal part to the nuclear forces is negligibly small as compared with the spatial part near the force range. While if it is expressed in the momentum space, the temporal part surpasses the spatial part in the high energy region.

But high energy virtual mesons contribute to the potential only inside of the cut off radius by the special role of the matrix γ_5 to be shown in the following, so that the contribution of the temporal part will still be inferior up to about half the force range.

Well, the spatial part of the $p\nu$ coupling contains the nucleon spin matrix σ , and so the nucleon can emit or absorb virtual mesons even at low energies by changing the spin orientation.

On the other hand, as the temporal part of the $p\nu$ coupling contains the queer matrix γ_5 , the nucleon cannot emit or absorb mesons unless the nucleon receives a large momentum transfer or a change of energy sign. Thus the uncertainty of energy caused by the temporal part will be much larger than that by the spatial part, i.e.

$$\Delta E_{\text{temp.}} \gg \Delta E_{\text{spat.}}$$

The ratio of the temporal to the spatial will be of the order of magnitude M/κ , so the lives of the virtual states caused by the temporal coupling are quite shorter than those by the spatial coupling.

In this way, the meson clouds induced by the coupling γ_5 shrink and are diluted in the outside region, namely the matrix γ_5 plays the role to shorten the force range of the nuclear forces. We shall illustrate this role of the matrix γ_5 by comparing the scalar and pseudoscalar potentials :

(1) Second order potentials

$$\begin{aligned}
 S(s) : & \quad - (f^2/4\pi) x (\tau^{(1)} \tau^{(2)}) \exp(-x)/x, & (x=xr) \\
 P_s(ps) : & \quad (f^2/4\pi) x (x/2M)^2 (\tau^{(1)} \tau^{(2)}) \left[\frac{1}{3} (\sigma^{(1)} \sigma^{(2)}) + \left(\frac{1}{3} + \frac{1}{x} + \frac{1}{x^2} \right) S_{12} \right] \frac{e^{-x}}{x}.
 \end{aligned} \tag{3}$$

Because of the factor $(x/2M)^2$ the scalar potential is much larger than the pseudoscalar one in the outside region where the potential does not show its singular feature yet. Since $S(s)$ and $P_s(ps)$ differ only by a matrix γ_5 , the role of the matrix γ_5 that it contracts the meson clouds is well illustrated by this example.

For the sake of completeness, we shall also write down the fourth order potentials:

(2) Fourth order potentials⁸⁾

$$\begin{aligned}
 S(s) : & \quad - \left(\frac{f^2}{4\pi} \right)^2 x (\tau^{(1)} \tau^{(2)}) \frac{4}{\pi} \frac{K_0(2x)}{x}, \\
 P_s(ps) : & \quad \left(\frac{f^2}{4\pi} \right)^2 x \left(\frac{x}{M} \right)^2 \cdot \frac{3}{8\pi} \frac{K_1(2x)}{x^2}.
 \end{aligned} \tag{4}$$

In this case, the situation is completely the same with the second order, and may also be valid in the higher orders.

Thus we have qualitatively justified the correctness of the omission of the temporal part outside of the cut off radius (about half the force range).

§ 3. Non-static effects

Next let us investigate the validity of the adiabatic potentials in which the velocity dependence is completely neglected.

So long as we bear our discussions on the meson theory, the potential inevitably becomes velocity dependent due to the nucleon recoil effects, so that there is the limit of applicability of the adiabatic approximation. The nuclear forces can be well approximated by the adiabatic potentials for comparatively large separations, but not for small separations because of the increase of the non-static effects due to the nucleon recoil.

According to Rosenfeld⁹⁾, we shall call this border separation as "the critical distance" and denote by R_c . We can employ following two methods to estimate this distance:

criterion (I): To search for the distance where the static approximation becomes inconsistent.

criterion (II): To examine the validity of the static approximation by comparing the magnitude of the static forces with that of the non-static forces.

Of course, it is clear that the latter criterion is more favourable than the former, but the latter requires the information on the velocity-dependent forces which is too complicated to compute, so that we shall be content with the former for the time being.

The non-static effects result mainly from the temporal part of the $p\nu$ coupling we have omitted, but since its contribution is shown to be small for comparatively large separations we may assume that the non-static forces caused by the temporal coupling are

masked by those of the spatial coupling for the separations we are concerned about. This viewpoint will be justified by the fact that both the force range of the temporal coupling and the critical distance are of the same order and about half the force range of the spatial coupling in the $P_s(pv)$ theory. But notice: "Even if it were shown by the criterion (I) that the static approximation would be valid for considerably small separations, we cannot, in general, conclude that the static approximation is surely correct in this case, for the criterion (II) is more essential."

Taking into consideration of the above arguments, we shall study how to estimate the non-static effects according to the criterion (I). In our former calculations, we have obtained velocity independent potentials by neglecting the time dependence of the bilinear forms of nucleon wave functions such as

$$\phi^* \sigma \phi \quad (\text{since } \sigma \text{ is an even matrix})$$

as compared with the time dependence of the meson wave functions. (Contrary to the above case, we can no more neglect the time dependence of the bilinear forms such as $\phi^* \gamma_5 \phi$, which therefore leads to essentially velocity dependent forces.) Thus our approximation is valid only to the regions where the precession frequencies of the nucleon spin σ are not larger than the meson frequency (energy divided by \hbar). As for the isotopic spin τ , we see completely the same state of affairs.

If we formulate the problem mathematically, we are to compute such a distance for which

$$|\Omega| \text{ or } |\omega| \sim x,$$

where Ω , ω are the precession angular velocities of the ordinary and isotopic spins, and x is the rest mass of the meson.

Now we shall expand the non-relativistic Hamiltonian of a two nucleon system in nucleon velocities.

$$\begin{aligned} H &= T + H_{\text{int}}, & (T: \text{kinetic energy of two nucleons}) \\ H_{\text{int}} &= H^0 + H^1 + \dots, \end{aligned} \tag{5}$$

where

$$H^0 = O\left(\left(\frac{v}{c}\right)^0\right), \quad H^1 = O\left(\left(\frac{v}{c}\right)^1\right), \dots$$

Then the equations of motion of spins are given by

$$\dot{\sigma}^{(1)} = i[H, \sigma^{(1)}], \quad \dot{\tau}^{(1)} = i[H, \tau^{(1)}]. \tag{6}$$

As we are interested in the distance where H^0 is dominant we may roughly replace H by H^0 , so the equations of motion in first approximation become

$$\dot{\sigma}^{(1)} = i[H^0, \sigma^{(1)}], \quad \dot{\tau}^{(1)} = i[H^0, \tau^{(1)}]. \tag{6'}$$

The above equations will be enough to do with the rough order estimation, and we can

take the adiabatic nuclear potential as H^0 . Then the equations (6') can be written in the following forms:

$$\dot{\sigma}^{(1)} = \Omega^{(1)} \times \sigma^{(1)}, \quad \dot{\tau}^{(1)} = \omega^{(1)} \times \tau^{(1)}. \quad (7)$$

Namely, σ and τ precess with the angular velocities Ω and ω respectively. Taking into account of the linearity of H^0 with respect to $\sigma^{(1)}$ and $\tau^{(1)}$, we can readily obtain the following relations regardless of the charge dependence of the potentials:

$$\Omega^{(1)} = 2 \frac{\partial H^0}{\partial \sigma^{(1)}}, \quad \omega^{(1)} = 2 \frac{\partial H^0}{\partial \tau^{(1)}}. \quad (8)$$

We now define the critical distance R_c as such a distance that for $r > R_c$, we have $x \gg |\Omega|, |\omega|$. This distance R_c indicates the critical separation that the velocity dependent forces turn out to be superior, and we can roughly estimate R_c if H^0 is given.

Such a criterion may be applicable even to the phenomenological potentials so long as we bear our interpretation of nuclear forces on the mesons.

In cases of charge independent theories, H^0 contains σ and τ only in the following forms:

$$(\sigma^{(1)} \sigma^{(2)}), \quad (\sigma^{(1)} r) (\sigma^{(2)} r), \quad (\tau^{(1)} \tau^{(2)}).$$

The contributions of the first and last terms to $\dot{\sigma}$ and $\dot{\tau}$ are given respectively by

$$\sigma^{(1)} \times \sigma^{(2)}, \quad \tau^{(1)} \times \tau^{(2)}.$$

These terms contribute nothing in the spin and charge singlet states respectively. Especially in the symmetrical theory, $\dot{\tau}$ identically vanishes in the charge singlet states because of the absence of tensor forces in the isotopic space.

We shall estimate the critical distance R_c for the following potentials:

(1) Symmetrical $P_S(pv)$ theoretical Potential

$$V = x \cdot \frac{g^2}{4\pi} (\tau^{(1)} \tau^{(2)}) \left[\frac{1}{3} (\sigma^{(1)} \sigma^{(2)}) + \left(\frac{1}{3} + \frac{1}{x} + \frac{1}{x^2} \right) S_{12} \right] \frac{e^{-x}}{x} \\ + x \cdot \left(\frac{g^2}{4\pi} \right)^2 [(\tau^{(1)} \tau^{(2)}) U_\tau + (\sigma^{(1)} \sigma^{(2)}) U_\sigma + S_{12} U_T].$$

As for the definitions of U_τ , U_σ and U_T , see ref. (5).

(2) Christian-Hart Potential⁽¹⁰⁾

$$^1V = -^1V_c \cdot \frac{1+P_x}{2} \frac{r_0}{r} e^{-r/r_0}, \quad (\text{for singlet states}) \\ ^3V = -^3V_c \cdot \frac{1+P_x}{2} \frac{r_0}{r} e^{-r/r_0} - ^3V_t (0.37 + 0.63 P_x) \frac{r_0}{r} e^{-r/r_0} \cdot S_{12}, \\ (\text{for triplet states})$$

where $r_0 = 1.35 \times 10^{-13}$ cm, $^1V_c = 35.3$ Mev., $^3V_c = 25.3$ Mev., $^3V_t = 48.2$ Mev.

(3) Christian-Noyes Potential⁽¹¹⁾

Here we use the parameters changed by Swanson⁽¹²⁾. Since $n-p$ potential is the same with the Christian-Hart potential, we shall only write down the proton-proton potential.

$${}^1V = -{}^1V_0 \frac{1+P_x}{2} \text{ for } r < r_1 \text{ and } {}^1V = 0 \text{ for } r > r_1, \text{ (singlet states)}$$

$${}^3V = -{}^3V_0 \frac{1+P_x}{2} \frac{r_2}{r} e^{-r/r_2} - {}^3V_t \frac{1+P_x}{2} \frac{r_2}{r} e^{-r/r_2} S_{12} - {}^3V_s \frac{1-P_x}{2} \left(\frac{r_3}{r} \right)^2 e^{-r/r_3} S_{12} \cdot$$

(triplet states)

$$r_1 = 2.615 \times 10^{-13} \text{ cm}, r_2 = 1.35 \times 10^{-13} \text{ cm}, r_3 = 1.6 \times 10^{-13} \text{ cm}, {}^1V_0 = 13.273 \text{ Mev},$$

$${}^3V_0 = 25.3 \text{ Mev}, {}^3V_t = 48.3 \text{ Mev}, {}^3V_s = 15.25 \text{ Mev}.$$

(4) Jastrow Potential⁽¹³⁾

$$V = - \left[\frac{1+P_x}{2} {}^3V + \frac{1-P_x}{2} {}^1V \right],$$

$${}^3V = \left\{ \frac{1+P_x}{2} + (0.3 + 0.7P_x)\gamma S_{12} \right\} {}^3V_0 e^{-r/r_t},$$

$${}^1V = \begin{cases} -\infty, & r < r_0, \\ {}^1V_0 \frac{1+P_x}{2} e^{-(r-r_0)/r_s}, & r > r_0, \end{cases}$$

$${}^3V_0 = 69 \text{ Mev}, r_t = 0.75 \times 10^{-13} \text{ cm},$$

$${}^1V_0 = 375 \text{ Mev}, r_s = 0.40 \times 10^{-13} \text{ cm}, r_0 = 0.6 \times 10^{-13} \text{ cm}.$$

The results are as follows:

(A) Singular Potentials ($P_S(pv)$, Christian-Noyes, Jastrow)

$$\alpha R_c \sim (0.4 \sim 0.5).$$

(B) Non Singular Potentials (Christian-Hart etc.)

$$\alpha R_c \sim (0.2 \sim 0.3).$$

For singular potentials R_c can sharply be determined because of their steepness, but rather vaguely for non-singular potentials.

By the way, we shall test the nuclear forces on the strong coupling approximation. In this case, the nuclear forces lose their spin (and isotopic spin) dependent character for small separations as the spins are frozen, and they become spin dependent only for large separations where forces are no more strong. Therefore we see always

$$|\mathcal{Q}|, |\omega| < \alpha,$$

and the critical distance does not exist in this case.

§ 4. Implication of the velocity-dependent forces in the interior region

Combining the discussions in the previous sections with the properties of proton-proton scatterings at high energies known from recent experiments, we shall investigate the approximation methods in the meson theory of nuclear forces. In section 3, we see that the self-consistency of the static approximation is guaranteed even for considerably small separations in the cases of non-singular potentials, while it breaks down at once in the cases of singular potentials. These results are due to the criterion (I).

At the first glance, nuclear forces on the strong coupling approximation seems superior to that on the weak coupling approximation provided that the static approximation will do for the whole region, but it is not true indeed, for we beared on the criterion (I) rather than the more definite criterion (II).

Then what can we conclude from the latter criterion?

To answer this question, we shall utilize experimental informations, as the theoretical prediction concerning the velocity dependent forces is absent.

Now it is well known that the singular potentials of the type (A) is more favourable in interpreting the data on the proton-proton scattering at high energies than the non-singular ones.

How shall we interpret this situation?

Well, the nuclear forces at high energies are strongly dependent on the character of the potentials of the interior region, about which we have no theoretical information at present since it is outside the frame of the present field theory, and so as a possible means to guess the character, we may estimate by extrapolating the outside potential into interior region.

In cases of non-singular potentials, it can be extrapolated into the interior region comparatively consistently. On the other hand, the extrapolation breaks down completely about half the force range if a singular potential is adopted.

The fact that experiments prefer type (A) rather than type (B) will be able to be interpreted only by assuming that the nuclear forces become essentially velocity dependent at distances less than about half the force range. There are following two reasons which seems to support the existence of the velocity dependent forces.

(1) The recent experiments on the proton-proton scattering at 345 Mev showed the inadequacy of the phenomenological static potentials hitherto used.¹²⁾

(2) The phenomenological static potentials determined to fit the scattering data cannot explain the saturation properties of nuclear forces. (Remember that a class of velocity dependent potentials can satisfy the saturation requirements.)*

From these results, we are inclined to think that the weak coupling approximation is superior to the strong coupling approximation at least at energies where various experiments are performed so far, and at large separations.

* Concerning this problem we must take into account many body effects and non-linear characters of meson fields too. (Schiff, Phys. Rev. **84** (1951) 1, 10; Kinoshita and Miyazawa, read at the annual meeting of Jap. Phys. Soc., October, 1951).

So we shall look back the weak coupling results. At separations near the force range, the lowest second order forces are not main, but the next fourth order forces contribute comparably with or even larger than the lowest order.

By the way, the static fourth order potential is of the following form :

$$V_4 \sim [O_L, O_{M'}]^{(1)} [O_L, O_M]^{(2)} G(r, r')_{r'=r}, \quad (9)$$

provided that the static second order potential is given by

$$V_2 \sim O_L^{(1)} O_L^{(2)} \frac{e^{-x}}{x},$$

from which we know that the fourth order forces are caused by purely quantum mechanical effects that the components of the spin (or isotopic spin) matrices are not commutative with each other.

In this sense, approximations in which σ and τ are treated as though classical quantities will be meaningless in the meson theory.

The fact that the contribution from the fourth order is larger than that from the second order even at the force range may indicate the possibility that further higher orders would also be large near the force range. Therefore it is desired to solve the problem of nuclear forces at comparatively large separations by means of the intermediate coupling approximation. In this case, we are suffered from the mathematical complexities, but for the sake of completeness we shall write down the Hamiltonian of a two nucleon system ;

$$H = \frac{1}{2} \sum_{L,\lambda} Q_{\lambda}^{(-)} (P_{L\lambda}^2 + Q_{L\lambda}^2 - 1) + \frac{1}{2} \sum_{L,\lambda} Q_{\lambda}^{(+)} (p_{L\lambda}^2 + q_{L\lambda}^2 - 1) \\ + \frac{1}{(2\pi)^{3/2}} \frac{g}{x} \sum_{L,\lambda} (O_{L\lambda}^{(+)} A_{\lambda}^{(-)} Q_{L\lambda} + O_{L\lambda}^{(-)} A_{\lambda}^{(+)} q_{L\lambda}). \quad (10)$$

The above Hamiltonian is obtained on the method developed by Watson and Hart¹⁴⁾, and the notations are as follows :

$$O_{L\lambda}^{(\pm)} = \sigma_{\lambda}^{(1)} \tau_L^{(1)} \pm \sigma_{\lambda}^{(2)} \tau_L^{(2)}, \\ Q_x^{(\pm)} = \frac{M_{(\pm)}}{A_{(\pm)}}, \quad Q_y^{(\pm)} = Q_z^{(\pm)} = \frac{N_{(\pm)}}{B_{(\pm)}}, \\ A_x^{(\pm)} = \frac{M_{(\pm)}}{(A_{(\pm)})^{1/2}}, \quad A_y^{(\pm)} = A_z^{(\pm)} = \frac{N_{(\pm)}}{(B_{(\pm)})^{1/2}},$$

where the x -axis is chosen parallel to the line joining two nucleons, and

$$A_{(\pm)} = \frac{2}{3} \int \frac{q^2 d\mathbf{q}}{q_0^3} \pm 8\pi \left\{ \frac{1}{r} \frac{d}{dr} + r^2 \left(\frac{1}{r} \frac{d}{dr} \right)^2 \right\} K_0(xr), \\ B_{(\pm)} = \frac{2}{3} \int \frac{\tilde{q}^2 d\mathbf{q}}{q_0^3} \pm 8\pi \frac{1}{r} \frac{d}{dr} K_0(xr), \\ M_{(\pm)} = \frac{2}{3} \int \frac{q^2 d\mathbf{q}}{q_0^3} \pm 4\pi \left\{ \frac{1}{r} \frac{d}{dr} + r^2 \left(\frac{1}{r} \frac{d}{dr} \right)^2 \right\} \frac{e^{-xr}}{r}, \\ N_{(\pm)} = \frac{2}{3} \int \frac{q^2 d\mathbf{q}}{q_0^3} \pm 4\pi \frac{1}{r} \frac{d}{dr} \frac{e^{-xr}}{r},$$

and

$$P_{Lx} = \left(\frac{A_{(-)}}{2} \right)^{1/2} (S_{Lx}^+ + S_{Lx}^-), \quad Q_{Lx} = i \left(\frac{A_{(-)}}{2} \right)^{1/2} (S_{Lx}^+ - S_{Lx}^-),$$

$$p_{Lx} = \left(\frac{A_{(+)}}{2} \right)^{1/2} (T_{Lx}^+ + T_{Lx}^-), \quad q_{Lx} = i \left(\frac{A_{(+)}}{2} \right)^{1/2} (T_{Lx}^+ - T_{Lx}^-),$$

as for y, z components, we have only to replace A by B , and S by T , S and T are operators such as

$$\begin{pmatrix} S_{Lx}^- \\ T_{Lx}^- \end{pmatrix} = \int \frac{d\mathbf{q}}{q_0^2} q_x (e^{-i\mathbf{q}x_1} \pm e^{-i\mathbf{q}x_2}) \phi_L^{\pm}(q),$$

(S^+, T^+ are Hermitian conjugates)

with the normalization

$$[\phi_L^-(q), \phi_M^+(q')] = -\delta_{LM} q_0 \delta(\mathbf{q} - \mathbf{q}').$$

§ 5. Remarks on the mesonic potential

In this section we shall investigate the reason why the concept of the mesonic potential cannot be obtained clearly. For this purpose we will compare the meson-nucleon interaction with the photon-electron interaction.

(a) First the meson has a non-vanishing rest mass.

In the case of electrodynamics, a subsidiary condition

$$(\text{Div } \mathbf{A})\Psi = 0 \quad (\text{Lorentz condition}) \quad (11)$$

appeared. This condition cannot be derived from field equations but is consistent with the field equations. If we review this situation from another viewpoint, the quantity $(\text{Div } \mathbf{A})$ can be regarded as an integral. So that confining ourselves only to such states as

$$(\text{Div } \mathbf{A})\Psi = 0 \cdot \Psi, \quad (11')$$

we arrive at the Coulomb potential in which longitudinal and scalar photons are eliminated and their effects are replaced by c -numbers, (i.e. quantum numbers).

But we have no similarity in the meson theory. Physically the non-vanishing rest mass of the meson causes the short range character of the nuclear forces in contrast to the long range character of the Coulomb forces.

(b) Second, that the meson has its charge necessarily requires the quantum mechanical treatment since we have no analogy in the classical theory, just as already been pointed out in the previous section.

(c) Third, the most characteristic differences appear in their couplings. That the meson-nucleon interaction is very strong requires some suitable approximation method other than the simplest perturbation.

But what is essentially important is the existence of the ordinary and isotopic spins in the interaction. The non-commutativity is the basic reason for the impossibility of the

separation of static potentials, and at the same time the reason for the occurrence of essentially velocity dependent forces as has been argued in section 3.

The electromagnetic field couples with the electron velocity which can be treated classically as well as quantally, while the meson field couples with the nucleon spin and charge which can never be treated classically as stressed in section 4.

Next the impossibility of eliminating divergences by the renormalization method is also a very severe limitation on our investigation as we are directly confronted with the lack of the present formalism of the field theory.

For derivative couplings, the perturbation method breaks down at high energies as has been pointed out by several authors¹⁵⁾. At low energies we cannot use Born approximations, and at high energies also we cannot employ the perturbation method and inevitably depend on the relativistic strong coupling theory. An attempt to treat the singular matrix γ_5 in the strong coupling theory has been performed¹⁶⁾, but we do not know its detailed results yet.

(d) Last we shall point out an energetic limitation on the concept of the meson potential which results from the non-vanishing of the meson rest mass.

The transition matrix of the nucleon-nucleon collision is not analytic with regard to the energy. The reason will be given in the following.

We define the transition matrix R by

$$S = 1 + R.$$

Then from the unitary character of the S matrix, we get

$$(NN|R|NN) + (NN|R^*|NN) + (NN|R^*|NN)(NN|R|NN) = 0. \quad (12)$$

below the threshold energy for the meson production. Here $(NN|R|NN)$ is the submatrix of R concerning the two nucleon scattering.

Above the threshold energy, we have

$$(NN|R|NN) + (NN|R^*|NN) + (NN|R^*|NN)(NN|R|NN) + (NN|R^*|NN\pi)(NN\pi|R|NN) = 0. \quad (13)$$

As is clear from the above two equations, the submatrix $(NN|R|NN)$ satisfies different forms of equations for different energies. Thus $(NN|R|NN)$ is not an analytic function of energy.

Next let us examine this situation on the theory of canonical transformations by which we have obtained the fourth order potentials.

We have utilized two Bloch-Nordsieck transformations $U_1[\sigma]$ and $U_2[\sigma]$ in the former calculations. Since our treatment was non-relativistic, we discarded relativistic processes such as

$$N + \bar{N} \rightleftharpoons \pi + \pi. \quad (N: \text{nucleon}, \bar{N}: \text{anti-nucleon}, \pi: \text{meson})$$

But above threshold energy, we need the knowledge about forces at small separations or synonymously the higher orders. If we intend to apply the third transformation in order to derive sixth order potentials, the difference becomes clearer. For instance, we must

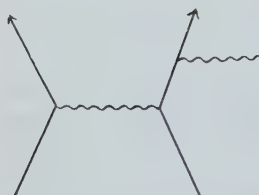


Fig. 1.

carefully doubt the terms like the one expressed by the Fig. 1 to drop or not to drop by the third Bloch-Nordsieck transformation. For energies below the characteristic threshold for the meson production, we must drop this term by the Bloch-Nordsieck transformation as the one describing the meson cloud around two nucleons since all mesons concerned are virtual.

On the contrary, for energies above threshold some of the mesons can be real, so that we cannot drop it.

Thus it is clear that we cannot describe the nuclear forces by an unified form of the potential valid at both low and high energies. Theoretically, the concept of the nuclear potential will lose its meaning above the threshold energy for the meson production, and perhaps the S matrix will be the only one to describe high energy phenomena.

At any rate the word "nuclear potential" is quite ambiguous and we feel keenly the powerlessness of the present meson field theory.

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Neutral-Meson Production by Gamma-Ray

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By means of the Lorentz-covariant perturbation and renormalization methods, the effective cross section for $\gamma\pi^0$ process in pseudoscalar meson theory has been obtained with the matrix elements calculated up to the order e^3 . The most conspicuous result, which could not be reached in the lowest order calculation, is that the cross section for neutral meson production turns out to be of the same order as that for charged meson, and that the angular distribution for that process is larger in the forward than in the backward direction. Further we have introduced a Pauli-type interaction with suitably chosen coupling constants in order to estimate the effect of the matrix elements of orders higher than e^3 , and gained a result similar to the above one, only with a smaller value for the coupling constant.

§ 1. Introduction and summary

The study of the electromagnetic properties of nucleon-meson system appears to be particularly interesting and, if we want to get insight into the dynamical behavior of this system, the $\gamma\pi$ process is one of the most important problems. The investigation about $\gamma\pi^\pm$ process including higher order effects has already been made by Koba, Kotani and Nakai,¹⁾ and we shall now study the $\gamma\pi^0$ process along the same line with these authors.

The existence of π^0 -meson has been predicted on various experimental and theoretical grounds, and recent experiments at Berkeley²⁾ have indeed confirmed this expectation directly. The facts which seem to have been established hitherto are as follows:

- (1) The neutral mesons are produced by γ -ray with the cross section comparable with those for the charged mesons.²⁾
- (2) The angular distribution for $\gamma\pi^0$ peaks in the forward, with the ratio $\sigma_{450}/\sigma_{900} \cong 2$, while that for $\gamma\pi^\pm$ is nearly isotropic in the laboratory system.²⁾

The absolute values of cross sections amount to:

$\sigma_{900}(\pi^0) \cong 3 \times 10^{-30} \text{ cm}^2/\text{sterad.}$, $\sigma_{900}(\pi^\pm) \cong 8 \times 10^{-30} \text{ cm}^2/\text{sterad.}$ at about 250 Mev of photon energy.³⁾

- (3) The neutral cross section rises much more steeply with energy than does the charged.⁴⁾

On the other hand, the calculation based on the lowest order perturbation⁵⁾ shows that the cross section for $P + \gamma \rightarrow P + \pi^0$ process is far smaller than that for the charged meson production and the angular distribution favors the backward rather than the forward direction. These contradictions with the experimental results seem at first sight to add a new difficulty to the current meson theory. In order to remove these faults, many studies have been done from various standpoints.^{6) 7) 8) 9) 10)} We also intend to investigate this problem

using weak coupling theory.

Thus, our object in this paper is to study the $\gamma\text{-}\pi^0$ process by carrying out a straightforward calculation of higher order effects* with the covariant perturbation method of Tomonaga-Schwinger-Feynman-Dyson. Making use of the formulation adopted in the calculation of $\gamma\text{-}\pi^\pm$ production,¹¹ we have classified numerous Feynman-Dyson diagrams into gauge-invariant classes and reduced the various terms of the matrix elements to four independent gauge-invariant fundamental expressions. Taking the neutral meson mass μ_0 equal to that of the charged meson μ for simplicity, we have calculated the cross section for $\gamma\text{-}\pi^0$ process at $K_0 = 2\mu = 276$ Mev of photon energy. With the value of the coupling constant adopted in the calculation of $\gamma\text{-}\pi^\pm$, the contribution from higher order terms to the cross section for $P + \gamma \rightarrow P + \pi^0$ is overwhelming and, especially for the case of P.S. coupling, causes a complete change in the angular distribution obtained by the lowest order perturbation. This fact makes our theoretical estimation agree fairly well with the experimental data.

Taking into consideration the circumstances that, the matrix element of ef^3 -order being large, we ought to examine the magnitude of the contribution from the terms higher than ef^3 , and that in our calculation the anomalous magnetic moments** of nucleons turn out unsatisfactory,¹² we have introduced in § 4 a Pauli-type interaction into the Hamiltonian and evaluated the matrix elements for this interaction also up to the order ef^3 . On the assumption that the contribution from all the orders higher than the fourth is represented by the second and fourth order effects of this Pauli-type interaction, we have proceeded to estimate the cross section not only for $\gamma\text{-}\pi^0$ but also for $\gamma\text{-}\pi^\pm$ *** in the case of pseudoscalar coupling. We have obtained, in this case, the good agreement with the experimental data for the neutral meson production as well as for the charged. It should be remarked that this conformity with the experimental facts in dynamical aspects is only guaranteed by adjusting the parameters entering into the theory in order to make the theoretical a.m.m., a static property, coincide with the empirical one. For the pseudovector coupling we are unable to carry out similar calculations owing to the appearance of unrenormalizable divergences.

§ 2. Matrix elements

According to Dyson,¹³ the S -matrix can be written as follows:

$$S \equiv U(-\infty, \infty) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n P(H(x_1), \dots, H(x_n)). \quad (1)$$

$H(x)$ is the interaction Hamiltonian density function in the interaction representation and

* Brueckner and Watson¹¹ have predicted that in the pseudoscalar meson theory these faults will be remedied by taking into account the contribution from higher orders.

** For short we shall write a.m.m. for the anomalous magnetic moment throughout this paper.

*** We thank to Messrs. Z. Koba, T. Kotani and S. Nakai for having shown us the matrix elements for $\gamma\text{-}\pi^\pm$ production.

is given by the following expression :

$$H = H^{mn} + H^{ne} + H^{me} + H^{nme} + H_{mass} + H_{coupl.}, \quad (2)$$

with

$$\begin{cases} H^{mn} = R^* \phi + R \phi^* + S_\mu (\partial \phi^* / \partial x_\mu) + S_\mu^* (\partial \phi / \partial x_\mu) + R^0 \phi^0 + S_\mu^0 (\partial \phi^0 / \partial x_\mu), \\ R = i f \bar{\psi} \gamma_5 \tau_{NP} \psi, & S_\mu = (i g / \mu) \bar{\psi} \gamma_5 \gamma_\mu \tau_{NP} \psi, \\ R^0 = i f^0 \bar{\psi} \gamma_5 \tau \psi, & S_\mu^0 = (i g^0 / \mu) \bar{\psi} \gamma_5 \gamma_\mu \tau \psi, \\ H^{ne} = -j_\mu A_\mu, & j_\mu = i e \bar{\psi} \gamma_\mu \tau_P \psi, \\ H^{me} = i e A_\mu [\phi^* (\partial \phi / \partial x_\mu) - (\partial \phi^* / \partial x_\mu) \phi], \\ H^{nme} = i e A_\mu (S_\mu \phi^* - S_\mu^* \phi). \end{cases}$$

The units are so chosen that $\hbar = c = 1$ and τ stands for τ_3 or $\tau_4 (=1)$ according to symmetrical or neutral theory. In order that the integrability condition may be satisfied, we must add certain terms which depend on the normal of the space-like surface to the above expression. One can, however, proceed to evaluate the S -matrix as if there were no such normal-dependent terms by modifying the definition of the P -symbol, as was the case with charged meson production.¹⁾

(I) First of all we consider the process $P + \gamma \rightarrow P + \pi^0$

(i) *P.S. coupling**

We write here the Feynman-Dyson diagrams and the corresponding "component" matrix elements in the second order.

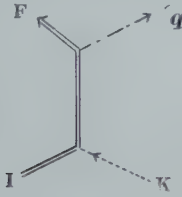


Fig. 1

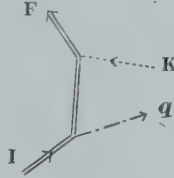


Fig. 2

\longrightarrow proton
 \longrightarrow neutron
 $\cdots \cdots \longrightarrow$ photon
 $\cdots \cdots \cdots \longrightarrow$ π^+ meson
 $\cdots \cdots \cdots \longrightarrow$ π^- meson
 $\cdots \cdots \cdots \longrightarrow$ π^0 meson

$$\left. \begin{aligned} U_1^{II} &= c^0 \bar{\psi}(F) i \gamma_5 \tau \frac{i \gamma(I+K) - M}{(I+K)^2 + M^2} \tau_P \gamma_\mu \psi(I) A_\mu(K), \\ U_2^{II} &= c^0 \bar{\psi}(F) \tau_P \gamma_\mu \frac{i \gamma(F-K) - M}{(F-K)^2 + M^2} i \gamma_5 \tau \psi(I) A_\mu(K), \end{aligned} \right\} \quad (3)$$

$$c^0 = -ef^0 / (4\pi)^2, \quad c = -ef / (4\pi)^2,$$

where I and F are the initial and the final four-momentum of the nucleon respectively, K represents the four-momentum of the incident photon, while q denotes that of the emitted π^0 meson.

* By the words P.S. coupling and P.V. coupling are meant the cases $g = g^0 = 0$ and $f = f^0 = 0$ respectively throughout this paper.

Following the formulation adopted in the calculation of $\gamma-\pi^\pm$ production, we have classified the numerous Feynman-Dyson diagrams in the fourth order into certain gauge-invariant classes and reduced the various terms of the component matrix elements to four independent gauge-invariant fundamental expressions, that is to say,

$$\begin{aligned} U_c &= \bar{\psi}(\mathbf{F}) \left\{ \frac{iF_u}{(\mathbf{F}\mathbf{K})} - \frac{iI_u}{(\mathbf{I}\mathbf{K})} \right\} i\gamma_5 \tau_P \psi(\mathbf{I}) A_\mu(\mathbf{K}), \\ U_m &= \bar{\psi}(\mathbf{F}) i\gamma_5 \tau \frac{\sigma_{\mu\nu} K_\nu \tau_P}{M^2} \psi(\mathbf{I}) A_\mu(\mathbf{K}), \\ U_A^{(1)} &= -\bar{\psi}(\mathbf{F}) i\gamma_5 \tau \frac{i\gamma(\mathbf{I}+\mathbf{K}) - M}{(\mathbf{I}+\mathbf{K})^2 + M^2} \frac{\sigma_{\mu\nu} K_\nu \tau_P}{2M} \psi(\mathbf{I}) A_\mu(\mathbf{K}), \\ \text{and} \\ U_A^{(2)} &= -\bar{\psi}(\mathbf{F}) \frac{\sigma_{\mu\nu} K_\nu \tau_P}{2M} \frac{i\gamma(\mathbf{F}-\mathbf{K}) - M}{(\mathbf{F}-\mathbf{K})^2 + M^2} i\gamma_5 \tau \psi(\mathbf{I}) A_\mu(\mathbf{K}), \end{aligned} \quad (4)$$

as for the physical meaning of these fundamental expressions, see the paper for $\gamma-\pi^\pm$ production.^{1)*}

Here we describe only the diagrams and the component matrix elements which yield finite observable contribution, after the renormalization of nucleon, meson mass and coupling constants is carried out. We write down beforehand the notation which will be used throughout this paper.

$$\begin{aligned} l_0 &= x^2 + \varepsilon(1-x), \\ l_v &= x^2 + \varepsilon(1-x) - ax(1-x)y - i\varepsilon, \\ l_s &= x^2 + \varepsilon(1-x) + \varepsilon x^2 y(y-1) - ax(1-x)y - i\varepsilon, \\ l_r &= x^2 + \varepsilon(1-x) - ax(1-x)y\bar{z} - i\varepsilon, \\ l_a &= x^2 + \varepsilon(1-x) + \varepsilon x^2 y(y-1) - ax(1-x)y\bar{z} - i\varepsilon, \\ l_{ad} &= x^2 + \varepsilon(1-x) + \varepsilon x^2 y(y-1) - ax(1-x)y(1-z) + dx^2 y(1-y)\bar{z} - i\varepsilon, \\ l_{ab} &= x^2 + \varepsilon(1-x) + \varepsilon x^2 y(y-1) - ax(1-x)y(1-z) + bx(1-x)(1-y)\bar{z} - i\varepsilon, \\ l_{ua} &= x^2 + \varepsilon(1-x) + \varepsilon x^2 y(y-1) - ax(1-x)y + (a+b)x(1-x)y\bar{z} - i\varepsilon, \\ l_{adu} &= l_{ad} + \{-ax(1-y) + b(1-xy)\} xy\bar{z}u, \\ l_{abu} &= l_{ab} + \{axy - b(1-x+xy)\} x(1-y)\bar{z}u, \end{aligned} \quad (5)$$

$$\begin{aligned} a &= -2(\mathbf{I}\mathbf{K})/M^2, \quad b = -2(\mathbf{F}\mathbf{K})/M^2, \quad d = -2(\mathbf{q}\mathbf{K})/M^2, \quad \varepsilon = \mu^2/M^2, \\ a &= f^2/4\pi, \quad a^0 = (f^0)^2/4\pi, \end{aligned}$$

$$\xi_I(a) = \int_0^1 dx \int_0^1 dy \frac{x^3}{l_v}, \quad \eta_I(a) = \int_0^1 dx \int_0^1 dy \frac{x^2(1-x)}{l_v},$$

* While the meson current in the case of charged meson production carries electric charge in the final state, the nucleon current takes its place in our case. So we have altered the definition of U_c and, moreover, modified U_m by a constant factor for convenience of calculation.

$$\xi_{II}(a) = \int_0^1 dx \int_0^1 dy \frac{x^2(1-x)y}{l_v}, \quad \eta_{II}(a) = \int_0^1 dx \int_0^1 dy \frac{x(1-x)^2y}{l_v},$$

$$\xi_{III}(a) = \int_0^1 dx \int_0^1 dy \frac{x^2}{l_s},$$

$$\lambda_I(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \left[\frac{x^2y \{1-2xy(1-z)\}}{2l_{ad}^2} - \frac{(x^4y + x^4y^2(1-z))}{l_{ad}^2} \right. \\ \left. + \frac{(\varepsilon/2)x^2y \{(1-x) + x(1-x)y(1-z) - x^2y(1-y)(1-z)\} - (a/2)x^3(1-x)y^2(1-z)}{l_{ad}^2} \right],$$

$$\lambda_{II}(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \cdot x^2(1-x)y \left[\frac{1}{l_a} + \frac{\varepsilon(1-x)}{2l_a^2} \right],$$

$$\lambda_{III}(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{x^4y^2}{l_{ad}^2}, \quad \lambda_{IV}(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{x^4y(1-y)}{l_{ad}^2},$$

$$\lambda_V(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{x^4y^2(1-y)z}{l_{ad}^2}, \quad \lambda_{VI}(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \cdot x^2(1-x)y \left[\frac{1}{l_r} + \frac{x^2}{l_r^2} \right],$$

$$\lambda_{VII}(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 du \cdot x^3y^2z \{ (xy-1)^2 - x^2(y-1)^2 \} \left[\frac{1}{l_{adc}^2} + \frac{\varepsilon(1-x)}{l_{adc}^3} \right],$$

$$\lambda_{VIII}(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \cdot x^2y \left[\frac{(1-2x)}{l_{ad}} - \frac{x \{ 2x + \varepsilon(1-xy) + (b-a)x(1-y) \}}{l_{ad}^2} \right],$$

$$\lambda_{IX}(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \cdot x^3y(1-y) \left[\frac{1}{l_{ad}} + \frac{\varepsilon(1-x)}{2l_{ad}^2} + \frac{x}{l_{ad}^3} \right], \quad (6)$$

$$\lambda_X(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \cdot x^2(1-x)y \left[\frac{1}{l_{wa}} + \frac{\varepsilon(1-x)}{2l_{wa}^2} \right],$$

$$\xi_I = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{x(1-x)}{l_{ab}}, \quad \xi_{II} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{x^3(1-x)}{l_{ab}^2},$$

$$\xi_{III} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{x^3(1-x)y}{l_{ab}^2}, \quad \xi_{IV} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{x^3(1-x)(1-y)}{l_{ab}^2},$$

$$\xi_V = \int_0^1 dx \int_0^1 dy \int_0^1 dz \cdot x^2(1-x)y \left[\frac{1}{l_{ab}} + \frac{\varepsilon(1-x)}{2l_{ab}^2} \right],$$

$$\xi_{VI} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 du \cdot x^3(1-x)y(1-y)z \left[\frac{1}{l_{abc}^2} + \frac{\varepsilon(1-x)}{l_{abc}^3} \right].$$

$\xi_I(-b)$, ..., $\lambda_X(-b)$ are respectively derived from those of (6) by the substitution $a \rightarrow (-b)$ and $b \rightarrow (-a)$.

First we enumerate the diagrams that take part in the static a.m.m. of the proton. They are shown in the Fig. (1.1), ..., Fig. (4.2).*

* Of the two numbers labelling the diagrams, the first one indicates the gauge-invariant class, to which they belong. Note that we do not mention the diagrams which give no observable contributions, as mentioned above, throughout this paper.



Fig. (1.1)



Fig. (1.2)



Fig. (2.1)



Fig. (2.2)



Fig. (3.1)



Fig. (3.2)



Fig. (4.1)



Fig. (4.2)

$$U_I^{IV} = -c^0(a^0/2\pi)(1/2)\xi_{II}(-b) \cdot U_m - c^0(a^0/2\pi)\xi_I(-b) \cdot U_A^{(2)}, \quad (7)$$

$$U_2^{IV} = -c^0(a^0/2\pi)(1/2)\xi_{II}(a) \cdot U_m - c^0(a^0/2\pi)\xi_I(a) \cdot U_A^{(1)}, \quad (8)$$

$$U_3^{IV} = c^0(a/2\pi)(1/2)\eta_{II}(a) \cdot U_m + c^0(a/2\pi)\eta_I(a) \cdot U_A^{(1)}, \quad (9)$$

$$U_4^{IV} = c^0(a/2\pi)(1/2)\eta_{II}(-b) \cdot U_m + c^0(a/2\pi)\eta_I(-b) \cdot U_A^{(2)}. \quad (10)$$

From the diagrams and the matrix elements, one can easily find the static a.m.m. to be

$$(a/2\pi)\eta_I(0) - (a^0/2\pi)\xi_I(0), \quad (11)$$

so that the expressions

$$(a/2\pi)[\eta_I(a) - \eta_I(0)] - (a^0/2\pi)[\xi_I(a) - \xi_I(0)] \quad (12)$$

and

$$(a/2\pi)[\eta_I(-b) - \eta_I(0)] - (a^0/2\pi)[\xi_I(-b) - \xi_I(0)] \quad (13)$$

have definite physical meaning from the point of view of perturbational treatment. They represent respectively the non-static parts of a.m.m. in the true sense of the word for the two types of $\gamma-\pi^0$ process, namely the one in which the nucleon absorbs incident photon first and the one in which the nucleon emits π^0 meson in the first step. We leave the question open as to the exact physical meaning of the possible appearance of imaginaries in the expression (12).

Next we shall proceed to examine the modification of γ_5 and so on.

$$\begin{aligned} U_5^{IV} = & c^0(a^0/2\pi)[\lambda_I(a) + \lambda_I(-b) + \lambda_{II}(a) + \lambda_{II}(-b)] \cdot U_m \\ & - c^0(a^0/2\pi)a[\lambda_{III}(a) + \lambda_{IV}(-b)] \cdot U_A^{(1)} + c^0(a^0/2\pi)b[\lambda_{III}(-b) + \lambda_{IV}(a)] \cdot U_A^{(2)} \\ & + c^0(a^0/2\pi)[-(ab/2)\{\lambda_V(a) + \lambda_V(-b) + \lambda_{VII}(a) + \lambda_{VII}(-b)\} \\ & + (a/2)\{\lambda_{II}(a) - \lambda_X(a)\} - (b/2)\{\lambda_{II}(-b) - \lambda_X(-b)\}] \cdot U_6, \end{aligned} \quad (14)$$

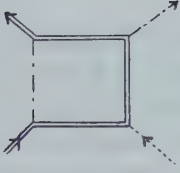


Fig. (5.1)

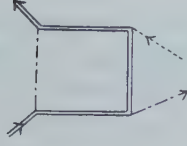


Fig. (5.2)

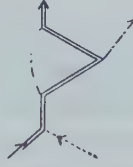


Fig. (5.3)

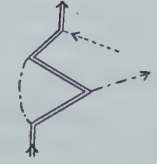


Fig. (5.4)



Fig. (6.1)

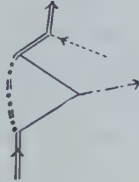


Fig. (6.2)

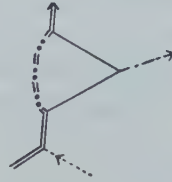


Fig. (6.3)

$$U_6^{IV} = \pm c^0(a/2\pi) [-\lambda_{II}(a) - \lambda_{II}(-b) + (1/2)\zeta_I - \zeta_{II}] \cdot U_m \mp c^0(a/2\pi) a\zeta_{III} \cdot U_A^{(1)} \\ \pm c^0(a/2\pi) b\zeta_{IV} \cdot U_A^{(2)} \pm c^0(a/2\pi) a [-\lambda_{II}(a) + \zeta_V] \cdot U_o. \quad (15)$$

The upper and lower signs correspond to the symmetrical and the (charged + neutral) theory respectively.*

The above evaluation of matrix element has been obtained by renormalizing the infinite coupling constant to the observed value. Such renormalization terms are

$$[-(a^0/2\pi)(Z^0 + Y^0 + X) - (a/2\pi)(Z - X)](U_1^{II} + U_2^{II}) \quad (16)$$

where Z^0 and Z originate from the diagrams with external proton lines modified by virtual neutral and charged meson respectively, Y_0 and X come from the diagrams with the free meson line and vertex part of γ_5 modified by virtual nucleon and virtual meson respectively. Their explicit expressions are:

$$\left. \begin{aligned} Z^0 &= \frac{1}{2\pi^2 i} \int_{-\infty}^{\infty} dt \int_0^1 dx \cdot x \frac{t^2 + 2M^2}{[t^2 + M^2 x^2 + \mu^2(1-x)]^3}, \\ Z &= Z^0 - 1/4, \\ Y^0 &= \frac{1}{2\pi^2 i} \int_{-\infty}^{\infty} dt \int_0^1 dx \cdot 4x \frac{t^2 + 2M^2}{[t^2 + M^2 - \mu^2 x(1-x)]^3}, \\ X &= \frac{1}{2\pi^2 i} \int_{-\infty}^{\infty} dt \int_0^1 dx \cdot 2x \frac{t^2 + \mu^2(1-x)}{[t^2 + M^2 x^2 + \mu^2(1-x) + \mu^2 x^2 y(y-1)]^3} - 1/4. \end{aligned} \right\} \quad (17)$$

The last terms of Z and X are derived from certain surface integrals which have been taken into account in our calculation.

* We use the double sign for this meaning and adopt, for convenience sake, the coupling constant f^0 equal to $f/\sqrt{2}$ in (charged + neutral) theory as in symmetrical theory throughout this paper.

(ii) *P.V. coupling*

When the pseudoscalar coupling is adopted, the surface integrals which appear in the evaluation of the matrix element have no effects upon the significant finite values. As for the pseudovector coupling, however, the situations are considerably different. We calculate the matrix elements for pseudovector coupling by separating the inequivalent terms according to the equivalence theorem. The surface integrals mentioned above have to do with the choice, whether we carry out such separation in the coordinate space or in the momentum space. Furthermore, the difference between the values obtained in these two ways is gauge-invariant, and we have as yet no criterion to remove this ambiguity for our $\gamma-\pi$ process. Although the result may be affected more or less by the difference of these two methods, at the moment we show here only the inequivalent terms obtained in the former way and that obtained in the latter way will be stated in another place.

(a) *symmetrical theory*

$$\begin{aligned}
 & (g')^3 c^0 [(a^0/2\pi) \{ (1/2) (\xi_I(a) + \xi_I(-b)) + \xi_{II}(a) + \xi_{II}(-b) \} \\
 & \quad + (a/2\pi) (1/4) \{ a\eta_{II}(a) - b\eta_{II}(-b) \}] \cdot U_m \\
 & + (g')^3 c^0 a [(a^0/2\pi) \xi_{II}(a) + (a/2\pi) (1/2) \eta_I(a)] \cdot U_A^{(1)} \\
 & - (g')^3 c^0 b [(a^0/2\pi) \xi_{II}(-b) + (a/2\pi) (1/2) \eta_I(-b)] \cdot U_A^{(2)}. \quad (18)
 \end{aligned}$$

(b) *charged and neutral theory*

$$\begin{aligned}
 & (g')^3 c^0 [(a^0/2\pi) \{ (1/2) (\xi_I(a) + \xi_I(-b)) + \xi_{II}(a) + \xi_{II}(-b) \} \\
 & \quad - (a/2\pi) \{ (1/2) (\eta_I(a) + \eta_I(-b)) + \eta_{II}(a) + \eta_{II}(-b) \}] \cdot U_m \\
 & + (g')^3 c^0 a [(a^0/2\pi) \xi_{II}(a) - (a/2\pi) \eta_{II}(a)] \cdot U_A^{(1)} \\
 & - (g')^3 c^0 b [(a^0/2\pi) \xi_{II}(-b) - (a/2\pi) \eta_{II}(-b)] \cdot U_A^{(2)}, \quad (19)
 \end{aligned}$$

where

$$g' = 2MJ/(\mu f).$$

(II) $N + \gamma \rightarrow N + \pi^0$ process

The non-vanishing matrix elements for this process appear first in the fourth order, and fortunately no divergent terms appear in our calculation when numerous diagrams are classified into certain gauge-invariant classes and an appropriate rule is adopted to remove the indefinite contributions. We write down the relevant matrix elements in the same way as (I). But, in this case, τ_P in (4) is replaced by τ_N .

(i) *P.S. coupling*

Fig. (7.1)

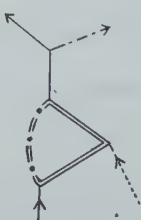


Fig. (7.2)



Fig. (8.1)



Fig. (8.2)

$$U_7^{IV} = -c^0(a/2\pi)(1/2) \{ \hat{\xi}_{II}(a) + \eta_{II}(a) \} \cdot U_m \\ - c^0(a/2\pi) \{ \hat{\xi}_I(a) + \eta_I(a) \} \cdot U_A^{(1)}, \quad (20)$$

$$U_8^{IV} = -c^0(a/2\pi)(1/2) \{ \hat{\xi}_{II}(-b) + \eta_{II}(-b) \} \cdot U_m \\ - c^0(a/2\pi) \{ \hat{\xi}_I(-b) + \eta_I(-b) \} \cdot U_A^{(2)}. \quad (21)$$

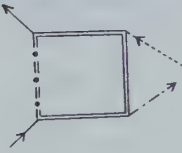


Fig. (9.1)

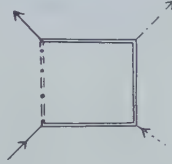


Fig. (9.2)

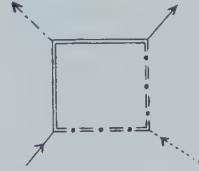


Fig. (9.3)

$$U_9^{IV} = \pm c^0(a/2\pi) [-\lambda_I(a) - \lambda_I(-b) - (1/2)\zeta_I + \zeta_{II}] \cdot U_m \\ \pm c^0(a/2\pi) a [\lambda_{IV}(-b) + \lambda_{III}(a) + \zeta_{III}] \cdot U_A^{(1)} \\ \mp c^0(a/2\pi) b [\lambda_{IV}(a) + \lambda_{III}(-b) + \zeta_{IV}] \cdot U_A^{(2)} \\ \pm c^0(a/2\pi) ab [(1/2) \{ \lambda_V(a) + \lambda_V(-b) \} + \zeta_{VI}] \cdot U_c. \quad (22)$$

(ii) *P.V. coupling*

In this case there are some divergent terms appearing in straightforward computation which we cannot remove by means of any renormalization methods, since the second order matrix element for the $N + \gamma \rightarrow N + \pi^0$ vanishes. Here, however, we may recall the general argument on gauge invariance by Koba, Mugibayashi and Nakai,¹⁴⁾ according to which it often happens, when a charge current is closed, that some non-gauge invariant terms come out of a class of diagrams which is gauge invariant in the coordinate space, if we first transform it into momentum representation. Under these circumstances such terms must be neglected by the requirement of gauge-invariance. Now, we find easily that the divergent terms which appeared in our calculation are just those non-gauge-invariant terms. By the reason mentioned above, we can remove the divergences and proceed to the estimation of matrix elements. The inequivalent terms are as follows:

(a) *symmetrical theory*

$$(g')^3 c^0(a/2\pi)(1/4) [-a \{ \hat{\xi}_{II}(a) + \eta_{II}(a) \} + b \{ \hat{\xi}_{II}(-b) + \eta_{II}(-b) \}] \cdot U_m \\ - (g')^3 c^0(a/2\pi)(a/2) [\hat{\xi}_I(a) + \eta_I(a)] \cdot U_A^{(1)} \\ + (g')^3 c^0(a/2\pi)(b/2) [\hat{\xi}_I(-b) + \eta_I(-b)] \cdot U_A^{(2)}. \quad (23)$$

(a) *charged and neutral theory*

$$(g')^3 c^0(a/2\pi) [(1/2) \{ \hat{\xi}_I(a) + \eta_I(a) + \hat{\xi}_I(-b) + \eta_I(-b) \} \\ + \{ \hat{\xi}_{II}(a) + \eta_{II}(a) + \hat{\xi}_{II}(-b) + \eta_{II}(-b) \}] \cdot U_m \\ + (g')^3 c^0(a/2\pi) a [\hat{\xi}_{II}(a) + \eta_{II}(a)] \cdot U_A^{(1)} \\ - (g')^3 c^0(a/2\pi) b [\hat{\xi}_{II}(-b) + \eta_{II}(-b)] \cdot U_A^{(2)}. \quad (24)$$

§ 3. Differential cross section

We summarize the matrix elements obtained in § 2 as follows:

$$c^0(a/2\pi)[(G_A(1)+iG'_A(1))U_A^{(1)}+(G_A(2)+iG'_A(2))U_A^{(2)} \\ + (G_m+iG'_m)U_m+(G_c+iG'_c)U_c], \quad (25)$$

where the $(G_A(1)+iG'_A(1))$, ..., $(G_c+iG'_c)$ terms have the following expressions.

(i) *P.S. coupling*

(I) $P+\gamma \rightarrow P+\pi^0$:

$$G_m+iG'_m=(-1/4)\{\xi_{II}(a)+\xi_{II}(-b)\}+(1/2)\{\eta_{II}(a)+\eta_{II}(-b)\} \\ + (1/2)\{\lambda_I(a)+\lambda_I(-b)+\lambda_{II}(a)+\lambda_{II}(-b)\} \\ \pm \{-\lambda_{II}(a)-\lambda_{II}(-b)+(1/2)\xi_I-\xi_{II}\}, \quad (26)$$

$$G_A^{(1)}+iG'_A(1)=(-1/2)\xi_I(a)+\eta_I(a)-(a/2)\{\lambda_{III}(a)+\lambda_{IV}(-b)\} \mp a\xi_{III}, \quad (27)$$

$$G_A(2)+iG'_A(2)=(-1/2)\xi_I(-b)+\eta_I(-b)+(b/2)\{\lambda_{III}(-b)+\lambda_{IV}(a)\} \pm b\xi_{IV}, \quad (28)$$

$$G_c+iG'_c=(-ab/4)\{\lambda_V(a)+\lambda_V(-b)+\lambda_{VII}(a)+\lambda_{VII}(-b)\}+(a/4)\{\lambda_{II}(a)-\lambda_X(a)\} \\ - (b/4)\{\lambda_{II}(-b)-\lambda_X(-b)\} \pm a\{-\lambda_{II}(a)+\xi_V\}. \quad (29)$$

(II) $N+\gamma \rightarrow N+\pi^0$:

$$G_m+iG'_m=(-1/2)\{\xi_{II}(a)+\eta_{II}(a)\}-(1/2)\{\xi_{II}(-b)+\eta_{II}(-b)\} \\ \pm \{-\lambda_I(a)-\lambda_I(-b)-(1/2)\xi_I+\xi_{II}\}, \quad (30)$$

$$G_A(1)+iG'_A(1)=-\{\xi_I(a)+\eta_I(a)\} \pm a\{\lambda_{IV}(-b)+\lambda_{III}(a)+\xi_{III}\}, \quad (31)$$

$$G_A(2)+iG'_A(2)=-\{\xi_I(-b)+\eta_I(-b)\} \mp b\{\lambda_{IV}(a)+\lambda_{III}(-b)+\xi_{IV}\}, \quad (32)$$

$$G_c+iG'_c=\pm ab[(1/2)\{\lambda_V(a)+\lambda_V(-b)\}+\xi_{VI}]. \quad (33)$$

(ii) *P.V. coupling*

Since we can easily obtain the similar terms for this case, by adding the corresponding inequivalent terms calculated in § 2 to above equivalent ones respectively, we do not here give the explicit expression to avoid complication.

For $P+\gamma \rightarrow P+\pi^0$, the matrix elements of second order are

$$c^0[(a-b)/ab \cdot U_m - U_c]. \quad (34)$$

This ef -term consists of two processes, in one of which the nucleon absorbs the incident photon in the first step and emits the π^0 meson in the second step, and in the other the photon absorption and meson production occur in the reversed order. Owing to the fact that contributions from the main terms in these two processes cancel out, the ef^3 -terms overwhelm the ef -terms in magnitude, if the accepted value of the coupling constant is adopted. Consequently, we have to take into account not only $(ef) \times (ef^3)$, but also

$(ef^3)^2$ -terms in the calculation of the cross section for the $P+\gamma\rightarrow P+\pi^0$ process. This, in turn, necessitates the computation of the cross terms $(ef)\times(ef^5)$, which, however, has not been still done. In these circumstances it is impossible to say anything rigorous, but, as the ef -terms are small, we may be allowed, for the present, to neglect the contribution from those cumbersome terms. In this way we calculate the cross section for $P+\gamma\rightarrow P+\pi^0$ up to the order e^2f^6 . Thus the differential cross section for this process is

$$\sigma d\Omega = (e^2/4\pi)(f^0)^2/4\pi \cdot (1/4MK_0)[I+II+III]d\Omega. \quad (35)$$

$$\begin{aligned} I &= (a-b)^2/a^2b^2 \cdot \chi_3 + \chi_1 - (a-b)/ab \cdot \chi_2, \\ II &= (a/2\pi)[(a-b)/ab \cdot G_A\chi_4 + \{(a-b)/ab \cdot G_c - G_m\} \chi_2 \\ &\quad + 2(a-b)/ab \cdot G_m\chi_3 - 2G_c\chi_1], \\ III &= (a/2\pi)^2[(G_m^2 + G_m'^2)\chi_3 + \{G_A^2(1) + G_A'^2(1) + G_A^2(2) + G_A'^2(2)\} \chi_5 \\ &\quad + (G_c^2 + G_c'^2)\chi_1 + (G_AG_m + G'_AG'_m)\chi_4 + \{G_A(1)G_A(2) + G'_A(1)G'_A(2)\} \chi_6 \\ &\quad + (G_mG_c + G'_mG'_c)\chi_2]. \end{aligned} \quad (36)$$

While, the differential cross section for $N+\gamma\rightarrow N+\pi^0$ is

$$\sigma d\Omega = (e^2/4\pi)(f^0)^2/4\pi \cdot (1/4MK_0)[III']d\Omega. \quad (37)$$

$$\begin{aligned} III' &= (a/2\pi)^2[(G_m^2 + G_m'^2)\chi_3 + \{G_A^2(1) + G_A'^2(1) + G_A^2(2) + G_A'^2(2)\} \chi_5 + (G_c^2 + G_c'^2)\chi_1 \\ &\quad + (G_AG_m + G'_AG'_m)\chi_4 + \{G_A(1)G_A(2) + G'_A(1)G'_A(2)\} \chi_6 + (G_mG_c + G'_mG'_c)\chi_2], \end{aligned} \quad (38)$$

where

$$\begin{aligned} G_A &= G_A(1) + G_A(2), & G'_A &= G'_A(1) + G'_A(2), \\ \chi_1 &= \rho \sum U_c^* U_c, \\ \chi_2 &= \rho \sum (U_c^* U_m + U_m^* U_c), \\ \chi_3 &= \rho \sum U_m^* U_m, \\ \chi_4 &= \rho \sum (U_m^* U_A^{(1)} + U_A^{(1)*} U_m) = \rho \sum (U_m^* U_A^{(2)} + U_A^{(2)*} U_m), \\ \chi_5 &= \rho \sum U_A^{(1)*} U_A^{(1)} = \rho \sum U_A^{(2)*} U_A^{(2)}, \\ \chi_6 &= \rho \sum (U_A^{(1)*} U_A^{(2)} + U_A^{(2)*} U_A^{(1)}), \\ \sum U_c^* U_A^{(1)} &= \sum U_A^{(1)*} U_c = \sum U_c^* U_A^{(2)} = \sum U_A^{(2)*} U_c = 0, \end{aligned} \quad (39)$$

ρ is the density factor and \sum means the average over polarizations of the incident photon and over initial spin states of the nucleon and the sum over final spin states of the nucleon. In order to see how the individual terms affect the resultant angular distribution, we give Fig. 3 the dependence of χ_1, \dots, χ_6 on the angle. The numerical values of constants which are used for the evaluation are $(e^2/4\pi)=1/137$, Compton wave length of nucleon $\lambda = (1/M) \simeq 2.1 \times 10^{-14}$ cm, $\epsilon = 1/44 \simeq 0.023$.

For the sake of convenience in comparing the values of cross section up to $e^2 f^2$ with those up to $e^2 f^6$, the following notations will be used.

- (1) ... $e^2 f^2$ -terms only are taken into account.
- (2) ... the same as above, but with a Pauli-type interaction.
- (3) ... up to $e^2 f^6$, our computation.

We show in Table 1, ..., 4 the differential cross sections in laboratory system at 276 Mev of photon energy. The value of coupling constant has been so chosen that it may not contradict with the result for charged meson production¹⁾ and also with the experimental value of cross section at 90° , 3×10^{-30} cm²/sterad.

The most remarkable fact in these tables is that, for the P.S. coupling in symmetrical theory, the value of (3) are considerably larger than that of (1) and the angular distribu-

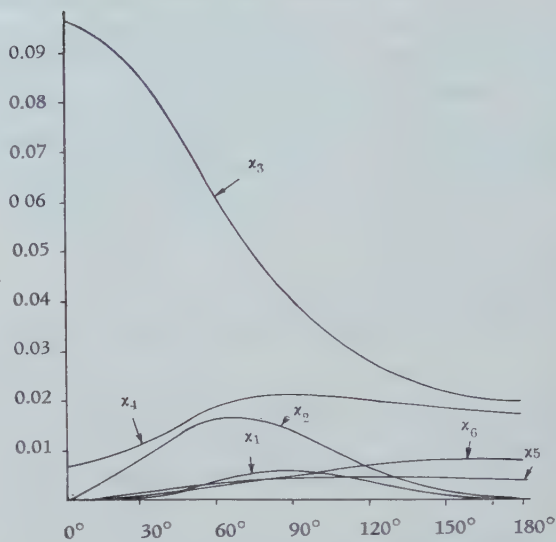


Fig. 3. The values of χ_1, \dots, χ_6 in laboratory system at 276 Mev of photon energy.

Table 1. σ_θ (10^{-30} cm²/sterad.) in the case of P.S. coupling of symmetrical theory: $a/4\pi=2$

		0°	30°	60°	90°	120°	150°	180°
$P+\gamma \rightarrow P+\pi^0$	(1)	0.016	0.033	0.15	0.33	0.46	0.51	0.52
	(2)	0.12	0.61	1.9	3.2	3.9	4.0	4.0
	(3)	12.4	10.9	7.6	4.4	2.4	1.4	1.0
$N+\gamma \rightarrow N+\pi^0$	(2)'	0.057	0.45	1.2	1.8	1.9	1.9	1.9
	(3)'	8.1	6.4	3.6	1.9	1.1	0.76	0.69

Table 2. σ_θ (10^{-30} cm²/sterad.) in the case of P.S. coupling of charged and neutral theory: $a/4\pi=1$

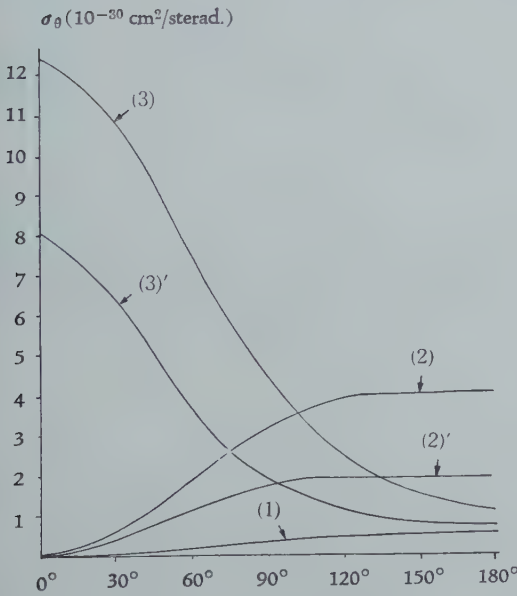
		0°	30°	60°	90°	120°	150°	180°
$P+\gamma \rightarrow P+\pi^0$	(1)	0.008	0.017	0.08	0.17	0.23	0.26	0.26
	(2)	0.06	0.31	1.0	1.6	2.0	2.0	2.0
	(3)	5.6	5.2	4.4	3.5	2.9	2.5	2.4
$N+\gamma \rightarrow N+\pi^0$	(2)'	0.029	0.23	0.6	0.9	1.0	1.0	1.0
	(3)'	20.1	19.1	16.7	13.6	11.0	9.4	8.7

Table 3. $\sigma_0(10^{-30} \text{ cm}^2/\text{sterad.})$ in the case of P.V. coupling of symmetrical theory: $(g')^2 a/4\pi=3$

		0°	30°	60°	90°	120°	150°	180°
$P+\gamma \rightarrow P+\pi^0$	(1)	0.024	0.05	0.23	0.5	0.69	0.77	0.78
	(2)	2.4	1.8	1.4	1.7	2.2	2.4	2.5
	(3)	0.25	0.41	0.91	1.4	1.7	1.6	1.6
$N+\gamma \rightarrow N+\pi^0$	(2)'	3.3	2.5	1.2	0.62	0.53	0.54	0.54
	(3)'	22.4	17.3	9.6	6.0	5.0	4.7	4.7

Table 4. $\sigma_0(10^{-30} \text{ cm}^2/\text{sterad.})$ in the case of P.V. coupling of charged and neutral theory: $(g')^2 a/4\pi=3$

		0°	30°	60°	90°	120°	150°	180°
$P+\gamma \rightarrow P+\pi^0$	(1)	0.024	0.05	0.23	0.5	0.69	0.77	0.78
	(2)	2.4	1.8	1.4	1.7	2.2	2.4	2.5
	(3)	0.94	0.69	0.48	0.74	1.3	1.6	1.6
$N+\gamma \rightarrow N+\pi^0$	(2)'	3.3	2.5	1.2	0.62	0.53	0.54	0.54
	(3)'	24.3	18.6	9.4	5.5	5.5	6.1	6.1

Fig. 4. Angular distribution for $\gamma-\pi^0$ in laboratory system at 276 Mev of photon energy when P.S. coupling is adopted.

- (1): $e^2 f^2$ -term only are taken into account.
- (2): the same as above, but with a Pauli-type interaction.
- (2)': the same above one for $N+\gamma \rightarrow N+\pi^0$.
- (3): up to $e^2 f^5$, our computation.
- (3)': up to $e^2 f^6$, our computation for $N+\gamma \rightarrow N+\pi^0$.

tion of (3) has a completely different from that of (1) or (2). These circumstances are shown in Fig. 4.

§ 4. Pauli-type interaction

The $\gamma-\pi^0$ process, of course, has to do with the a.m.m. of nucleons, namely the lowest order term of the perturbation series which gives rise to a.m.m. makes a contribution to our fourth order calculation for $\gamma-\pi^0$. In the calculation of the second order for this static a.m.m. of nucleons, any of the present theories cannot give satisfactory values. But Nakabayasi and Sato¹⁵⁾ have shown that one can get rid of these difficulties to some extent when one proceeds to the computations of the next order. As for the matrix elements for $\gamma-\pi^0$, these circumstances will become clear, if we carry out the calculations up to the order ϵ^f , but here we shall go an easier way and make up the lack in § 3 making use of the Pauli-type interaction.*

If we carry out the calculation for static a.m.m. of nucleons, the resulting form will be

$$\begin{aligned} & (u/2\pi)l_0[1 + (u/2\pi)l_1 + (u/2\pi)^2l_2 + \dots]** \quad \text{for proton} \\ \text{and} \quad & (u/2\pi)n_0[1 + (u/2\pi)n_1 + (u/2\pi)^2n_2 + \dots] \quad \text{for neutron,} \end{aligned} \quad (40)$$

$(u/2\pi)l_0$ and $(u/2\pi)n_0$ being the values given by Case.¹²⁾ Similarly, when we carry out the computations for $P+\gamma \rightarrow P+\pi^0$,*** the matrix elements of the four kinds will come out to be of the forms.

$$\epsilon^0[(u/2\pi)l_0(1 + (u/2\pi)l_1 + \dots) + (u/2\pi)l'_0(1) + (u/2\pi)^2l'_1(1)(1 + (u/2\pi)l'_2(1) + \dots)] \cdot U_A^{(1)}, \quad (41)$$

$$\epsilon^0[(u/2\pi)l_0(1 + (u/2\pi)l_1 + \dots) + (u/2\pi)l'_0(2) + (u/2\pi)^2l'_1(2)(1 + (u/2\pi)l'_2(2) + \dots)] \cdot U_A^{(2)}, \quad (42)$$

$$\epsilon^0[(u/2\pi)D + (u/2\pi)^2D_1(1 + (u/2\pi)D_2 + \dots)] \cdot U_m, \quad (43)$$

$$\epsilon^0[(u/2\pi)E + (u/2\pi)^2E_1(1 + (u/2\pi)E_2 + \dots)] \cdot U_v. \quad (44)$$

In (41) and (42), the static and non-static part of a.m.m. are written separately, the sign (') indicating the non-static one. (1) and (2) in $l'_1(1)$, $l'_1(2)$, ..., correspond to the process in which nucleon absorbs the incident photon at the first and at the last step respectively; D , E , $l'_0(1)$ and $l'_0(2)$ are what we have obtained in § 2 and are connected with (25) as follows.

$$\begin{aligned} D &= G_m + iG'_m, \\ E &= G_e + iG'_e, \\ l_0 + l'_0(1) &= G_A(1) + iG'_A(1), \\ l_0 + l'_0(2) &= G_A(2) + iG'_A(2). \end{aligned} \quad (45)$$

* What we aim at by introducing the Pauli-type interaction is to take into account the effects of higher orders in a certain sense. Thus our standpoint differs from that of Prof. S. Sakata, who recognizes no *a priori* difference between a current interaction and a Pauli-type interaction except for the renormalizability condition.

** $a = f^2/4\pi$

*** This argument holds for $N+\gamma \rightarrow N+\pi^0$ process with necessary changes.

The presumption we are going to make is as follows; the second and fourth order effects of Pauli-type interaction stand for the contributions from all of the static a.m.m. (containing the lowest order) and all the orders higher than the fourth in regard to the non-static part of a.m.m. and the matrix elements of U_m and U_c classes.*

The reason why such a presumption including a cumbersome fourth order calculation for Pauli-type must be employed can be seen from the following consideration. If one assumes that the higher order calculation for the static a.m.m. of nucleons in field theory ultimately yields the experimental values, one might be inclined to introduce a Pauli-type interaction into Hamiltonian,

$$H_P = \left(-1/2 \right) \left(\epsilon/2M \right) \bar{\psi}(x) \left(\mu'_P \tau_P + \mu'_N \tau_N \right) \sigma_{\mu\nu} \psi(x) F_{\mu\nu}(x), \quad (46)$$

$$F_{\mu\nu}(x) = \partial A_\nu(x)/\partial x_\mu - \partial A_\mu(x)/\partial x_\nu, \quad (47)$$

$$\mu'_P = (u/2\pi) l_0 (1 + (u/2\pi) l_1 + \dots), \quad (48)$$

$$\mu'_N = (u/2\pi) n_0 (1 + (u/2\pi) n_1 + \dots),$$

setting the values of μ'_P and μ'_N equal to the measured ones respectively, and stop at the lowest order perturbation. In this case, indeed, the contributions from higher orders for static a.m.m. would be taken into account, but those from non-static a.m.m. and the matrix elements of U_m and U_c classes would not be taken into consideration. Therefore we had better regard μ'_P and μ'_N as mere parameters at the moment and proceed to estimate the matrix elements for that interaction up to the order ϵf^3 .

After the renormalization of μ'_P and μ'_N is executed, the contributions from the static a.m.m. to $P + \gamma \rightarrow P + \pi^0$ and $N + \gamma \rightarrow N + \pi^0$ are expressed respectively as follows:

$$c^0(\mu'_P + (u/2\pi)\beta\mu'_N)(U_A^{(1)} + U_A^{(2)}),$$

$$c^0(\mu'_N + (u/2\pi)\beta\mu'_P)(U_A^{(1)} + U_A^{(2)}),$$

where β is a definite number. In consequence $(\mu'_P + (u/2\pi)\beta\mu'_N)$ and $(\mu'_N + (u/2\pi)\beta\mu'_P)$ are, in our case, the static a.m.m. of proton and neutron respectively, so that, making these values equal to those obtained from measurement we can determine the μ'_P and μ'_N which have been left arbitrary.

$$\begin{aligned} \mu'_P + (u/2\pi)\beta\mu'_N &= \mu_P (= 1.79), \\ \mu'_N + (u/2\pi)\beta\mu'_P &= \mu_N (= -1.91). \end{aligned} \quad (49)$$

Since we have taken the static a.m.m. derived from second and fourth order terms for Pauli-type interaction in place of those derived from the ordinary calculation and moreover determined μ'_P and μ'_N so as to make them agree with the experimental values, the effects of the static a.m.m. are probably accounted for correctly.

Let us further examine the non-static parts of a.m.m. and the matrix elements of U_m and U_c classes. These terms that appear for the first time at the calculation of fourth

* We are informed by Mr. Kotani that Prof. S. Tomonaga has a similar idea rather akin to our's.

order in this coupling have the following forms.

$$\begin{aligned}
 & c^0[(a/2\pi)\mu'_P l'_{1P}(1) + (a/2\pi)\mu'_N l''_{1P}(1)] \cdot U_A^{(1)} \\
 & + c^0[(a/2\pi)\mu'_P l'_{1P}(2) + (a/2\pi)\mu'_N l''_{1P}(2)] \cdot U_A^{(2)} \\
 & + c^0[(a/2\pi)\mu'_P G'_{mP} + (a/2\pi)\mu'_N G'_{mP}] \cdot U_m \\
 & + c^0[(a/2\pi)\mu'_P G'_{eP} + (a/2\pi)\mu'_N G'_{eP}] \cdot U_e.
 \end{aligned} \quad (50)^*$$

As we can see easily from (47) and (48), μ'_P and μ'_N correspond to terms of the order of f^2 (or a) in the usual calculation and so we write (50) as follows for the sake of convenience.

$$\begin{aligned}
 & c^0[(a/2\pi)^2 A(1)] \cdot U_A^{(1)} + c^0[(a/2\pi)^2 A(2)] \cdot U_A^{(2)} \\
 & + c^0[(a/2\pi)^2 B] \cdot U_m + c^0[(a/2\pi)^2 C] \cdot U_e.
 \end{aligned} \quad (51)$$

We now go back to the expressions (41), (42), (43) and (44). The correct values of $l'_1(1)(1 + (a/2\pi)l'_2(1) + \dots)$, $l'_1(2)(1 + (a/2\pi)l'_2(2) + \dots)$, $D_1(1 + (a/2\pi)D_2 + \dots)$ and $E_1(1 + (a/2\pi)E_2 + \dots)$ are of course yet unknown, but it would not be at all unreasonable to assume the following relations, if one compares (41), (42), (43) and (44) with (51) and takes notice of the order in the coupling constant f .

$$\begin{aligned}
 & l'_1(1)(1 + (a/2\pi)l'_2(1) + \dots) = A(1), \\
 & l'_1(2)(1 + (a/2\pi)l'_2(2) + \dots) = A(2), \\
 & D_1(1 + (a/2\pi)D_2 + \dots) = B, \\
 & E_1(1 + (a/2\pi)E_2 + \dots) = C.
 \end{aligned} \quad (52)$$

This approximation corresponds from the standpoint of Feynman-Dyson formalism to neglecting a part of the diagrams which come out for the first time in the orders higher than the fourth.

On the above assumption we can express the total matrix element which would be obtained, when the higher order calculations for the ordinary interaction are carried out successively, as follows:

$$\begin{aligned}
 & c^0({}^T G_m + i{}^T G'_m) \cdot U_m + c^0({}^T G_A(1) + i{}^T G'_A(1)) \cdot U_A^{(1)} \\
 & + c^0({}^T G_A(2) + i{}^T G'_A(2)) \cdot U_A^{(2)} + c^0({}^T G_e + i{}^T G'_e) \cdot U_e.
 \end{aligned} \quad (53)$$

Here

$$\begin{aligned}
 & {}^T G_m + i{}^T G'_m = (a-b)/ab + (a/2\pi)[(G_m + iG'_m) + \mu'_P G'_{mP} + \mu'_N G'_{mP}], \\
 & {}^T G_A(1) + i{}^T G'_A(1) = \mu_P + (a/2\pi)[G_A(1) + iG'_A(1) - l_0 + \mu'_P l'_{1P}(1) + \mu'_N l''_{1P}(1)], \\
 & {}^T G_A(2) + i{}^T G'_A(2) = \mu_P + (a/2\pi)[G_A(2) + iG'_A(2) - l_0 + \mu'_P l'_{1P}(2) + \mu'_N l''_{1P}(2)], \\
 & {}^T G_e + i{}^T G'_e = -1 + (a/2\pi)[G_e + iG'_e + \mu'_P G'_{eP} + \mu'_N G'_{eP}].
 \end{aligned} \quad (54)$$

As the whole static a.m.m. are included in μ_P , we remove the static part of a.m.m. from $G_A(1)$ and $G_A(2)$ obtained by (27) and (28). G'_{mP} , G'_{mP} , $l'_{1P}(1)$, $l''_{1P}(1)$, $l'_{1P}(2)$,

* The suffix P indicates the contribution from the Pauli-type interaction.

$L''_{1P}(2)$, G'_{cP} and G''_{cP} are complex values* with will be calculated in § 5.

§ 5. The matrix element of the fourth order for the Pauli-type interaction and the differential cross sections taking into account of the contributions from the orders higher than the fourth

We describe only the diagrams and the component matrix elements which yield finite observable contributions arranging them along the same line with the way in § 2, though each diagram is individually gauge invariant.

(i) *P.S. coupling*

(I) $P + \gamma \rightarrow P + \pi^0$

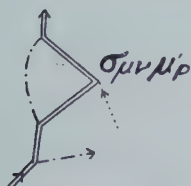


Fig. (1p. 1)

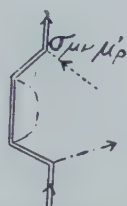


Fig. (1p. 2)

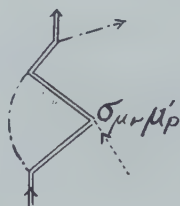


Fig. (2p. 1)

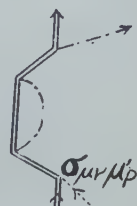


Fig. (2p. 2)



Fig. (3p. 1)

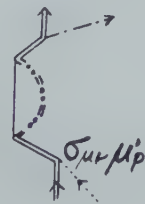


Fig. (3p. 2)

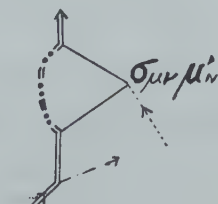


Fig. (4p. 1)

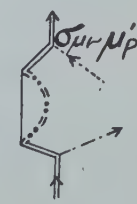


Fig. (4p. 2)

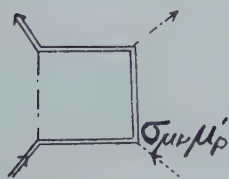


Fig. (5p. 1)

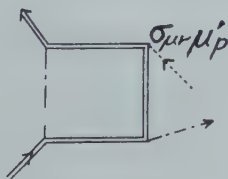


Fig. (5p. 2)



Fig. (5p. 3)

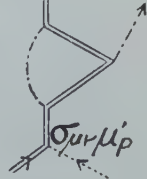


Fig. (5p. 4)

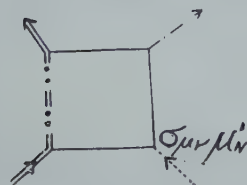


Fig. (6p. 1)

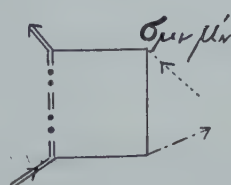


Fig. (6p. 2)



Fig. (6p. 3)



Fig. (6p. 4)

* Note that all the values of G'_{mP} , ..., G'_{cP} for the processes of $N + \gamma \rightarrow N + \pi^0$, $P + \gamma \rightarrow N + \pi^+$ and $N + \gamma \rightarrow P + \pi^-$ which will be mentioned below are of course also complex numbers.

$$U_{1P}^{IV} = -c^0(a^0/2\pi)\mu'_P(1/2)\eta_I(-b) \cdot U_m \\ + c^0(a^0/2\pi)\mu'_P[(-1/2)\{\xi_I(-b) - \xi_I(0)\} + (b/2)\lambda_{VI}(-b)] \cdot U_A^{(2)}, \quad (55)$$

$$U_{2P}^{IV} = -c^0(a^0/2\pi)\mu'_P(1/2)\eta_I(a) \cdot U_m \\ + c^0(a^0/2\pi)\mu'_P[(-1/2)\{\xi_I(a) - \xi_I(0)\} - (a/2)\lambda_{VI}(a)] \cdot U_A^{(1)}, \quad (56)$$

$$U_{3P}^{IV} = -c^0(a/2\pi)(\mu'_P + \mu'_N)(1/4)\eta_I(a) \cdot U_m \\ + c^0(a/2\pi)[-(\mu'_N/2)\xi_I(a) - (a/2)\mu'_P\lambda_{VI}(a)] \cdot U_A^{(1)}, \quad (57)$$

$$U_{4P}^{IV} = -c^0(a/2\pi)(\mu'_P + \mu'_N)(1/4)\eta_I(-b) \cdot U_m \\ + c^0(a/2\pi)[-(\mu'_N/2)\xi_I(-b) + (b/2)\mu'_P\lambda_{VI}(-b)] \cdot U_A^{(2)}, \quad (58)$$

$$U_{5P}^{IV} = c^0(a^0/2\pi)\mu'_P[(-1/4)\{\xi_{III}(a) + \xi_{III}(-b)\} + (1/4)\{\lambda_{VII}(a) + \lambda_{VII}(-b)\}] \cdot U_m \\ - c^0(a^0/2\pi)\mu'_Pa[\lambda_{II}(a) + (1/2)\{\lambda_{III}(a) + \lambda_{IV}(a)\} + \lambda_{IX}(-b)] \cdot U_A^{(1)} \\ + c^0(a^0/2\pi)\mu'_Pb[\lambda_{II}(-b) + (1/2)\{\lambda_{III}(-b) + \lambda_{IV}(-b)\} + \lambda_{IX}(a)] \cdot U_A^{(2)} \\ - c^0(a^0/2\pi)\mu'_P(ab/4)[\lambda_{IV}(a) + \lambda_{IV}(-b)] \cdot U_c, \quad (59)$$

$$U_{6P}^{IV} = \pm c^0(a/2\pi)[(\mu'_P/4)\{\xi_{III}(a) + \xi_{III}(-b)\} - (\mu'_N/4)\{\lambda_{VII}(a) + \lambda_{VII}(-b)\}] \cdot U_m \\ \pm c^0(a/2\pi)a[\mu'_P\lambda_{II}(a) + \mu'_N(1/2)\{\lambda_{III}(a) + \lambda_{IV}(a)\} + \mu'_N\lambda_{IX}(-b)] \cdot U_A^{(1)} \\ \mp c^0(a/2\pi)b[\mu'_P\lambda_{II}(-b) + \mu'_N(1/2)\{\lambda_{III}(-b) + \lambda_{IV}(-b)\} + \mu'_N\lambda_{IX}(a)] \cdot U_A^{(2)} \\ \pm c^0(a/2\pi)\mu'_N(ab/4)[\lambda_{IV}(a) + \lambda_{IV}(-b)] \cdot U_c. \quad (60)$$

When we rewrite the terms of $-(a/2\pi)(\mu'_N/2)\xi_I(a)$ in (57) and $-(a/2\pi)(\mu'_N/2)\xi_I(-b)$ in (58) as follows:

$$-(a/2\pi)(\mu'_N/2)\{\xi_I(a) - \xi_I(0)\} - (a/2\pi)(\mu'_N/2)\xi_I(0), \\ -(a/2\pi)(\mu'_N/2)\{\xi_I(-b) - \xi_I(0)\} - (a/2\pi)(\mu'_N/2)\xi_I(0),$$

then, the first and the second terms correspond respectively to the non-static and static a.m.m., and they appear first in the calculation of fourth order. Especially the latter term, so to speak, has a meaning of the contribution from the static a.m.m. of virtual neutron. The term $(a/2\pi)\mu'_N\beta$ in (49) represents just this one.

Thus, we can write down G'_{mP} , G''_{mP} , ..., G'_{oP} , G''_{oP} in (54) as follows:

$$G'_{mP} = (-1/2)\{\eta_I(a) + \eta_I(-b)\} - (1/8)\{\xi_{III}(a) + \xi_{III}(-b)\} \\ + (1/8)\{\lambda_{VII}(a) + \lambda_{VII}(-b)\} \pm (1/4)\{\xi_{III}(a) + \xi_{III}(-b)\}, \\ G''_{mP} = (-1/4)\{\eta_I(a) + \eta_I(-b)\} \mp (1/4)\{\lambda_{VII}(a) + \lambda_{VII}(-b)\}, \\ I'_{1P}(1) = (-1/4)\{\xi_I(a) - \xi_I(0)\} - (3/4)a\lambda_{VI}(a) - (a/2)[\lambda_{II}(a) + (1/2) \\ \{\lambda_{III}(a) + \lambda_{IV}(a)\} + \lambda_{IX}(-b)] \pm a\lambda_{II}(a), \\ I''_{1P}(1) = (-1/2)\{\xi_I(a) - \xi_I(0)\} \pm a[(1/2)\{\lambda_{III}(a) + \lambda_{IV}(a)\} + \lambda_{IX}(-b)],$$

$$\begin{aligned}
l'_{1P}(2) &= (-1/4) \{ \hat{\xi}_I(-\delta) - \hat{\xi}_I(0) \} + (3/4) \delta \lambda_{VI}(-\delta) + (\delta/2) [\lambda_{II}(-\delta) \\
&\quad + (1/2) \{ \lambda_{III}(-\delta) + \lambda_{IV}(-\delta) \} + \lambda_{IX}(a)] \mp \delta \lambda_{II}(-\delta), \\
l''_{1P}(2) &= (-1/2) \{ \hat{\xi}_I(-\delta) - \hat{\xi}_I(0) \} \mp \delta [(1/2) \{ \lambda_{III}(-\delta) + \lambda_{IV}(-\delta) \} + \lambda_{IX}(a)], \\
G'_{cP} &= (-ab/8) \{ \lambda_{IV}(a) + \lambda_{IV}(-\delta) \}, \\
G''_{cP} &= \pm (ab/4) \{ \lambda_{IV}(a) + \lambda_{IV}(-\delta) \}.
\end{aligned} \tag{61}$$

$$(II) \quad N + \gamma \rightarrow N + \pi^0$$

The total matrix element is

$$\begin{aligned}
&c^0 ({}^T G_m + i {}^T G'_m) \cdot U_m + c^0 ({}^T G_A(1) + i {}^T G'_A(1)) \cdot U_A^{(1)} \\
&+ c^0 ({}^T G_A(2) + i {}^T G'_A(2)) \cdot U_A^{(2)} + c^0 ({}^T G_c + i {}^T G'_c) \cdot U_c,
\end{aligned} \tag{62}$$

where

$$\begin{aligned}
{}^T G_m + i {}^T G'_m &= (a/2\pi) [G_m + i G'_m + \mu'_P G'_{mP} + \mu'_N G''_{mP}], \\
{}^T G_A(1) + i {}^T G'_A(1) &= \mu_N + (a/2\pi) [G_A(1) + i G'_A(1) - n_0 + \mu'_P l'_{1P}(1) + \mu'_N l''_{1P}(1)], \\
{}^T G_A(2) + i {}^T G'_A(2) &= \mu_N + (a/2\pi) [G_A(2) + i G'_A(2) - n_0 + \mu'_P l'_{1P}(2) + \mu'_N l''_{1P}(2)], \\
{}^T G_c + i {}^T G'_c &= (a/2\pi) [G_c + i G'_c + \mu'_P G'_{cP} + \mu'_N G''_{cP}].
\end{aligned} \tag{63}$$

$(G_m + i G'_m)$, ..., $(G_c + i G'_c)$ are given by (30), (31), (32), (33) and G'_{mP} , G''_{mP} , ..., G'_{cP} , G''_{cP} , are respectively given by the substitution the sign $(') \rightarrow (')$ and the sign $(') \rightarrow (')$ in (61).

(ii) *P.V. coupling*

For the pseudovector coupling we are unable to carry out the calculations owing to the appearance of unrenormalizable divergences. It may be necessary to assume some kind of new interactions, and we shall discuss about them in § 6.

It is very interesting to examine what results will be obtained in the case of charged meson production by the same way as above. Utilizing much of the valuable results obtained by Koba, Kotani and Nakai and introducing again the Pauli-type interaction, as in the case of neutral meson production, in order to take into account of higher order effect, we obtain the total matrix elements for charged meson production in the following expressions.

$$(III) \quad P + \gamma \rightarrow N + \pi^+$$

$$\begin{aligned}
&c ({}^T G_m + i {}^T G'_m) \cdot U_m + c ({}^T G_A(1) + i {}^T G'_A(1)) \cdot U_A^{(1)} \\
&+ c ({}^T G_A(2) + i {}^T G'_A(2)) \cdot U_A^{(2)} + c ({}^T G_c + i {}^T G'_c) U_c,
\end{aligned} \tag{64}^*$$

where

$$\begin{aligned}
{}^T G_m + i {}^T G'_m &= (-1/a) + (a/2\pi) [(G_m + i G'_m) + \mu'_P G'_{mP} + \mu'_N G''_{mP}], \\
{}^T G_A(1) + i {}^T G'_A(1) &= \mu_P + (a/2\pi) [(G_A(1) + i G'_A(1) - l_0) + \mu'_P l'_{1P}(1) + \mu'_N l''_{1P}(1)],
\end{aligned}$$

* In this case, $\tau\tau_P$ in (4) is replaced by τ_{NP} or τ_{PN} according to π^+ or π^- meson production. Note further that our definitions of U_m and U_c are different from those of Koba, Kotani and Nakai.

$$\begin{aligned}
{}^T G_A(2) + i G'_A(2) &= \mu_N + (a/2\pi) [(G_A(2) + i G'_A(2) - n_0) + \mu'_P l'_{1P}(2) + \mu'_N l''_{1P}(2)], \\
{}^T G_c + i {}^T G'_c &= b/(a-b) + (a/2\pi) [(G_c + i G'_c) + \mu'_P G'_{cP} + \mu'_N G''_{cP}], \\
G_m + i G'_m &= (1/2) [-\xi_{II}(-b) - \eta_{II}(-b) + \eta_{II}(a) - (1/2)\xi_{II}(a) \mp (1/2)\bar{\lambda}_{II}(a)], \\
G_A(1) + i G'_A(1) &= \eta_I(a) - (1/2)\xi_I(a) \pm (a/2)\lambda_{III}(a), \\
G_A(2) + i G'_A(2) &= -\xi_I(-b) - \eta_I(-b) \mp (b/2)\lambda_{IV}(a), \\
G_c + i G'_c &= \pm (1/2)b\bar{\lambda}_I(a),
\end{aligned} \tag{65}$$

(66)*

and

$$\begin{aligned}
G'_{mP} &= (-1/4)\eta_I(-b) - (1/2)\eta_I(a) \mp (1/8)\lambda_{VII}(a) \pm (1/8)\xi_{III}(a), \\
G''_{mP} &= (-1/4)\eta_I(a) - (1/2)\eta_I(-b) \mp (1/8)\lambda_{VII}(-b) \pm (1/8)\xi_{III}(-b), \\
l'_{1P}(1) &= (-1/4)[\xi_I(a) - \xi_I(0)] - (3a/4)\lambda_{VI}(a) \pm (a/2)\lambda_{II}(a) \\
&\quad \pm (a/4)\{\lambda_{III}(a) + \lambda_{IV}(a)\}, \\
l''_{1P}(1) &= (-1/2)[\xi_I(a) - \xi_I(0)] \pm (a/2)\lambda_{IX}(-b), \\
l'_{1P}(2) &= (-1/2)[\xi_I(-b) - \xi_I(0)] \mp (b/2)\lambda_{IX}(a), \\
l''_{1P}(2) &= (-1/4)[\xi_I(-b) - \xi_I(0)] + (3b/4)\lambda_{VI}(-b) \mp (b/2)\lambda_{II}(-b) \\
&\quad \mp (b/4)\{\lambda_{III}(-b) + \lambda_{IV}(-b)\}, \\
G'_{cP} &= \pm (ab/8)\lambda_{IV}(a), \\
G''_{cP} &= \pm (ab/8)\lambda_{IV}(-b).
\end{aligned} \tag{67}$$

$$(IV) \quad N + \gamma \rightarrow P + \pi^-$$

$$\begin{aligned}
&c({}^T G_m + i {}^T G'_m) \cdot U_m + c({}^T G_A(1) + i {}^T G'_A(1)) \cdot U_A^{(1)} \\
&+ c({}^T G_A(2) + i {}^T G'_A(2)) \cdot U_A^{(2)} + c({}^T G_c + i {}^T G'_c) \cdot U_c,
\end{aligned} \tag{68}$$

where

$$\begin{aligned}
{}^T G_m + i {}^T G'_m &= (1/b) + (a/2\pi) [(G_m + i G'_m) + \mu'_P G'_{mP} + \mu'_N G''_{mP}], \\
{}^T G_A(1) + i {}^T G'_A(1) &= \mu_N + (a/2\pi) [(G_A(1) + i G'_A(1) - n_0) + \mu'_P l'_{1P}(1) + \mu'_N l''_{1P}(1)], \\
{}^T G_A(2) + i {}^T G'_A(2) &= \mu_P + (a/2\pi) [(G_A(2) + i G'_A(2) - l_0) + \mu'_P l'_{1P}(2) + \mu'_N l''_{1P}(2)], \\
{}^T G_c + i {}^T G'_c &= -a/(a-b) + (a/2\pi) [(G_c + i G'_c) + \mu'_P G'_{cP} + \mu'_N G''_{cP}], \\
G_m + i G'_m &= (1/2) [-\xi_{II}(a) - \eta_{II}(a) + \eta_{II}(-b) - (1/2)\xi_{II}(-b) \mp (1/2)\bar{\lambda}_{II}(-b)], \\
G_A(1) + i G'_A(1) &= -\xi_I(a) - \eta_I(a) \pm (a/2)\lambda_{III}(-b), \\
G_A(2) + i G'_A(2) &= \eta_I(-b) - (1/2)\xi_I(-b) \mp (b/2)\lambda_{IV}(-b), \\
G_c + i G'_c &= \mp (a/2)\bar{\lambda}_I(-b).
\end{aligned} \tag{69}$$

(70)

G'_{mP} , G''_{mP} , ..., G'_{cP} , G''_{cP} are respectively expressions obtained by exchanging the signs ('') and (') in (67).

*. $\bar{\lambda}_I(a)$ and $\bar{\lambda}_{II}(a)$ are equal to $\lambda_I(a)$ and $\lambda_{II}(a)$ in (6), but equal to those of Koba, Kotani and Nakai.¹⁾

In terms of these total matrix elements* we can express the differential cross sections for $\gamma \rightarrow \pi$ as follows:

$$\sigma d\Omega = (\epsilon^2/4\pi) \cdot (f^0)^2/4\pi \cdot (1/4MK_0) \cdot III d\Omega \quad \text{for neutral meson production,} \\ \sigma d\Omega = (\epsilon^2/4\pi) \cdot (f^2/4\pi) \cdot (1/4MK'_0) \cdot III d\Omega \quad \text{for charged meson production,} \quad (71)$$

$$III = ({}^TG_m^2 + {}^TG_m'^2)\chi_3 + \{ {}^TG_A^2(1) + {}^TG_A'^2(1) + {}^TG_A^2(2) + {}^TG_A'^2(2) \}\chi_5 + ({}^TG_c^2 + {}^TG_c'^2)\chi_1 \\ + ({}^TG_m {}^TG_A + {}^TG_m' {}^TG_A')\chi_4 + ({}^TG_m {}^TG_c + {}^TG_m' {}^TG_c')\chi_2 \\ + ({}^TG_A(1) {}^TG_A(2) + {}^TG_A'(1) {}^TG_A'(2))\chi_6, \quad (72)$$

$${}^TG_A = {}^TG_A(1) + {}^TG_A(2), \quad {}^TG_A' = {}^TG_A'(1) + {}^TG_A'(2). \quad (73)$$

We show in Table 5. and Table 6 the differential cross sections in laboratory system at 276 Mev of photon energy.

Table 5. $\sigma_\theta(10^{-30} \text{ cm}^2/\text{sterad.})$ in the case of P.S. coupling of symmetrical theory: $a/4\pi = 1.194^{**}$ ($a=15$)

	0°	30°	60°	90°	120°	150°	180°
$P + \gamma \rightarrow P + \pi^0$	14.9	12.7	7.9	4.1	2.0	1.0	0.71
$N + \gamma \rightarrow N + \pi^0$	15.1	12.7	8.0	4.2	2.3	1.4	1.2
$P + \gamma \rightarrow N + \pi^+$	15.1	10.9	9.6	7.3	5.2	4.1	3.7
$N + \gamma \rightarrow P + \pi^-$	17.4	13.1	12.6	10.6	8.5	7.2	6.8

Table 6. $\sigma_\theta(10^{-30} \text{ cm}^2/\text{sterad.})$ in the case of P.S. coupling of charged and neutral theory: $a/4\pi = 1.194$

	0°	30°	60°	90°	120°	150°	180°
$P + \gamma \rightarrow P + \pi^0$	6.2	5.7	4.7	3.9	3.4	3.1	3.0
$N + \gamma \rightarrow N + \pi^0$	11.4	10.8	9.3	7.6	6.2	5.2	4.9
$P + \gamma \rightarrow N + \pi^+$	22.8	14.4	11.8	9.6	7.6	6.4	6.0
$N + \gamma \rightarrow P + \pi^-$	18.0	11.9	11.5	10.4	8.8	7.8	7.3

Looking at these tables we note the following features: (1) The cross section for neutral meson production is of the same order as that for charged. Especially in the case of symmetrical theory, (2) the angular distributions for $\gamma \rightarrow \pi^0$ are larger in forward than in backward direction, (3) the differential cross sections at 90° both for the neutral meson production and for the charged agree, within a tolerable range, with the experimental data with the same suitably chosen coupling constant, and (4) $\sigma_{90^\circ}(P + \gamma \rightarrow P + \pi^0)/\sigma_{90^\circ}(P + \gamma \rightarrow N + \pi^+) \cong 0.56$, $\sigma_{90^\circ}(N + \gamma \rightarrow N + \pi^0)/\sigma_{90^\circ}(N + \gamma \rightarrow P + \pi^-) \cong 0.4$, $\sigma_{91.0^\circ}/\sigma_{90^\circ} \cong 1.45$. These circumstances are shown in Fig. 5.

* Cf. (53), (62), (64) and (68).

** For this value of coupling constant, we obtain $\mu_P' = 1.045$, $\mu_N' = -1.321$ from (49).

§ 6. Discussions

Looking back upon the results obtained, we find that P.S. coupling in symmetrical theory seems to be the most promising, and that the matrix elements of U_m class among the four independent gauge-invariant fundamental expressions in (4) play the most important role for our $\gamma-\pi^0$ process and determine the character of the angular distribution for this process.

But the results in § 3 are rather open to doubt since we have calculated the cross section for $\gamma-\pi^0$ up to the order $e^2 f^6$ without the estimation of the cross terms of (ef) and (ef^6) . If the ef^5 -term is small compared with ef , there will be no difficulty, but if contrariwise that term is large, the cross term of (ef^3) and (ef^6)

becomes also considerable, and we are compelled then to examine the ef^7 , ..., terms. Thus more detailed investigations into the convergence nature of perturbation theory will become necessary. It, however, seems that the evaluations of higher order terms cannot be easily done so long as more powerful methods are not available.

Although our method in § 4 and § 5 is more or less doubtful, we may hope it represents perhaps a fair approximation for $\gamma-\pi$, conjecturing from the results in § 5. Under this presumption, it may be stated that the contribution from the orders higher than the fourth is considerably large, so that it should be difficult to see the aspects for $\gamma-\pi^0$ process only from the fourth order correction. In this meaning, it will not be possible to say anything about P.V. coupling from our calculation with the matrix element up to the fourth order only. But the cross sections for $\gamma-\pi^0$ process in P.S. coupling up to the order $e^2 f^6$ have shown the same tendency as that obtained in § 5. Has this fact happened by chance? If the theoretical ground which gives rise to these circumstances is founded by some methods, we shall be able to get insight into the problem with the matrix elements calculated up to the fourth order. We, of course, can not always consider the weak coupling approximation in the nucleon-meson system to be valid and we do not strongly insist that our results showing good agreement with the experiments are theoretically sound, but we should like to emphasize that there are some cases in which the results obtained by taking into account of higher order effect show considerably different properties from those yielded by the calculation of the lowest order, though in such cases perturbation method will lose its original meaning.

Moreover, we have to note that there are other important problems for $\gamma-\pi^0$ process,

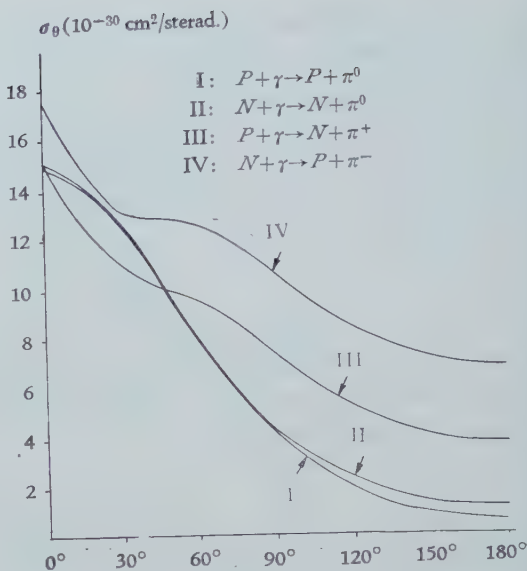


Fig. 5. Angular distribution for $\gamma-\pi$ in laboratory system at 276 Mev of photon energy when symmetrical theory is adopted.

for example, the dependence of cross section upon the incident photon energy, for which the study is now in progress. Within the range treated in this paper also, some problems have been left untouched, for instance, the unrenormalizable divergences, as stated in § 5, in the calculation of fourth order for Pauli-type interaction when P.V. coupling is adopted. In order to get rid of this difficulty we might assume, as a trial, the following interactions :

$$(g^0/\mu)(e/2M)\bar{\psi}(x)i\gamma_5\tau(\mu'_P\tau_P+\mu'_N\tau_N)\sigma_{\mu\nu}\psi(x)F_{\mu\nu}(x)\phi^0(x) \quad (74)$$

for the neutral meson,

$$\begin{aligned} & (1/2)(g/\mu)(e/2M)\bar{\psi}(x)(i\gamma_5\tau_{PN}\mu'_N\tau_N+\mu'_P\tau_P i\gamma_5\tau_{PN})\sigma_{\mu\nu}\psi(x)F_{\mu\nu}(x)\phi(x) \\ & + (1/2)(g/\mu)(e/2M)\bar{\psi}(x)(i\gamma_5\tau_{NP}\mu'_P\tau_P+\mu'_N\tau_N i\gamma_5\tau_{NP})\sigma_{\mu\nu}\psi(x)F_{\mu\nu}(x)\phi^*(x) \end{aligned} \quad (75)$$

for the charged meson.

When (charged + neutral) theory is adopted, by introducing these interactions the divergencies can be eliminated, but in symmetrical theory they can not still be removed. If these interactions (74) and (75) turn out to be necessary, not only $\gamma-\pi$ process but also the other ones will be influenced by them. For instance, in the lowest order calculation for $\gamma-\pi^0$ process, taking into account of the a.m.m. of nucleons, the equivalence comes into existence and the P.V. coupling also gives rise to the angular distribution with backward predominance.

In conclusion, the author wishes to express his sincere thanks to Prof. K. Husimi and Asst. Prof. Z. Koba for their guidances and to Messrs. S. Nakai, T. Kotani and N. Mugibayashi for their valuable discussions. Moreover, he wishes to express his gratitudes for the financial aid from the Yomiuri Yukawa Fellowship to him.

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Meson Reactions in Deuterium and Meson-Nucleon Scattering

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Various meson reactions in deuterium are investigated on the basis of impulse approximation. Then one obtains the close connection between deuteron data and nucleon data, which is shown to be useful to get an insight into the spin- and charge-dependences of meson-nucleon interaction. The brief discussions of available results of experiments are also given.

§ 1. Introduction

In the previous paper¹⁾ we have investigated in greater detail all possible charge dependence of meson-nucleon scattering. Therefore, the spin-dependence is the next problem that must be examined. In this note we want to give the concise expressions for meson induced reactions of deuteron, which may serve to see the spin- and charge-dependence of meson-nucleon scattering if combined with meson-nucleon data. They are derived from the "impulse approximation" which is often used to analyze various phenomena particularly concerning deuteron.

§ 2. Theoretical

When fast π^+ -meson beam is bombarded on deuterium, the following five reactions are possible :

$$\pi^+ + d \rightarrow \begin{cases} p + p \cdots \cdots \cdots \text{nuclear capture,} & \text{(i)} \\ p + p + \gamma \cdots \cdots \cdots \text{radiative capture,} & \text{(ii)} \\ p + p + \pi^0 \cdots \cdots \cdots \text{charge exchange scattering,} & \text{(iii)} \\ d + \pi^+ \cdots \cdots \cdots \text{elastic scattering,} & \text{(iv)} \\ p + n + \pi^+ \cdots \cdots \cdots \text{inelastic scattering.} \end{cases}$$

Quite similar arguments hold for π^- -meson as for π^+ -meson. Therefore, let us confine our discussion to π^+ -meson case.

Many experimental researches^{2),3)} have been performed concerning the nuclear capture and its inverse since Cheston⁴⁾ has pointed out that these reactions are quite useful to determine the spin of π^\pm -meson. The total cross section of $\pi^\pm - d$ reactions was measured by Steinberger⁵⁾ while the partial cross section of charge exchange scattering by Wilson and Perry⁶⁾.

Transition matrices As for the transition matrices we assume the following type (omitting irrelevant factors):

$$\begin{array}{lll}
 \pi^+ + n \rightarrow p + \gamma & (K\sigma + L)\tau_+ & d\sigma_n^{\gamma}/d\omega \\
 \pi^+ + p \rightarrow p + \pi^+ & A_p\sigma + a_p & d\sigma_p^{\sigma\pi d}/d\omega \propto |A_p|^2 + |a_p|^2 \\
 \pi^+ + n \rightarrow n + \pi^+ & A_n\sigma + a_n & d\sigma_n^{\sigma\pi d}/d\omega \propto |A_n|^2 + |a_n|^2 \\
 \pi^+ + n \rightarrow p + \pi^0 & \sqrt{2}(B_n\sigma + b_n)\tau_+\omega_- & d\sigma_n^{\pi\pi}/d\omega \propto 2(|B_n|^2 + |b_n|^2)
 \end{array} \quad (1.1)$$

where τ_+ and ω_- mean the charge operators giving rise to the transitions:

$$\begin{cases} \tau_+ : & (\text{neutron}) \rightarrow (\text{proton}), \\ \omega_- : & (\text{positive meson}) \rightarrow (\text{neutral meson}). \end{cases}$$

$K, L; A_p, \dots, b_n$ are in general the (complex) functions of momenta of particles involved. The differential cross sections $d\sigma/d\omega$ for these processes are also given above (again discarding the common factors independent of spin and charge). The so-called charge independence* is established if and only if

$$\left. \begin{aligned} a_p - a_n &= 2b_n & (\equiv 2b), \\ A_p - A_n &= 2B_n & (\equiv 2B). \end{aligned} \right\} \quad (1.2)$$

Furthermore we use the notations:

$$\begin{aligned} a_p + a_n &= 2a, \\ A_p + A_n &= 2A. \end{aligned}$$

Next we are going to examine individual modes of reactions.

(i) Nuclear capture

The detailed balancing consideration leads to the famous relationship

$$\frac{d\sigma_{cap}}{d\Omega} = \frac{2}{3} \frac{p^2}{q^2} \frac{d\sigma_{prod}}{d\Omega} \quad (i.1)$$

between the capture and production cross sections (in the c. m. system)**, where p and q are the proton and meson momentum, respectively. As for the total cross sections, it may be expected that $\sigma_{prod} \propto q^3$ because of the p -wave feature of produced meson ($d\sigma_{prod}/d\Omega \propto \cos^2 \theta$, where θ is an angle between the incident proton beam and meson); and thus one gets a crude guess $\sigma_{cap} \propto q$. However, the new experiments of Steinberger et al.⁽³⁾ showed stronger energy dependence than predicted here: $\sigma_{cap} \propto q^2$ or steeper. As was emphasized by Chew et al.⁽⁷⁾, this reaction provides an important check for nuclear forces.

* More precisely see ref. 1). The nucleons are treated non-relativistically throughout this paper.

** The c. m. cross sections will be marked by capital $d\Omega$, while the laboratory cross sections by small $d\omega$ in this paper.

(ii) Radiative capture

At first let us compare two reactions induced by π^+ -meson with the same energy:

$$\begin{cases} \pi^+ + n \rightarrow p + \gamma: & d\sigma_n^\pi/d\omega, \\ \pi^+ + d \rightarrow p + p + \gamma: & d\sigma_d^\pi/d\omega. \end{cases}$$

As for the former we refer other papers, e.g., ref. 8). In the latter we are interested only in the direction of photon *irrespective of its energy*. If the energy of π^+ is sufficiently great to permit the use of the impulse (and target at rest) approximation⁹⁾, we readily find

$$\frac{d\sigma_d^\pi}{d\omega} = \frac{d\sigma_n^\pi}{d\omega} [D(\theta) - \epsilon^\pi I(\theta)], \quad (\text{ii} \cdot 1)$$

$$D(\theta) = \frac{\int d\mathbf{p} k^2 dk \delta\left(\sqrt{\mu^2 + q^2} - E_d - \frac{p^2}{M} - \frac{|\mathbf{k} - \mathbf{q}|^2}{4M} - k\right) \phi_d^2\left(\mathbf{p} + \frac{\mathbf{k} - \mathbf{q}}{2}\right)}{\int d\mathbf{p} k^2 dk \delta\left(\sqrt{\mu^2 + q^2} - \frac{p^2}{2M} - k\right) \delta(\mathbf{p} + \mathbf{k} - \mathbf{q})}, \quad (\text{ii} \cdot 1')$$

$$I(\theta) = \frac{\int d\mathbf{p} k^2 dk \delta\left(\sqrt{\mu^2 + q^2} - E_d - \frac{p^2}{M} - \frac{|\mathbf{k} - \mathbf{q}|^2}{4M} - k\right) \phi_d\left(\mathbf{p} + \frac{\mathbf{k} - \mathbf{q}}{2}\right) \phi_d\left(\mathbf{p} - \frac{\mathbf{k} - \mathbf{q}}{2}\right)}{\int d\mathbf{p} k^2 dk \delta\left(\sqrt{\mu^2 + q^2} - \frac{p^2}{2M} - k\right) \delta(\mathbf{p} + \mathbf{k} - \mathbf{q})};$$

where ϵ^π is a phase factor discussed below, \mathbf{q} and \mathbf{k} means the meson and photon momentum ($q = |\mathbf{q}|$, $k = |\mathbf{k}|$), μ and M denote the meson and nucleon mass, respectively; E_d is a binding energy of deuteron (2.23 Mev) and $\phi_d(\mathbf{p})$ is a Fourier transform of deuteron function:

$$\phi_d(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \int \psi_d(\mathbf{r}) e^{i\mathbf{p}\mathbf{r}} d\mathbf{r}.$$

The integration over photon momentum must be performed keeping its direction at $d\omega$ ($\angle(\mathbf{q}, \mathbf{k}) = \theta$). The so-called closure approximation gives the following estimates of (ii · 1'):

$$D^c(\theta) = 1, \quad I^c(\theta) = \int d\mathbf{r} |\phi_d(\mathbf{r})|^2 \frac{\sin(|\mathbf{k} - \mathbf{q}|r)}{|\mathbf{k} - \mathbf{q}|r}, \quad (\text{ii} \cdot 1'')$$

where \mathbf{k} is determined from the free nucleon case,

$$\begin{cases} \sqrt{\mu^2 + q^2} = k + \frac{p^2}{2M}, \\ \mathbf{q} = \mathbf{k} + \mathbf{p}. \end{cases}$$

ϵ^π depends on the spin-dependence of photo-mesonic process. Assuming the "target at rest" approximation one finds

$$\epsilon^\gamma = \frac{\frac{1}{3} |\mathbf{K}|^2 + |\mathbf{L}|^2}{|\mathbf{K}|^2 + |\mathbf{L}|^2}, \quad (\text{ii} \cdot 2)$$

evidently

$$1/3 \leq \epsilon^\gamma \leq 1.$$

The radiative capture process of π^+ -meson by deuteron leads to the di-proton final state, which suffers an appreciable effect of the exclusion principle. The second term $\epsilon^\gamma I(\theta)$ of (ii.1) is just responsible for this effect, and will make σ_d^γ smaller than σ_n^γ (except near threshold). It is clear from the closure estimates (ii.2'') that the effect considered predominates in the forward emission of photon while it is small in the backward emission. In the latter case deuteron data can be regarded as the direct source of the neutron case, $\pi^+ + n \rightarrow p + \gamma$, without any ambiguities due to the phase factor ϵ^γ .

Furthermore the similar process $\gamma + d \rightarrow \pi + (\text{di-nucleon})$ will be discussed in another paper⁽¹⁰⁾.

It seems worth while to write down the detailed balancing relation between two elementary processes :

$$\gamma + p \rightleftharpoons n + \pi^+ \quad \left\{ \frac{d\sigma_{prod}^\gamma/d\Omega}{d\sigma_{cap}^\gamma/d\Omega} \right\} \quad \frac{d\sigma_{cap}^\gamma}{d\Omega} = 2 \frac{k^2}{q^2} \frac{d\sigma_{prod}^\gamma}{d\Omega}, \quad (\text{ii} \cdot 3)$$

where k and q denote the photon and meson momentum (in the c. m. system), respectively, and a factor 2 is originated from a number of possible polarizations of light quantum. We can readily estimate the radiative capture cross section from observed production one⁽⁸⁾.

(iii) Charge exchange scattering

This process is inevitably inelastic, because of the lack of bound di-proton. Within the impulse approximation its cross section $d\sigma_d^{ex}/d\omega$ (irrespective of energy) is related to $d\sigma_n^{ex}/d\omega$ as follows :

$$\frac{d\sigma_d^{ex}}{d\omega} = \frac{d\sigma_n^{ex}}{d\omega} [D(\theta) - \epsilon^{ex} I(\theta)], \quad (\text{iii} \cdot 1)$$

where $D(\theta)$ and $I(\theta)$ are the appropriate integrals which can be obtained from (ii.1') by replacing photon by scattered π^0 -meson. The phase factor ϵ^{ex} is

$$\epsilon^{ex} = \frac{\frac{1}{3} |\mathbf{B}_n|^2 + |\mathbf{b}_n|^2}{|\mathbf{B}_n|^2 + |\mathbf{b}_n|^2}, \quad (\text{iii} \cdot 2)$$

and is limited within

$$1/3 \leq \epsilon^{ex} \leq 1.$$

The second term of (iii.1) represents an effect of the exclusion principle.

- (iv) **Ordinary scattering**, (elastic $\pi^+ + d \rightarrow d + \pi^+$
plus inelastic $\pi^+ + d \rightarrow p + n + \pi^+$)

For energetic π^+ -mesons, one gets quite similarly,

$$\frac{d\sigma_a^{ord}}{d\omega} = \left(\frac{d\sigma_p^{ord}}{d\omega} + \frac{d\sigma_n^{ord}}{d\omega} \right) [D(\theta) + \epsilon^{ord} I(\theta)], \quad (\text{iv} \cdot 1)$$

where $d\sigma_a^{ord}/d\omega$ denotes the cross section of ordinary scattering of π^+ by deuteron (irrespective again of the energy of scattered π^+ ; i.e. *elastic plus inelastic*) while $d\sigma_p^{ord}/d\omega$ or $d\sigma_n^{ord}/d\omega$ is that for free proton or neutron; $D(\theta)$ and $I(\theta)$ are the same as in (iii.1) ignoring the small mass difference between π^\pm and π^0 . The phase factor ϵ^{ord} is now given by

$$\epsilon_{ord} = \frac{\frac{1}{3} (A_p^* A_n + A_n^* A_p) + a_p^* a_n + a_n^* a_p}{|A_p|^2 + |a_p|^2 + |A_n|^2 + |a_n|^2}, \quad -1 \leq \epsilon^{ord} \leq 1. \quad (\text{iv} \cdot 2)$$

As was in (ii), from the meson scattered in the backward direction one obtains the estimates of $d\sigma_n/d\omega$ from directly observed $d\sigma_a/d\omega$ and $d\sigma_p/d\omega$. The fraction of elastically scattered meson (i.e., $\pi^+ + d \rightarrow d + \pi^+$) may provide further insight into the spin-dependence of meson scattering, though for this purpose at least the elastic scattering must be examined in a more refined manner.

Similar arguments were also given by Fernbach et al.⁽¹¹⁾, but they gave another expression for $d\sigma_a/d\omega$:

$$\frac{d\sigma_a^{ord}}{d\omega} = \left(\frac{d\sigma_p^{ord}}{d\omega} + \frac{d\sigma_n^{ord}}{d\omega} \right) D(\theta) + 2 \cos \delta \left(\frac{d\sigma_p^{ord}}{d\omega} \frac{d\sigma_n^{ord}}{d\omega} \right)^{\frac{1}{2}} I(\theta).$$

Although they left $\cos \delta$ undetermined, we can give its form as

$$2 \cos \delta = \frac{\frac{1}{3} (A_p^* A_n + A_n^* A_p) + a_p^* a_n + a_n^* a_p}{\sqrt{(|A_p|^2 + |a_p|^2)(|A_n|^2 + |a_n|^2)}}; \quad -1 \leq \cos \delta \leq 1.$$

.....

In the above we have described final di-nucleon and meson by plane waves, which is undoubtedly not a good approximation. However the former may be justifiable from the fact that we considered only the cross sections irrespective of meson energy.

Finally it is noted that many authors⁽¹²⁾ gave the analyses of these reactions on the basis of meson theories.

§ 3. Comparison with experiments

The total cross section of π^- -hydrogen scattering was measured by Steinberger⁽⁵⁾ and Anderson⁽¹³⁾ over a considerably wide range of meson energy. It varies as roughly proportional to the square of incident meson energy. This strong energy dependence should be closely connected with the similar dependence of nuclear capture of mesons by deuterium (§ 2, (i)) and of photo- π^0 -mesons from hydrogen⁽⁸⁾⁽¹⁴⁾.

It seems reasonable to assume that the cross sections for $\pi^- + p \rightarrow p + \pi^-$ and $\rightarrow n + \pi^0$ are equal to those of $\pi^+ + n \rightarrow n + \pi^+$ and $\rightarrow p + \pi^0$, respectively, ignoring the

Coulomb effect which is presumably unimportant for fast mesons. Let us now estimate the partial cross sections of deuteron reactions for separately with the help of the relations summarized in the preceding section.

(i) **Nuclear capture** Its cross section was measured experimentally³⁾ and is shown in Fig. 1.

(ii) **Radiative capture**

σ_n^r is calculated from the observed inverse and illustrated in Fig. 2. (where we assume the total cross section for $r+p \rightarrow n+\pi^+$ at 250 Mev as 0.15 mb).

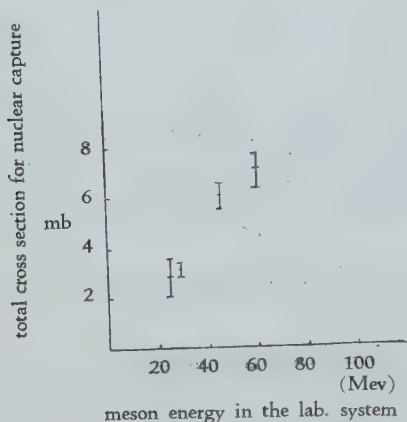


Fig. 1 The total cross section for
 $\pi^+ + d \rightarrow p + p$
(Steinberger et al., ref 3))

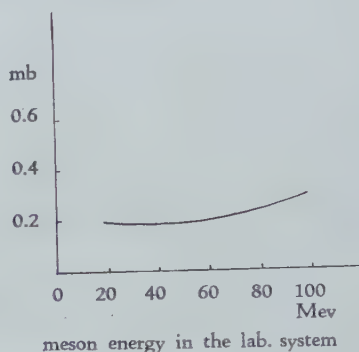


Fig. 2 The cross section for
 $\pi^+ + n \rightarrow p + \gamma$

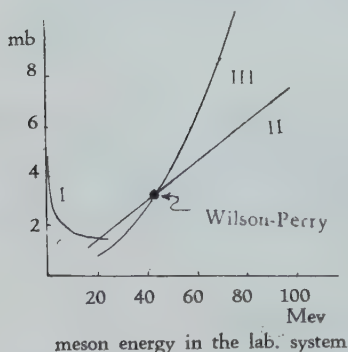


Fig. 3 The cross section for
 $\pi^+ + n \rightarrow p + \gamma$

I: eq. (3.1)

II: σ_n^{exoc} (meson energy)

III: σ_n^{exoc} (meson energy)²

As for σ_n^r , for instance, a rough estimate is (cf. (ii.1) and (ii.1''))

$$\sigma_n^r \approx \sigma_n^r (1 - 0.44\epsilon^r) \quad \text{at } \sim 50 \text{ Mev,}$$

where we assume that $d\sigma_n^r/d\omega$ is not dependent on θ^* , and thus from Fig. 2

$$\sigma_n^r \approx 0.21 \sim 0.32 \text{ mb} \quad \text{at } \sim 50 \text{ Mev.}$$

Even in the worst case, we may safely regard σ_n^r as $\lesssim 0.5$ mb at ~ 50 Mev.

(iii) **Charge exchange scattering**

At very low meson energy, the total cross section of charge scattering is estimated as

$$\sigma_n^{ex} \approx \frac{v_0}{v_+} 1.4 \text{ mb,} \quad \text{at very low meson energy} \quad (3.1)$$

* Considering the angular distribution of photo π^+ -mesons from hydrogen, this seems to be approximately correct.

where v_+ and v_0 is the velocities of the incident π^+ and scattered π^0 , respectively. This result is derived from the two data:

- (a) the slow π^- -capture by hydrogen (the capture rate of $\pi^- + p \rightarrow n + \gamma$ is equal to that of $\pi^- + p \rightarrow n + \pi^0$),
and (b) the observed cross section for $\gamma + p \rightarrow n + \pi^+$.

According to Wilson and Perry⁶⁾ (who assumed the isotropy of scattered π^0), the total cross sections for three nuclei at 44 Mev π^+ are as follows.

$$\sigma_d^{ex} = 1.5 \text{ mb for deuterium,}$$

$$\sigma_{Be}^{ex} = 3.7 \text{ mb for beryllium,}$$

and $\sigma_o^{ex} = 0.8 \text{ mb for oxygen.}$

If we assume the alpha-particle model, which may be valid for light nuclei, we can interpret

$$\sigma_{Be}^{ex} \text{ as } \sigma_{Be}^{ex} = 2\sigma_\alpha^{ex} + \sigma_n^{ex}$$

and $\sigma_o^{ex} \text{ as } \sigma_o^{ex} = 4\sigma_\alpha^{ex},$

where σ_α^{ex} means the total cross section of charge exchange scattering for an alpha-particle. Then we get*

$$\left. \begin{array}{l} \sigma_n^{ex} \approx 3.3 \text{ mb} \\ \sigma_\alpha^{ex} \approx 0.2 \text{ mb} \end{array} \right\} \text{ at 44 Mev } \pi^+. \quad (3.2)$$

Again assuming the isotropy of $d\sigma_n^{ex}/d\omega$ and using the closure estimates, we find (cf. (iii.1))

$$\sigma_d^{ex} \approx \sigma_n^{ex} (1 - 0.64\epsilon^{ex}) \text{ at } \sim 50 \text{ Mev } \pi^+.$$

Combining these results, we see

$$\epsilon^{ex} \approx 0.8.$$

Therefore, if our assumptions are valid, the spin independent transition should predominate in the charge exchange scattering:

$$|\delta_n|^2 \approx 3|B_n|^2 \text{ at } \sim 44 \text{ Mev } \pi^+.$$

We have not as yet direct experimental informations concerning the energy dependence of charge exchange scattering. Nevertheless we can suppose a rather strong raise of this cross section with increasing meson energy as compared with the total cross section of π^- -scattering. The other evidence for it lies in the analysis of emulsion data¹⁵⁾ (see Appendix I).

The total cross section σ_n^{ex} is shown in Fig. 3.

(iv) Ordinary scattering

One can readily obtain the total cross section for ordinary scattering from the total cross section for $\pi^+ - d$ reactions and the partial cross sections described in (i), (ii) and

* As for σ_α^{ex} , see Appendix II.

(iii). The preliminary deuteron data (at 56 Mev) of Steinberger⁵⁾ combined with Wilson-Perry's result seem to suggest that the charge independence is no more valid assuming the isotropy of scattered meson. But this situation can be improved so as to be valid the charge independence, if we assume the scattering mechanism e.g. in the meson theories (the effect of exclusion principle in the intermediate di-neutron state)¹²⁾. More complete discussions will be given after the publication of *accurate* experimental results.

Appendix I.

Energy dependence of charge exchange scattering

Since Bernardini et al.¹⁵⁾ did not report the separate data of "sudden stops" into zero prong stars and exchange scatterings, we must suitably estimate each of them. For this purpose we demand the following two assumptions which may be regarded as supported by some experimental evidences :

Firstly we assume that the prong number distribution of stars of light nuclei (C, N and O) caused by fast π^- -capture is the same as that of so-called σ -stars of light nuclei¹⁶⁾. This assumption may be justified from two reasons ; (1) the kinetic energy of π^- may not change appreciably the momentum transfer accompanying capture because of the rather large rest energy of π -meson ; therefore the features of nuclear disruption due to π^- -capture are practically insensitive to the kinetic energy of π^- ; (2) there are strong evidences suggesting that the general features of light nucleus-stars are independent of the agent (i.e., the bombarding particle on light nuclei)¹⁷⁾.

Secondly we assume that the nuclear capture cross section is proportional to nuclear area ; in fact we can show⁸⁾¹⁸⁾ that the capture cross section is close to the geometrical area if we use the observed deuteron data and the "quasi-deuteron" model of Heidmann¹⁹⁾ for a nucleus.

Upon the basis of these two assumptions we can derive the prong number distribution of heavy nucleus-stars caused by fast mesons, which in turn is sufficient to find out the average number of prongs and the probable number of zero prong stars in a self-consistent way. (We can safely assume that the light nuclei do not contribute to the zero prong stars, as might be seen from the alpha-particle model and empirical evidences.) The errors of these results are now estimated from the deviation of prong distribution from the theoretical expectation or Poisson's one²⁰⁾. Our estimates are shown in Table AI. The general trend of average number of prongs *vs.* incident meson energy is in satisfactory agreement with the statistical evaporation theory providing that a some parts of

Table AI

energy of incident meson (Mev)	no. of stars	no. of stops	estimated no. of zero prong stars	average no. of prongs	estimated no. of charge exchange scatterings
σ -stars*	—	—	—	1.1	—
30~50	49	8	8($\times 2$)	1.7 \pm 0.6	0 \pm 4
60~110	139	26	13($\times 2$)	2.1 \pm 0.7	13 \pm 13

* heavy nuclei only.

total π^- -energy are carried away by fast neutron without heating up the whole nucleus as in the case of σ -stars¹⁶⁾. However if we assume *all* stops to be zero prong stars we can see that this satisfactory situation is lost. Thus our final results may be regarded as the justification of rather complicated procedures that have been used.

Finally we get the probable number of charge exchange scattering from the difference (stops) — (zero prong stars), which is also shown in Table AI. Our results seem again to show the strong energy dependence of charge exchange scattering.

Appendix II.

Meson-nucleus scattering

Since more complete analyses will be given elsewhere, only the results of closure approximation are stated here. Moreover we want to use the alpha-particle model for a nucleus.

The meson scattering cross sections by alpha-particle (free or inside the nucleus) are as follows :

$$\frac{d\sigma_{\alpha}^{ord}}{d\omega} \propto 4 [|a|^2 + |A|^2 + |b|^2 + |B|^2] + 4 [3|a|^2 - |A|^2 - |b|^2 - |B|^2] \left\langle \frac{\sin |\mathbf{q} - \mathbf{q}'| r}{|\mathbf{q} - \mathbf{q}'| r} \right\rangle_{av}$$

$$\left(\begin{array}{ll} 2a = a_p + a_n, & 2A = A_p + A_n, \\ 2b = a_p - a_n, & 2B = A_p - A_n. \end{array} \right)$$

for ordinary scattering,

$$\frac{d\sigma_{\alpha}^{ex}}{d\omega} \propto 4 [|\dot{a}_n|^2 + |B_n|^2] - 4 [|\dot{b}_n|^2 + |B_n|^2] \left\langle \frac{\sin |\mathbf{q} - \mathbf{q}'| r}{|\mathbf{q} - \mathbf{q}'| r} \right\rangle_{av}$$

for charge exchange scattering,

where \mathbf{q} and \mathbf{q}' mean the momenta of incident and scattered meson, respectively; $\langle \rangle_{av}$ denotes an average on the square of alpha-particle wave functioning. The requirement of small meson nucleus scattering^{16) 21) 22)} * should demand that

(a) a must be zero, or in other words, $a_p = -a_n$,

and (b) $\left\langle \frac{\sin |\mathbf{q} - \mathbf{q}'| r}{|\mathbf{q} - \mathbf{q}'| r} \right\rangle_{av}$ must be close to unity;

the latter suggests the strong degree of correlation in nuclear structure.

* Also see (3.2).

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Meson Production by γ -Rays from Deuterium

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Photo- π^\pm and π^0 -mesons from deuterium, which are expected to give us a definite clue to photo-mesons from neutrons and spin- and charge-dependences of photo-mesonic processes, are discussed phenomenologically, adopting the so-called impulse approximation. Energy spectra of mesons at three angles (0, 45 and 90 degrees) are given in figures (Figs. 9.1—9.18) for incident γ -rays with 200, 250, 300 and 350 Mev. Our results will be useful for direct comparison with observed data.

§ 1. Introduction and summary

In order to complete our discussions which appeared in the previous papers¹⁾²⁾, we want to investigate extensively the photo-meson production from deuterium, i.e.,

$$\gamma + d \rightarrow N + N + \pi. \quad (1.1)$$

As was pointed out in the preceding paper²⁾, this process is expected to give some definite clue to the dependency of matrix elements on charge and spin which are responsible for photo-mesonic processes and also the photo-mesons from neutrons:

$$\gamma + n \rightarrow \begin{cases} p + \pi^-, \\ n + \pi^0. \end{cases} \quad (1.2)$$

It is, therefore, highly important to discuss (1.1) in detail and it is just the purpose of this paper.

This problem has already been worked out by many authors³⁾⁴⁾⁵⁾, but we think that our analyses are more general than the existing ones (refs. 3 and 4).

In § 2 we will state some preliminary terms. § 3 will be devoted to discuss the inevitable situation adopting the so-called impulse approximation. § 4 and § 5 contain the arguments concerning the choice of wave functions and transition matrices. The analytic expressions for differential cross sections are given in § 6 and their physical meaning is intuitively interpreted in § 7 and § 8. The numerical results are illustrated in § 9 and those may be useful to get insight concerning the spin- and charge-dependences of photo-meson production from observed data.

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§ 2. Preliminary considerations

At first let us list down all possible processes caused by γ -ray bombardment on deuterium :

$$\text{simple disintegration} \quad \gamma + d \rightarrow p + n \quad (2.1)$$

$$\text{meson production} \quad \gamma + d \rightarrow \begin{cases} n + n + \pi^+, \\ p + n + \pi^0, \quad d + \pi^0, \\ p + p + \pi^-. \end{cases} \quad (2.2)$$

The multiple meson production may occur for very energetic γ -rays. But we restrict ourselves to single meson production, experiments of which are now available in the laboratory.

Among these processes, the simple photo-disintegration (2.1) has been well examined so far⁶⁾. Thus it seems unnecessary to develop further analysis for this process, though there remain some unsolved problems such as the role of exchange or interaction moment⁷⁾.

The total cross section of (2.1) is, as is well-known, given by^{8)*}

$$\sigma_{\text{dis}} = 20.4 \left[1 - \left(\frac{k}{29E_d + k} \right)^2 \right] \left(\frac{k - E_d}{E_d} \right)^{3/2} \left(\frac{E_d}{k} \right)^3 \text{ mb}, \quad (2.3)$$

k = incident photon energy,

E_d = binding energy of deuteron = γ^2/M ,

M = nucleon mass,

where we take into account only the electric dipole transition. More trustful estimates of the cross section, for which higher multipoles are taken into account, have been given by Schiff and Marshal-Guth⁶⁾. On the other hand, the total yield of photo- π^+ or $-\pi^0$ from hydrogen is known experimentally as²⁾

$$\sigma_p^+ \approx 0.15 (k - E_{th}) / (250 \text{ Mev} - E_{th}) \text{ mb} \quad (2.4)$$

or

$$\sigma_p^0 \approx 0.05 (k - E_{th})^{3/2} / (250 \text{ Mev} - E_{th})^{3/2} \text{ mb},$$

where E_{th} means the threshold energy for photo-meson production. Assuming the photo-meson yields from deuterium do not differ appreciably from (2.4) (see § 8), we can show that the meson production (2.2) occur much more likely rather than the simple disintegration (2.1) for high energy photons (i.e., with energy greater than, say, 200 Mev). This forms quite a contrast to the meson production by nucleon-nucleon collision, which is only minor parts as compared with the frequency of elastic scattering at the energy of a few hundred Mev of incident nucleon.

§ 3. Fundamental assumptions of our calculations

For high energy γ -ray ($\gtrsim 200 \text{ Mev}$) its wave length is considerably smaller than the deuteron radius $1/\gamma = 4.3 \times 10^{-13} \text{ cm}$. Thus we can suppose that in the process of photo-meson production from deuteron one of the two nucleons in the deuteron ("target nucleon") contributes to meson production while the other keeps ignorantly its own motion.

* We use natural units ($\hbar=c=1$) throughout this paper. Moreover we treat mesons *relativistically* while nucleons *non-relativistically*.

This picture is nothing but the concept of impulse approximation⁹⁾, which we want to adopt throughout this paper. But there may be of course possible effects as the so-called exchange currents or others. If such effects are no more negligible, the impulse approximation will become rather less acceptable. Nevertheless, since we have not any other powerful and trustful methods of calculation, we are compelled to use this crude approximation.

Next we must know the wave functions of nucleonic system and of produced meson. For brevity, let us assume the plane wave for meson wave function, ignoring a small deformation expected from the Coulomb and meson-nucleon interactions (e.g., meson-nucleon or nucleus scattering¹⁰⁾ reveal the existence of the latter). The Hulthén's type is adopted for deuteron function in order to simplify our calculations. Furthermore, as we shall see later, the relative angular momentum of final nucleonic system is not limited within lower quantum numbers (this is due to the largeness of deuteron radius). Thus we use, too, the plane wave for final nucleonic system. Since this is not a good approximation for low energy region concerning nucleonic relative motion or the higher side of meson-energy spectra, we shall also investigate the case of distorted *S*-wave for final nuclear state (also see ref. 1)). Of course if we use the nuclear potential proposed by Christian et al. or some others¹¹⁾, we can perform more refined calculations within the impulse approximation. We shall, however, not enter into their details because of their complexity and crudeness of our investigation.

§ 4. The wave functions for nucleonic system

Next let us summarize the necessary terms.

For convenience, we want to use the isotopic spin formalism. Thus the total wave function $\Psi(1, 2)$ describing the two nucleon system should be given by antisymmetrized products of following three partial wave functions:

(a) charge function,

$$\begin{aligned} \text{two protons:} \quad & {}^3(\tau)_1 = p(1) \cdot p(2), \\ \text{one proton and one neutron:} \quad & \begin{cases} {}^3(\tau)_0 = \frac{1}{\sqrt{2}} \{ p(1)n(2) + n(1)p(2) \}, \\ {}^1(\tau)_0 = \frac{1}{\sqrt{2}} \{ p(1)n(2) - n(1)p(2) \}, \end{cases} \\ \text{two neutrons:} \quad & {}^3(\tau)_{-1} = n(1)n(2), \end{aligned}$$

where $p(j)$ and $n(j)$ denote the proton and neutron state:

$$\tau_3^{(j)} p(j) = p(j), \quad \tau_3^{(j)} n(j) = -n(j)$$

and $\tau^{(j)}$ is a isotopic spin of j th nucleon.

(b) spin function,

$$\text{triplet state:} \quad {}^3(\sigma)_m = \begin{cases} \alpha(1)\alpha(2) & (m=1), \\ \frac{1}{\sqrt{2}} \{ \alpha(1)\beta(2) + \beta(1)\alpha(2) \} & (m=0), \\ \beta(1)\beta(2) & (m=-1), \end{cases}$$

singlet state : $^1(\sigma)^0 = \frac{1}{\sqrt{2}} \{a(1)\beta(2) - \beta(1)a(2)\},$

where $\sigma_s^{(j)} a(j) = a(j), \quad \sigma_s^{(j)} \beta(j) = -\beta(j).$

and $\sigma^{(j)}$ means a spin of j th nucleon.

(c) spatial function (r =relative coordinate, $r=|r|$),

(i) deuteron state :

$$\psi_d(r) = A \frac{e^{-\gamma r} - e^{-\beta r}}{r}, \quad A^2 = \frac{\gamma\beta(\gamma + \beta)}{2\pi(\gamma - \beta)^2},$$

where $\gamma = \sqrt{ME_d}$ and β must be chosen so as to give the correct effective range for triplet- n - p -state. We ignored the effect of tensor force (*i.e.*, the small D -mixture).

(ii) distorted S -state in continuum :

$$\psi_s(r) = \frac{1}{pr} \{ \sin(pr + \delta) - e^{-\eta r} \sin \delta \},$$

where p is the relative momentum and $p=|p|$; δ means the phase shift. The second term is a correction to make the above function as good an approximation to the exact one inside the force range as possible. For that it is sufficient to fix η so as to give the correct effective range at $p=0$. Moreover we assume, for brevity and clearness, that the nuclear force is charge independent. This assumption has reality at least in low energy regions and may be checked from the comparison with observed data as in the case of slow π^- -meson capture by deuterium²⁾.

(iii) the plane wave, for convenience we discriminate its even and odd parts :

$$\phi_{\pm}(r) = \frac{1}{2} \{ e^{iPr} \pm e^{-iPr} \}.$$

It is noted that $\phi_{-}(r)$ is a correct wave function if we adopt the Serber type nuclear potential. If a final state consists of a di-proton, we must use the complicated Coulomb function for its description. As it is somewhat troublesome to treat Coulomb function, we use here the "neutral wave function" also for the di-proton, and the Coulomb effect may be taken into account, if necessary, by the so-called Coulomb penetration factor.

In order to get the complete spatial wave function, we must add the motion of the center of mass :

$$e^{iPR}$$

where P and R are the total momentum and the coordinate of center of mass of di-nucleon.

For example, the complete wave function of initial deuteron (which is assumed to be at rest) is

$$\Psi_d(1, 2) = {}^1(\tau)_0 {}^3(\sigma)_m \psi_d(r).$$

§ 5. The transition matrices

We describe the j th nucleon by $\Psi_I(j)$ and $\Psi_F(j)$, where I and F stand for initial and final states, respectively. Following Lax and Feshbach¹²⁾, the transition matrix which is responsible to the π^\pm -meson production from nucleon (j) bombarded by a photon with

momentum \mathbf{k} ($k=|\mathbf{k}|$) can be written as

$$\int \Psi_{\mathbf{r}}^{*}(j) \{ \mathbf{K}_{\pm}^{(j)} \sigma^{(j)} + L_{\pm}^{(j)} \} \tau_{\mp}^{(j)} \frac{e^{-i\mathbf{q}\cdot\mathbf{r}_j}}{\sqrt{2}q_0} \frac{e^{i\mathbf{k}\cdot\mathbf{r}_j}}{\sqrt{2}k} \Psi_1(j), \quad (5.1)$$

where \mathbf{q} is a meson momentum and

$$q=|\mathbf{q}|, \quad q_0=\sqrt{q^2+\mu^2}, \quad \mu=\text{meson mass},$$

and

$$\tau_{\pm}^{(j)} p(j) = n(j), \quad \tau_{\pm}^{(j)} n(j) = p(j).$$

As for π^0 -production $\{ \mathbf{K}_{\pm}^{(j)} \sigma^{(j)} + L_{\pm}^{(j)} \} \tau_{\pm}^{(j)}$ must be replaced by

$$\begin{aligned} & \{ \mathbf{K}_p^{(j)} \sigma^{(j)} + L_p^{(j)} \} \frac{1+\tau_3^{(j)}}{2} + \{ \mathbf{K}_n^{(j)} \sigma^{(j)} + L_n^{(j)} \} \frac{1-\tau_3^{(j)}}{2} \\ &= \left\{ \frac{\mathbf{K}_p^{(j)} + \mathbf{K}_n^{(j)}}{2} \sigma^{(j)} + \frac{L_p^{(j)} + L_n^{(j)}}{2} \right\} \quad (1\text{-part}) \\ &+ \left\{ \frac{\mathbf{K}_p^{(j)} - \mathbf{K}_n^{(j)}}{2} \sigma^{(j)} + \frac{L_p^{(j)} - L_n^{(j)}}{2} \right\} \tau_3^{(j)} \quad (\tau_3\text{-part}). \end{aligned} \quad (5.2)$$

In these expressions vector \mathbf{K} 's and scalar L 's have a dimension of length and may generally depend on \mathbf{q} , \mathbf{p} and \mathbf{k} . We adopt here these rather phenomenological description which is quite insensitive to any details of meson theories, since current meson theories give us as yet no good answers at the present stage. Available data of $\gamma+p \rightarrow n+\pi^+$ (or $\gamma+p \rightarrow p+\pi^0$) shows that \mathbf{K}_+ and L_+ are nearly constant (or \mathbf{K}_p and L_p are nearly proportional to q)²⁾.

In the case of photo-meson from deuterium, it only suffices to evaluate the matrix elements (we use here the "impulse approximation")

$$\begin{aligned} & \int \Psi_{\mathbf{r}}^{*}(1, 2) \left\{ \sum_j (\mathbf{K}_{\pm}^{(j)} \sigma^{(j)} + L_{\pm}^{(j)}) \tau_{\mp}^{(j)} \frac{e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}_j}}{2\sqrt{k}q_0} \right\} \Psi_d(1, 2), \\ & \int \Psi_{\mathbf{r}}^{*}(1, 2) \left[\sum_j \left\{ \left(\frac{\mathbf{K}_p^{(j)} + \mathbf{K}_n^{(j)}}{2} \sigma^{(j)} + \frac{L_p^{(j)} + L_n^{(j)}}{2} \right) \right. \right. \\ & \quad \left. \left. + \left(\frac{\mathbf{K}_p^{(j)} - \mathbf{K}_n^{(j)}}{2} \sigma^{(j)} + \left(\frac{L_p^{(j)} - L_n^{(j)}}{2} \right) \tau_3^{(j)} \right) \frac{e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}_j}}{2\sqrt{k}q_0} \right\} \right] \Psi_p(1, 2) \end{aligned}$$

for π^0 -production.

Furthermore, we may adopt here the "target at rest" approximation in accordance with the use of impulse approximation. We can, therefore, drop the affices (j) of \mathbf{K} 's and L 's, which are now replaced by the corresponding quantities for free nucleon (of course initially at rest).

In the π^0 -production there is a marked feature. That is, the 1-part of (5.2) preserves the charge multiplet while τ_3 -part gives rise to merely singlet \rightarrow triplet transition because of the charge singlet character of deuteron. (Note that, if simultaneously $\mathbf{K}_p = \mathbf{K}_n$ and $L_p = L_n$ (or $\mathbf{K} = -\mathbf{K}_n$ and $L = -L_n$), 1- (or τ_3 -) part vanishes.) In general some parts of final triplet even state (charge singlet) may be attributed to deuteron state, which may easily be checked by, say, magnetic device, and thus the fraction of

$$\gamma + d \rightarrow d + \pi^0$$

gives a good measure for charge-dependence of photo- π^0 -meson production. In addition, if we know the spin state of final nucleonic system, we can get an idea about the spin-dependence of photo-mesons. These facts make the photo-mesons from deuterium of great importance.

§ 6. Differential cross sections

If we substitute an appropriate wave functions in (5.3) and perform integrations, we can easily find the differential cross sections (in the laboratory system) for photo-production of meson emitted in $d\Omega$ and with energy between q_0 and $q_0 + dq_0$. The results are as follows :

$$\frac{d\sigma}{d\Omega dq_0} = \frac{1}{8\pi^2} \sum \cdot A^2 \cdot F(k, q) \cdot C \quad (6.1)$$

where \sum (spin- and charge-dependent factor) is tabulated in Table 6.1, C is a correction factor (Coulomb penetration factor) which takes into account the effect of Coulomb interaction between final two protons (see Table 6.2), and F is given by

Table 6.1

produced meson	final state of nucleonic system	\sum	charge state
π^+	nn singlet even	$\frac{1}{3} K_+ ^2$	charge triplet
	nn triplet odd	$\frac{2}{3} K_+ ^2 + L_+ ^2$	charge triplet
π^-	pp singlet even	$\frac{1}{3} K_- ^2$	charge triplet
	pp triplet odd	$\frac{2}{3} K_- ^2 + L_- ^2$	charge triplet-
π^0	pn singlet even	$\frac{1}{6} K_p - K_n ^2$	charge triplet
	pn triplet odd	$\frac{1}{3} K_p - K_n ^2 + \frac{1}{2} L_p - L_n ^2$	charge triplet
π^0	pn singlet odd	$\frac{1}{6} K_p + K_n ^2$	charge singlet
	pn triplet even	$\frac{1}{3} K_p + K_n ^2 + \frac{1}{2} L_p + L_n ^2$	charge singlet

Table 6.2
Coulomb penetration factors

	C
S -state	$2\pi\eta/(e^{2\pi\eta}-1)$
P -state	$\{(1+\eta^2)/9\} \cdot 2\pi\eta/(e^{2\pi\eta}-1)$

where

$$\eta = \frac{e^2}{v} = \frac{e^2 M}{p}$$

$$F_{\pm} = \frac{Mpq}{2k} \left[\frac{4}{(p^2 - l^2)^2 + 2(p^2 + l^2)\beta^2 + \beta^4} + \frac{4}{(p^2 - l^2)^2 + 2(p^2 + l^2)\gamma^2 + \gamma^4} \right. \\ \mp \frac{2}{\{2(p^2 + l^2) + \beta^2 + \gamma^2\}pl} \ln \left\{ \frac{(p+l)^2 + \beta^2}{(p-l)^2 + \beta^2} \cdot \frac{(p+l)^2 + \gamma^2}{(p-l)^2 + \gamma^2} \right\} \\ \pm \frac{2}{(\beta^2 - \gamma^2)pl} \ln \left\{ \frac{(p+l)^2 + \gamma^2}{(p-l)^2 + \gamma^2} \cdot \frac{(p-l)^2 + \beta^2}{(p+l)^2 + \beta^2} \right\} \\ \left. \pm \frac{1}{(p^2 + l^2 + \beta^2)pl} \ln \left\{ \frac{(p+l)^2 + \beta^2}{(p-l)^2 + \beta^2} \right\} \pm \frac{1}{(p^2 + l^2 + \gamma^2)pl} \ln \left\{ \frac{(p+l)^2 + \gamma^2}{(p-l)^2 + \gamma^2} \right\} \right] \\ \text{for } \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \text{ part of plane wave } \left(\begin{matrix} \text{upper} \\ \text{lower} \end{matrix} \text{ sign} \right) \quad (6.2\pm)$$

and

$$F_s = \frac{Mq}{2kp} \frac{\sin^2 \delta}{l^2} \left[\frac{1}{2} \cot \delta \ln \left\{ \frac{(p+l)^2 + \gamma^2}{(p-l)^2 + \gamma^2} \cdot \frac{(p-l)^2 + \beta^2}{(p+l)^2 + \beta^2} \right\} \right. \\ \left. + \tan^{-1} \frac{2\gamma l}{p^2 - l^2 + \gamma^2} - \tan^{-1} \frac{2\beta l}{p^2 - l^2 + \beta^2} + 2 \tan^{-1} \frac{(\beta - \gamma)l}{(\gamma + \eta)(\beta + \eta) + l^2} \right]^2 \\ \text{for distorted S-wave,} \quad (6.2S)$$

where

$$l = \frac{1}{2} |\mathbf{k} - \mathbf{q}|$$

and

$$k = \frac{1}{4M} |\mathbf{k} - \mathbf{q}|^2 + \frac{p^2}{M} + q_0 + E_d.$$

In the latter formula (6.2S) we must, of course, take suitable δ and η for each spin state (singlet or triplet).

Especially the differential cross section for $\gamma + d \rightarrow d + \pi^0$ is

$$\frac{d\sigma}{d\Omega} = \frac{1}{8\pi^2} \sum A^2 \cdot G, \\ G = \frac{4\pi^2 A^2}{kq_0 l^2} \left\{ \tan^{-1} \frac{l}{2\gamma} + \tan^{-1} \frac{l}{2\beta} - 2 \tan^{-1} \frac{l}{\beta + \gamma} \right\} \frac{q^2}{\frac{q - k \cos \theta}{2M} + \frac{q}{q_0}}; \quad (6.2d)$$

where θ means the angle between \mathbf{k} and \mathbf{q} (or the direction $d\Omega$) and G is a quantity to be compared with $\int F d\Omega_0$.

For caution's sake we write down the differential cross sections (in the laboratory system) for photo-meson production from free nucleon as follows:

$$\left(\frac{d\sigma}{d\Omega} \right)_{free} = \frac{1}{16\pi^2} \{ |\mathbf{K}|^2 + |\mathbf{L}|^2 \} \frac{1}{kq_0} \frac{q^2}{\frac{q - k \cos \theta}{M} + \frac{q}{q_0}}, \quad (6.1F)$$

where θ is the angle between \mathbf{k} and \mathbf{q} (or the direction $d\Omega$) and \mathbf{K} or \mathbf{L} stands for

\mathbf{K}_+ or \mathbf{L}_+	for	$\gamma + p \rightarrow n + \pi^+$,
\mathbf{K}_- or \mathbf{L}_-	for	$\gamma + n \rightarrow p + \pi^-$,
\mathbf{K}_p or \mathbf{L}_p	for	$\gamma + p \rightarrow p + \pi^0$,
\mathbf{K}_n or \mathbf{L}_n	for	$\gamma + n \rightarrow n + \pi^0$.

§ 7. Physical interpretations

In order to get a physical idea about the results obtained in the preceding section, let us here consider a simple model than the real case: the photo-meson production from a bound proton trapped in the potential of central force. $\psi(r)$ denotes the wave function of this proton;

$$\psi(r) = \frac{2}{(2\pi)^{3/2}} \int \varphi(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{r}} d\mathbf{p}.$$

The cross section for π^+ -production from a bound proton is readily found as follows (the wave function of final neutron is assumed to be plane wave):

$$\frac{1}{16\pi^2} \int \frac{|\mathbf{K}_+|^2 + |\mathbf{L}_+|^2}{q_0 k} |\varphi(\mathbf{k} - \mathbf{q} - \mathbf{p})|^2 \delta\left(\frac{\mathbf{p}^2}{2M} + E_b + q_0 - k\right) d\mathbf{q} d\mathbf{p}, \quad (7.1)$$

where E_b means the binding energy of proton. If we replace $|\varphi(\mathbf{k} - \mathbf{q} - \mathbf{p})|^2$ by $\delta(\mathbf{k} - \mathbf{q} - \mathbf{p})$ and put $E_b = 0$ in (7.1), we get the free nucleon cross section. Assuming

$$\psi(r) = \sqrt{\frac{\gamma}{2\pi}} \frac{e^{-\gamma r}}{r}, \quad \varphi(\mathbf{p}) = \frac{\sqrt{\gamma}}{\pi} \frac{1}{\gamma^2 + \mathbf{p}^2}, \quad E_b = \frac{\gamma^2}{M}, \quad (7.2)$$

we find the differential cross section with respect to meson energy and angle:

$$\frac{d\sigma}{d\Omega dq_0} = \frac{1}{4\pi^3} M \gamma \frac{pq}{k} \frac{|\mathbf{K}_+|^2 + |\mathbf{L}_+|^2}{(\gamma^2 + \mathbf{p}^2 + \mathbf{l}^2)^2 - 4\mathbf{p}^2 \mathbf{l}^2}, \quad \mathbf{l} = |\mathbf{k} - \mathbf{q}|. \quad (7.3)$$

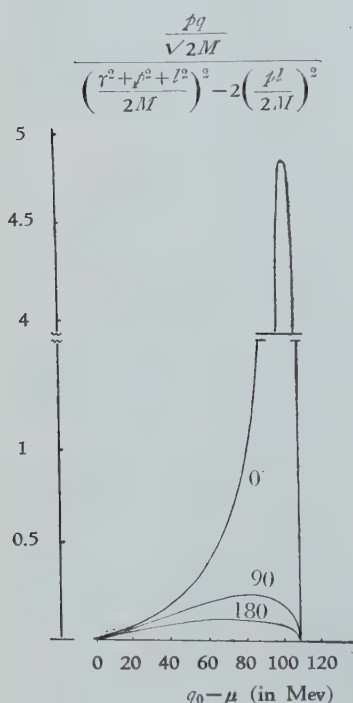


Fig. 7.1 ($k = 250$ Mev)

Fig. 7.1 shows meson distributions at some emission angles, assuming that $|\mathbf{K}_+|^2 + |\mathbf{L}_+|^2$ does not depend on q and the angle and E_b is equal to 2.2 Mev.

As is seen from this result, the energy and angular distribution of mesons are mainly determined by the nucleon wave function of initial bound state; the strong forward peak is a direct consequence of assumed $\varphi(\mathbf{p})$, (7.2), which is larger than the smaller one as in the case of true deuteron function. In the present processes of photo-mesons from deuteron, the coexistence of another nucleon makes the expressions more complicated (i.e., the energy and momentum balance must be modified and now the interference effect appears due to the exclusion principle); but one can easily see that the angular distribution of mesons is, too, most sensitive to the initial wave function of bound state or deuteron.

Next we can see from (7.1) that the total yield of photo-charged mesons from a bound nucleon is usually smaller than that from a free nucleon. For instance, if we assume (7.2) and constant $|\mathbf{K}_+|^2 + |\mathbf{L}_+|^2$, we find the total π^+ -yield from a free proton is 1.7 times larger

than that from a bound one (i.e., the integral of (7.3)) at the incident energy of 250 Mev. (In actual process there exists further reduction of π^\pm -yields due to the exclusion principle; see next section.)*

§ 8. Alternative descriptions

In order to bring the results of § 6 in more concise form, it is better to express the deuteron cross section (irrespective of the nucleon state and meson energy) in terms of free nucleon cross sections following the standard manner of impulse approximation. To perform it, let us first define differential cross sections for free nucleon:

$$\begin{aligned} \gamma + p &\rightarrow \begin{cases} n + \pi^- & : & d\sigma_p^+ / d\Omega, \\ p + \pi^0 & : & d\sigma_p^0 / d\Omega, \end{cases} \\ \gamma + n &\rightarrow \begin{cases} p + \pi^- & : & d\sigma_n^- / d\Omega, \\ n + \pi^0 & : & d\sigma_n^0 / d\Omega, \end{cases} \end{aligned}$$

which may be dependent on the photon energy and meson angle. Then the photo-meson cross sections from deuterium are now written as follows:

$$\begin{aligned} \frac{d\sigma_d^+}{d\Omega} &= \frac{d\sigma_p^+}{d\Omega} [D(\theta) - \varepsilon^+ I(\theta)] && \text{for } \pi^+-\text{meson,} \\ \frac{d\sigma_d^-}{d\Omega} &= \frac{d\sigma_n^-}{d\Omega} [D(\theta) - \varepsilon^- I(\theta)] && \text{for } \pi^--\text{meson;} \end{aligned} \quad (8.1)$$

where

$$\begin{aligned} D(\theta) &= \frac{\int d\mathbf{p} q^2 dq \delta\left(q_0 + \frac{p^2}{M} + \frac{|\mathbf{k}-\mathbf{q}|^2}{4M} + E_d - k\right) \varphi_d^2\left(\mathbf{p} + \frac{\mathbf{q}-\mathbf{k}}{2}\right)}{\int d\mathbf{p} q^2 dq \delta\left(q_0 + \frac{p^2}{2M} - k\right) \delta(\mathbf{p} + \mathbf{q} - \mathbf{k})}, \\ I(\theta) &= \frac{\int d\mathbf{p} q^2 dq \delta\left(q_0 + \frac{p^2}{M} + \frac{|\mathbf{p}-\mathbf{q}|^2}{4M} + E_d - k\right) \varphi_d\left(\mathbf{p} + \frac{\mathbf{q}-\mathbf{k}}{2}\right) \varphi_d\left(\mathbf{p} - \frac{\mathbf{q}-\mathbf{k}}{2}\right)}{\int d\mathbf{p} q^2 dq \delta\left(q_0 + \frac{p^2}{2M} - k\right) \delta(\mathbf{p} + \mathbf{q} - \mathbf{k})}; \end{aligned}$$

(θ = angle between \mathbf{k} and \mathbf{q}
(the direction of \mathbf{q} must be kept in the integration))

and $\varphi_d(\mathbf{p})$ is Fourier transform of deuteron function

$$\varphi_d(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \int \psi_d(\mathbf{r}) e^{i\mathbf{p}\cdot\mathbf{r}} d\mathbf{r}.$$

and is normalized as

$$\int d\mathbf{p} |\varphi_d(\mathbf{p})|^2 = 1.$$

(8.1) are easily found from (6.1) where plane wave is used for the final di-nucleon system.

* Except very near threshold.

$\epsilon^\pm I(\theta)$ represents a correction factor for the exclusion effect in the final nucleonic system and ϵ^\pm is a phase factor concerning the spin-dependence of photo-meson (as for the notations, see (5.1)) :

$$\epsilon^\pm = \frac{\frac{1}{3} |\mathbf{K}_\pm|^2 + |\mathbf{L}_\pm|^2}{|\mathbf{K}_\pm|^2 + |\mathbf{L}_\pm|^2} \quad \therefore \left(\frac{1}{3} \leq \epsilon^\pm \leq 1 \right) \quad (8.3)$$

and thus $d\sigma_n^+/d\Omega$ and $d\sigma_n^-/d\Omega$ are smaller than $d\sigma_p^+/d\Omega$ and $d\sigma_n^-/d\Omega$, respectively, except very near threshold.

Quite similarly, for π^0 -mesons from deuteron we can get from (6.1)

$$\frac{d\sigma_n^0}{d\Omega} = \left(\frac{d\sigma_p^0}{d\Omega} + \frac{d\sigma_n^0}{d\Omega} \right) (D(\theta) + \epsilon^0 I(\theta)). \quad (8.4)$$

Using (5.2)

$$\epsilon^0 = \frac{\frac{1}{3} (\mathbf{K}_p^* \mathbf{K}_n + \mathbf{K}_n^* \mathbf{K}_p) + L_p^* L_n + L_n^* L_p}{|\mathbf{K}_p|^2 + |\mathbf{L}_p|^2 + |\mathbf{K}_n|^2 + |\mathbf{L}_n|^2}, \quad (8.5)$$

In eq. (8.4) the second term represents the interference effect between neutron and proton.

These expressions give only the angular distribution of mesons but quite concisely and will valid for high photon energy or except for the forward direction because of the plane wave approximation in their derivation.

Finally one may obtain the approximate expressions for $D(\theta)$ and $I(\theta)$ using the so-called closure approximation :

$$D^c(\theta) = 1, \quad I^c(\theta) = \int d\mathbf{r} |\psi_p(r)|^2 \frac{\sin(|\mathbf{k}-\mathbf{p}|r)}{|\mathbf{k}-\mathbf{q}|r}, \quad (8.6)$$

where \mathbf{q} is defined, for a given angle θ , by the free particle energy-momentum conservation law. (8.6) shows that $I^c(\theta)$, and thus $I(\theta)$, will be much smaller than unity except for the forward regions (where θ is small). Thus we can obtain good measures for free nucleon cross sections from deuteron data at not too small θ , even if we do not know any details of phase factors ϵ^\pm and ϵ^0 .

§ 9. Differential cross sections.....numerical results.

Finally in this section we illustrate the results of our numerical calculations. In our calculations we adopt the following values for parameters ;

$$M = \text{nucleon mass} = 939 \text{ Mev}$$

$$\mu = \text{meson mass} = 141 \text{ Mev}$$

(We discarded the mass difference between neutron and proton, and charged and neutral mesons.)

$$E_d = \text{binding energy of deuteron} = 2.227 \text{ Mev,}$$

$$\gamma = \sqrt{ME_d}, \quad 1/\gamma = 4.314 \times 10^{-13} \text{ cm.}$$

As for the distorted S -wave, the phase shift δ is calculated from the shape independent

formula¹³⁾

$$p \cot \delta = -\frac{1}{a} + \frac{1}{2} r_0 p^2,$$

where the scattering length a and effective range r_0 is determined from the newest experiments¹⁴⁾:

We calculated only the factors F 's and G ($6.2 \pm$), ($6.2S$) and ($6.2d$). The results are shown in Figs. 9.1—9.19. The statement of § 7 and § 6 are readily seen from these figures.

	a	r_0
triplet	$5.378 \times 10^{-13} \text{cm}$	$1.71 \times 10^{-13} \text{cm}$
singlet	$-26.68 \times 10^{-13} \text{cm}$	$2.55 \times 10^{-13} \text{cm}$

If we know K 's and L 's from other data, we can get the deuteron cross sections from (6.1) and (6.3) using these figures. Conversely the deuteron data combining with the values of F 's or G 's just obtained give more refined informations about photo-mesons from nucleon than in the case of § 8. Detailed comparison with experimental results must be postponed as the future problem because of the lack of sufficient data¹⁵⁾.

Interpretation of figures

Numerical values of G and P 's are illustrated in Fig. 9.2 and Figs. 9.2—9.19. θ is the angle between the meson emission and incident photon beam, and $q_0 - \mu$ is the kinetic energy of mesons: Both of them are measured in the *laboratory system*.

Figs. 9.1—9.2 concern the process $\gamma + d \rightarrow d + \pi^0$, while Figs. 9.3—9.19 concern the processes $\gamma + d \rightarrow n + n + \pi^+$, $p + n + \pi^0$ and $p + p + \pi^-$. The incident photon energy is assigned for each curve in Figs. 9.1—9.14; each figure corresponds to the given angle of meson emission and individual state (1S , 3S , odd and

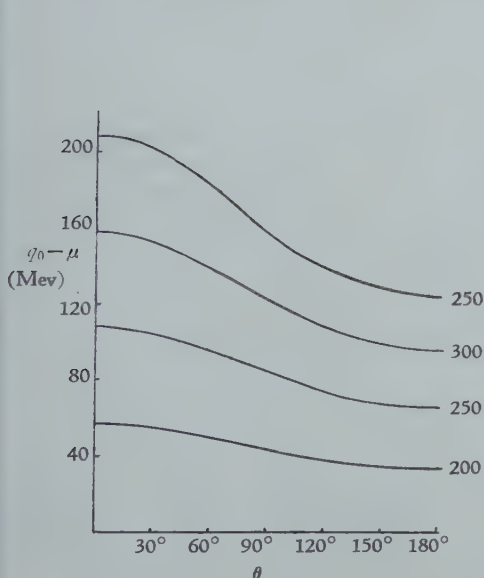


Fig. 9.1
 $\gamma + d \rightarrow d + \pi^0$
angle-energy relation of produced π^0 .

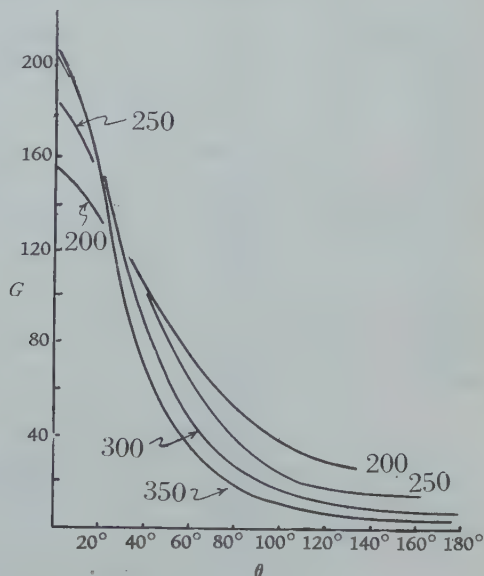
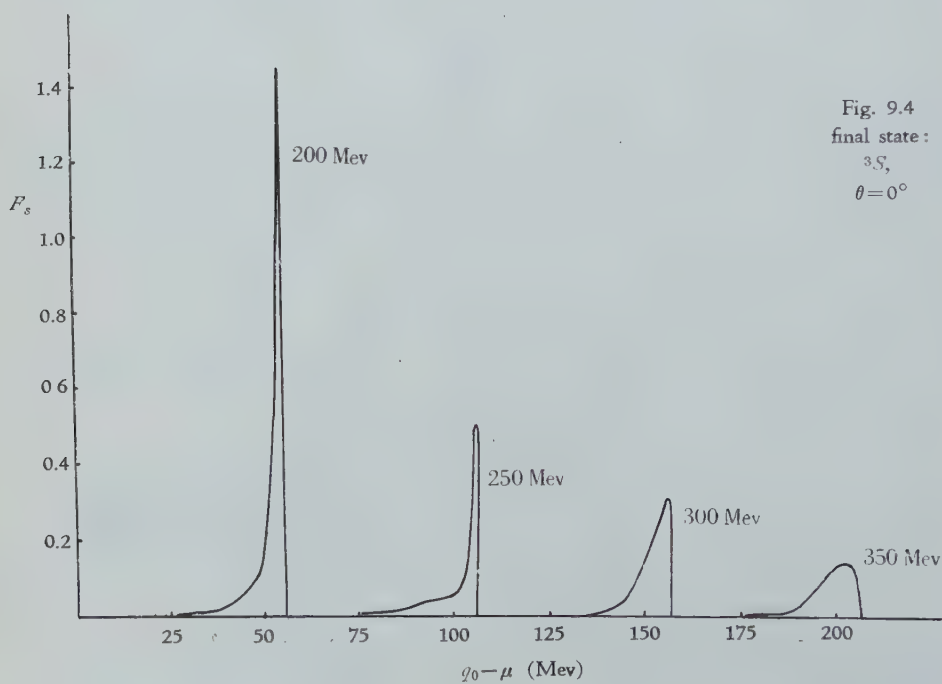
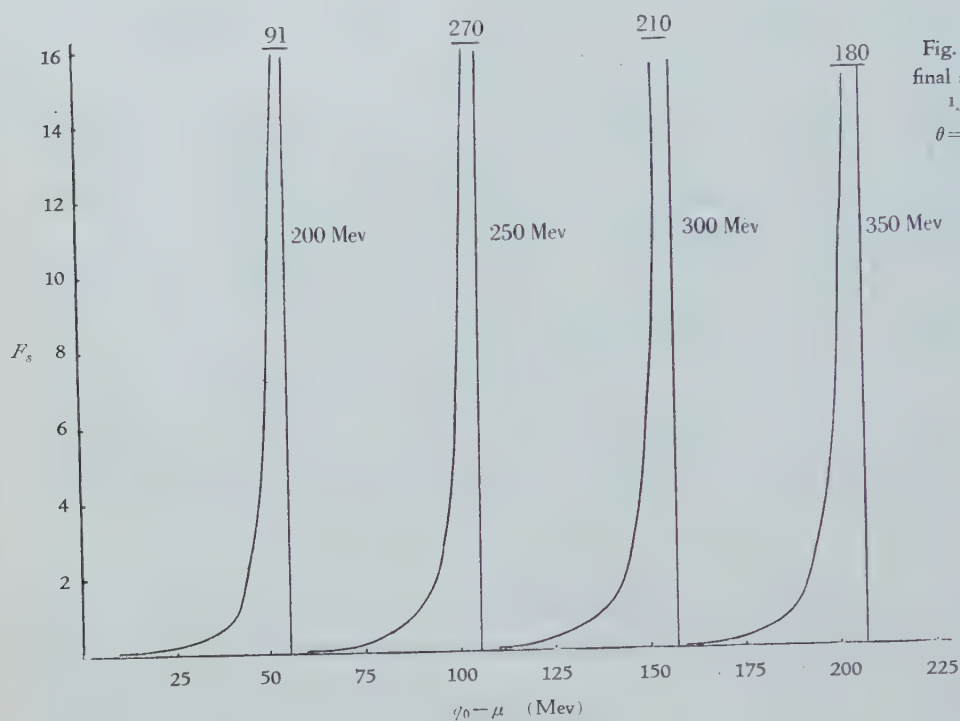


Fig. 9.2
 $\gamma + d \rightarrow d + \pi^0$
The values of G vs. θ



even-state) of the final di-nucleon. Figs. 9.15--9.18 show examples of angular distribution of mesons at $k=250$ Mev. Finally we have compared the cotributions from various final states of di-nucleon at $k=350$ Mev in Fig. 9.19.

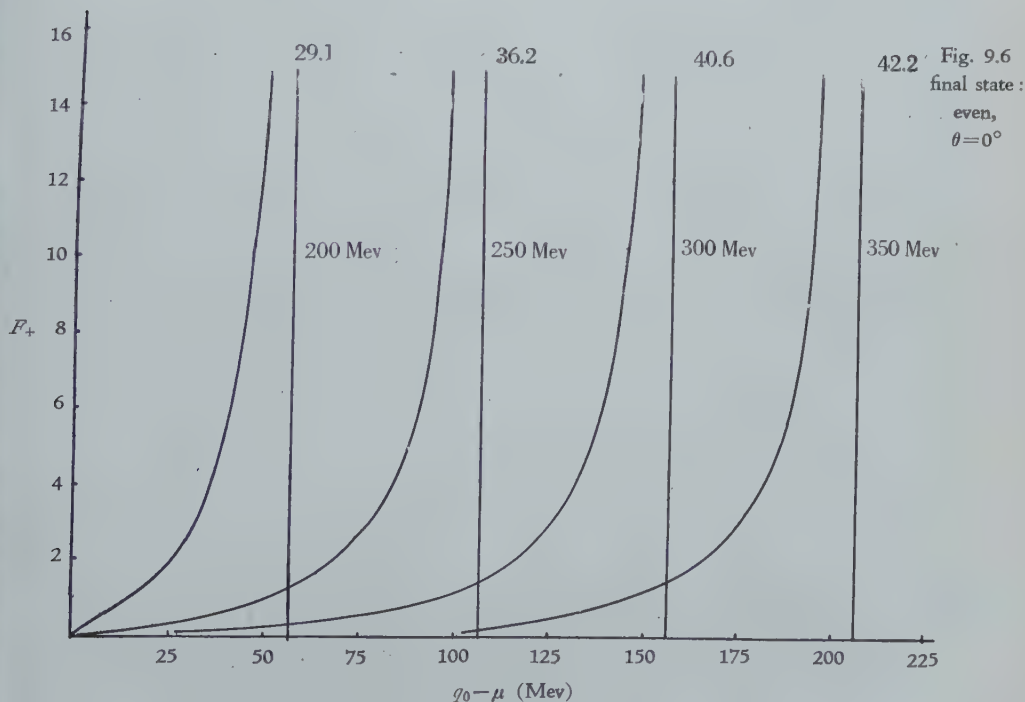
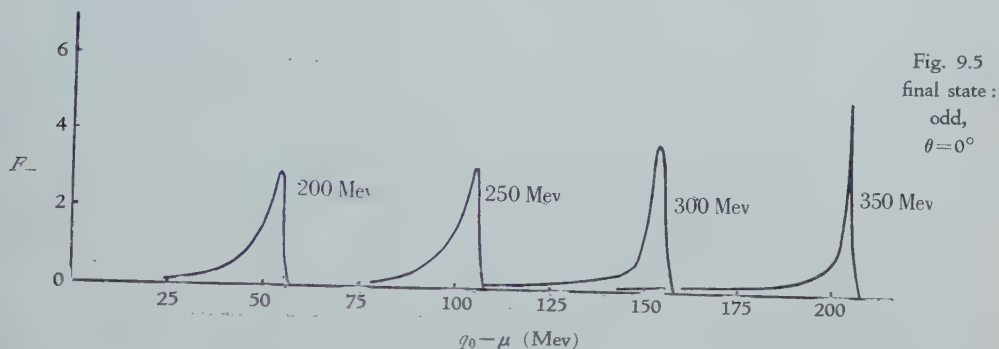
It is noted that F_+ (plane wave result!) must be compared with F_s (singlet) or F_s (triplet) plus G .

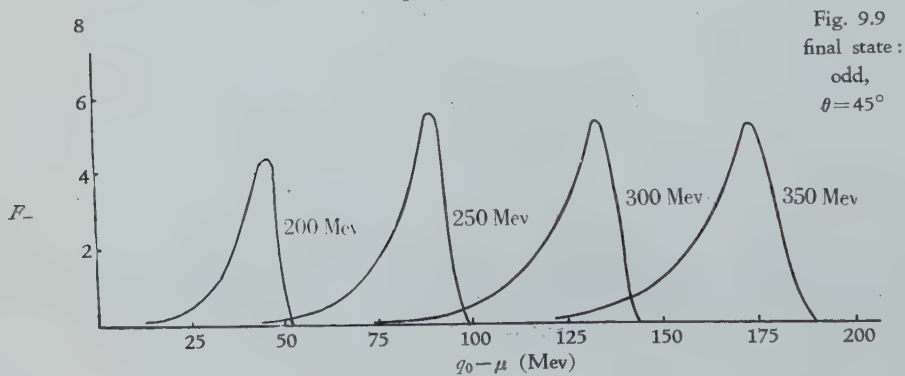
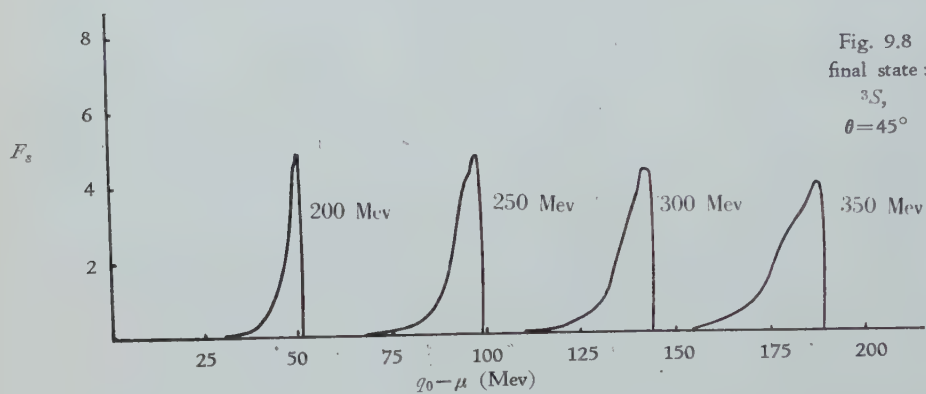
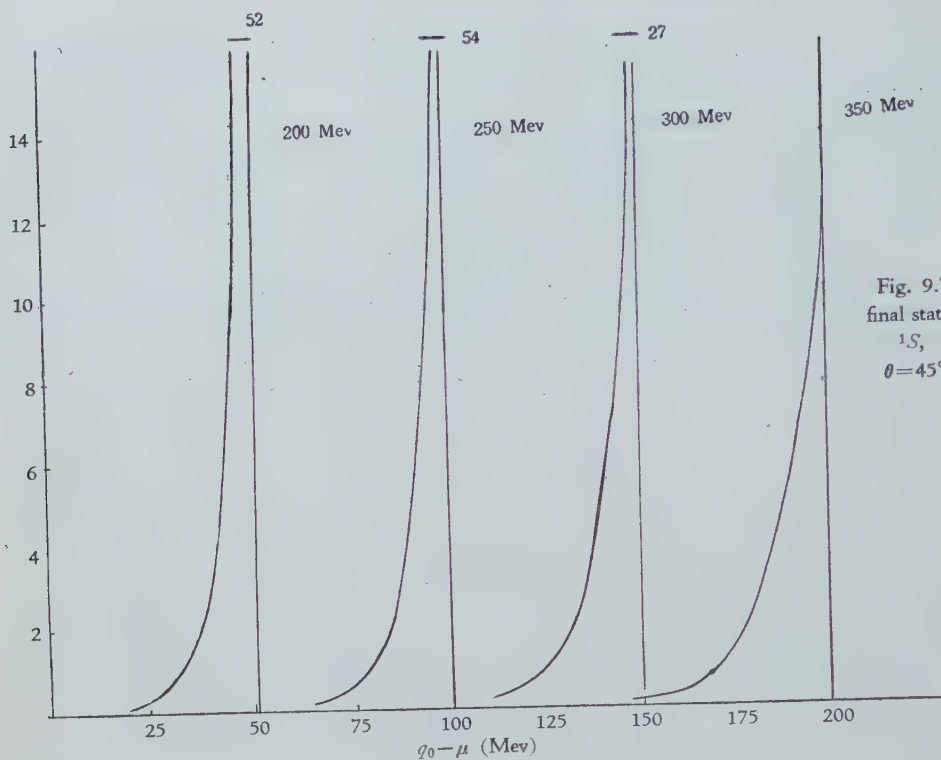
The yield of photo- π 's by bremsstrahlung X-rays can be obtained from these values of F 's and G 's, if the energy dependence of K and L is known.

The maximum values of F 's are shown near the tops of the curves in Figs. 9.6, 9.7, 9.11, 9.15 and 9.18.

Absolute cross sections

G must be compared with $\frac{1}{2}Fdq_0$ where q_0 is measured in Mev. The absolute cross sections (in $\text{cm}^2/\text{sterad}$ or $\text{cm}^2/\text{sterad Mev}$) can be obtained if the numerical values of F or G shown in these figures and $A^2=0.001085$ are used and $|K|$ and $|L|$ measured in cm.





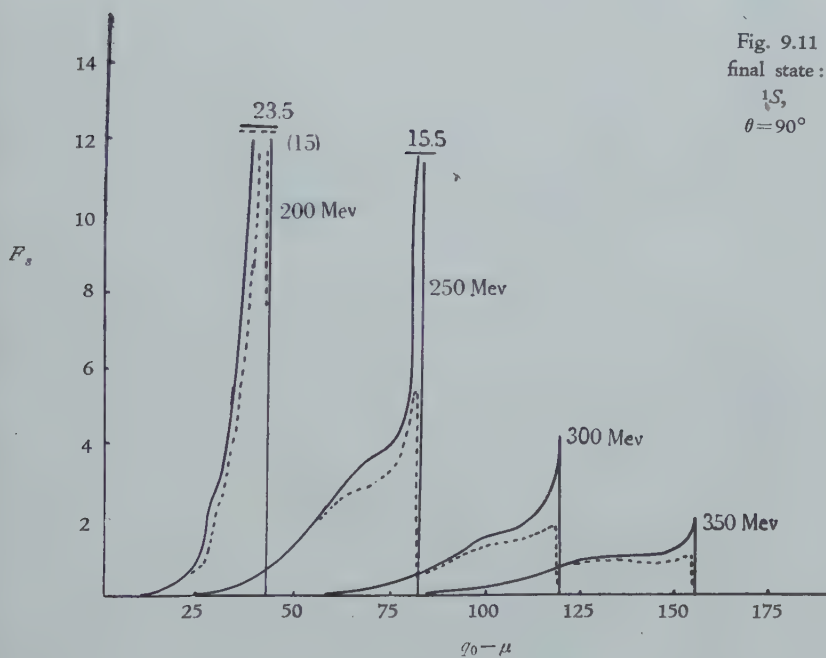
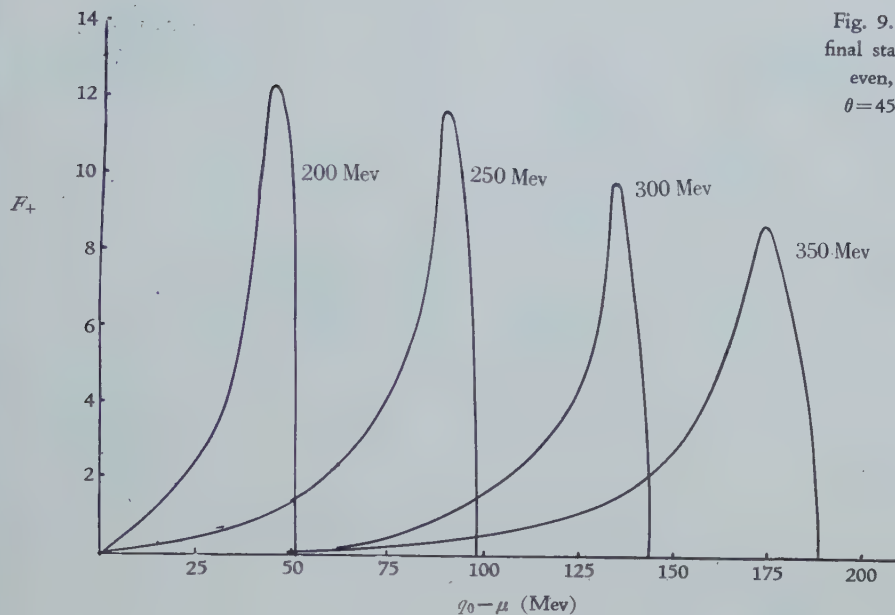


Fig. 9.11

Multiplying the Coulomb penetration factor (cf. Table 6.2) one gets the dotted curves (which are the energy spectra of π^- -mesons, $\gamma + d \rightarrow p + p + \pi^-$) from the solid curves (energy spectra of π^0 - and π^+ -mesons). As is seen from this figure, the Coulomb effect is not so appreciable, we neglect it in other figures.

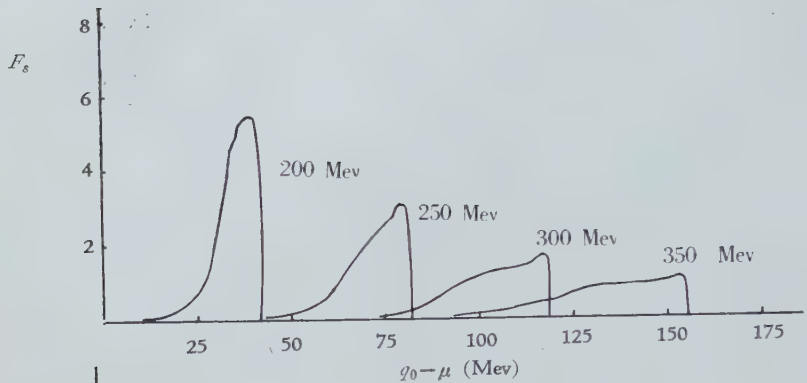


Fig. 9.12
final state :
 3S_1
 $\theta = 90^\circ$

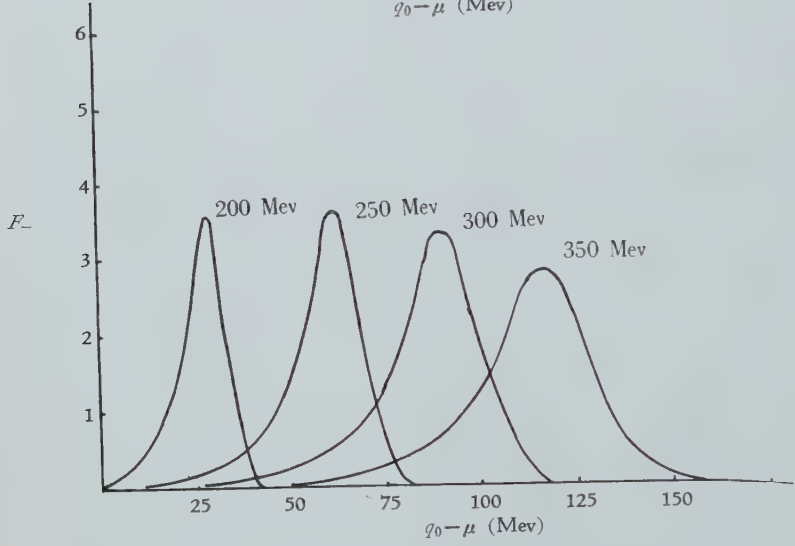


Fig. 9.13
final state :
odd,
 $\theta = 90^\circ$

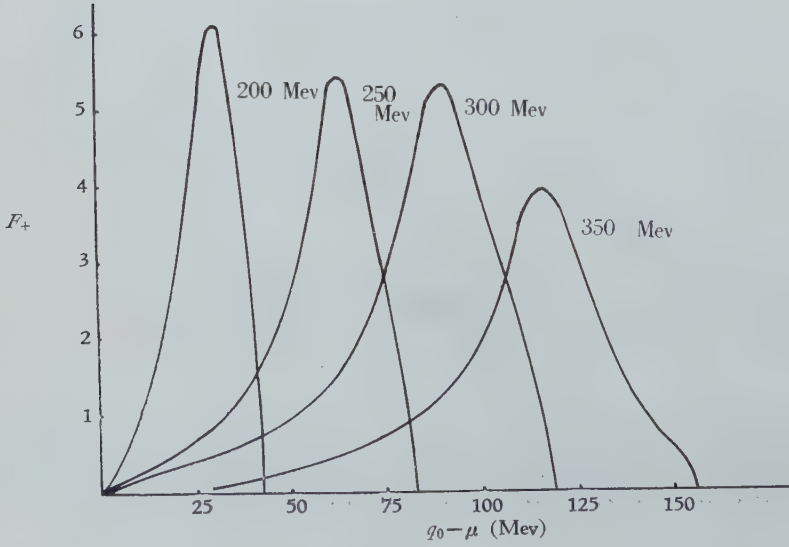


Fig. 9.14
final state :
even,
 $\theta = 90^\circ$

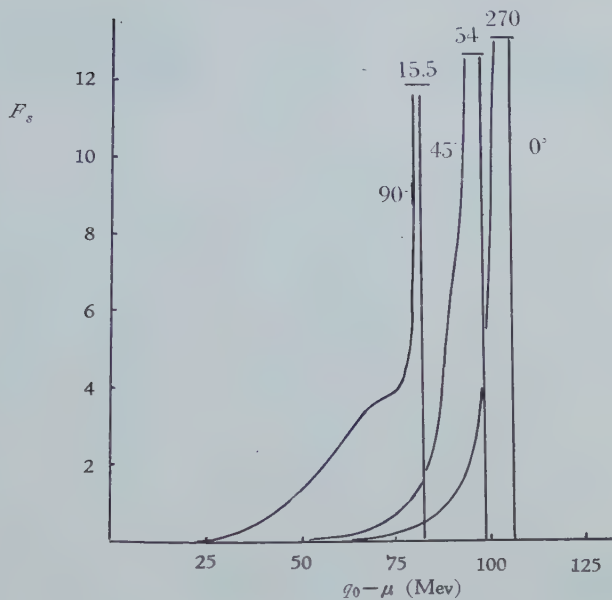


Fig. 9.15
angular distribution
(final state: $1S$,
 $k=250$ Mev)

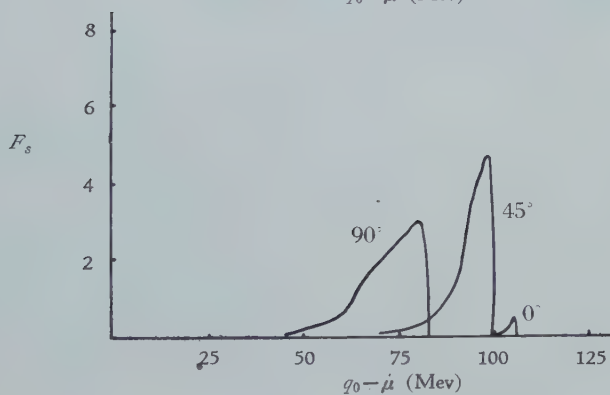


Fig. 9.16
angular distribution
(final state: $3S$
 $k=250$ Mev
("Deuteron
peak" is not
presented;
see Fig. 9.2))

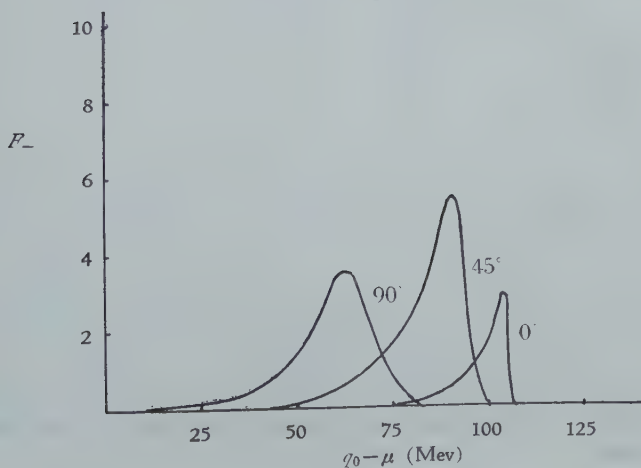


Fig. 9.17
angular distribution
(final state: odd
 $k=250$ Mev)

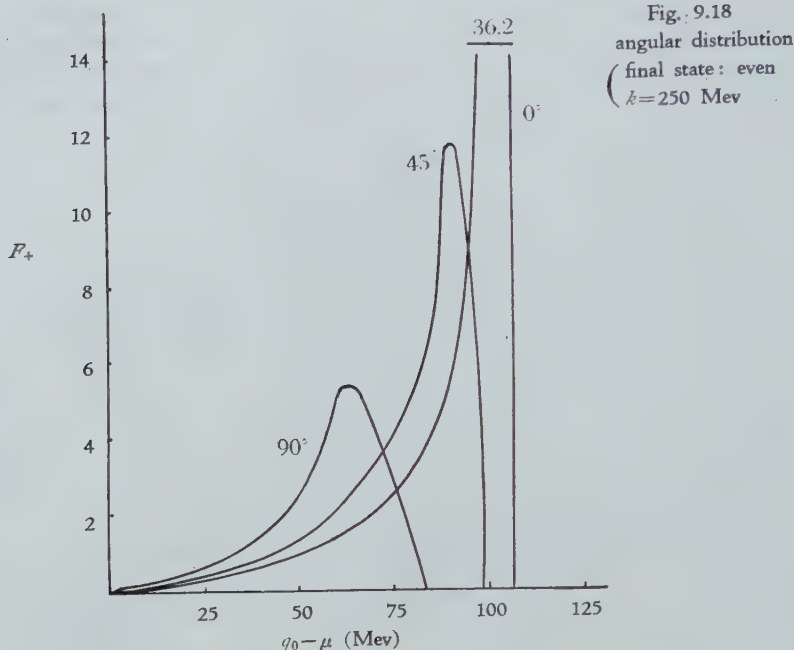
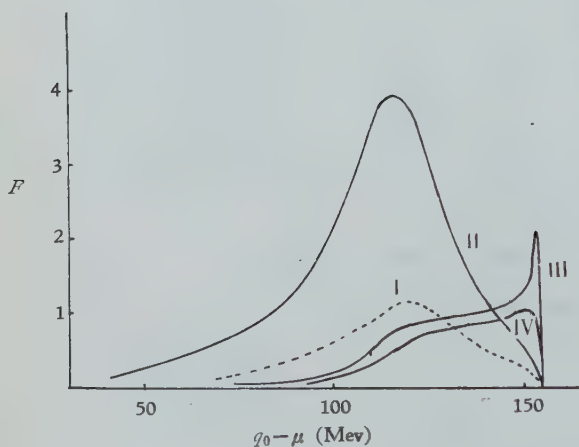


Fig. 9.19



	final nucleonic system
I	free S -wave
II	even part of plane wave
III	distorted S -wave for singlet state
IV	distorted S -wave (continuum state only) of triplet state

($k=350$ Mev
 $\theta=90^\circ$)

Reliability of our results As was often emphasized, our results have less reliability at very low energy of incident photons because of impulse approximation. The results for distorted S -wave (7.2S) are acceptable only for great meson energies. Furthermore it must be kept in mind that our results for $\gamma + d \rightarrow d + \pi^0$ are the most trustful among our calculations, because we have used the most reliable deuteron function under the ignorance of tensor force and this function has appeared also in the final state.

Evidently careful investigations of our process should also give additional informations about nuclear potential as in the cases of nucleonic production of mesons¹⁶⁾ and the slow π^- -meson capture by deuteron²⁾.

Acknowledgement. We are indebted to Messrs T. Dodo, Y. Kakudoh and M. Yokoyama for their kind assistance in numerical calculations, and to Messrs S. Machida and T. Tamura for discussions and communication of their results prior to publication.

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Note added in proof. Recently similar discussions have been published by G. F. Chew and H. W. Lewis, *Phys. Rev.* **82** (1951), 779.

Letters to the Editor

The Burst Production by the μ -Meson

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September 20, 1951

The μ -meson spectrum at sea level is intimately correlated with the size-frequency curve of the μ -meson burst. The μ -meson spectrum has not yet been determined directly at sea level in the high energy region. Accordingly, it must be obtained by the analysis of the intensity-depth curve of the cosmic-ray underground. In such analysis, the interaction of μ -meson with matter plays an essential role. Christy-Kusaka¹⁾ have taken the ionization and bremsstrahlung. Fujimoto-Hayakawa have obtained considerable deviation from the experimental results of the μ -meson burst,²⁾ taking into account of the fact that the energy loss of the μ -meson is also caused by bremsstrahlung and by pair-creation. Recently, it is found that $\mu-\pi$ process gives considerable influence on the cosmic-ray phenomena.³⁾ This process may be accompanied with the additive energy loss of the μ -meson.

In these situations, it might be necessary to investigate whether the recently found process gives considerable modification to the cosmic-ray model supported up to now, or not. In this note, we study the μ -meson burst under such intention.

The μ -meson spectrum at sea level is obtained from the intensity depth curve of the hard component of the cosmic-ray under-

ground,⁴⁾ being taken into account ionization bremsstrahlung, pair creation and $\pi-\mu$ process. The energy loss of the $\mu-\pi$ process is already calculated.⁵⁾ On the $\mu-\pi^0$ process, another remarks are necessary. Since π^0 decays into photons, $\mu-\pi^0$ process influence the μ -meson burst directly. This process however, is found to give only small correction on the μ -meson burst. The initiative process of the burst is mostly bremsstrahlung.

Since our analysis is on same line as Christy-Kusaka, detail of the calculation is not presented. Our result is presented in Fig. 1. A few remarks must be retained about the plot of the experimental result by Schein-Gill.⁶⁾

As the critical energy loss β , we take $\beta=18$ Mev according to Lapp.⁷⁾ Nishimura-

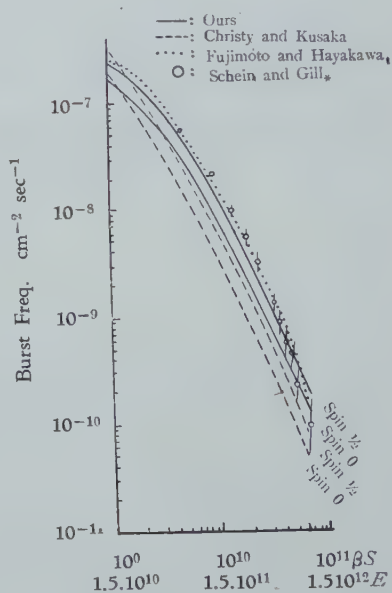


Fig. 1. The burst frequency against the size and energy, at sea level.

Ida⁸⁾ have pointed out that the size of the burst must be multiplied by 3.5 in the experimental size frequency curve, according to their analysis of the shower spread. Further, considering Greisen's remark,⁹⁾ we have increased the size of the burst by a factor 2.5.

From Fig. I, we can conclude that the accepted model of cosmic-ray up to now needs not any modification so far as μ -meson burst is concerned. That is, μ -meson's spin is $1/2$ or 0 .

Though obtained μ -meson spectrum at sea level becomes something flatter in high energy region, it is contradictory to the explanation that the μ -mesons at sea level are resulted from $\pi-\mu$ decay in upper atmosphere.¹⁰⁾ Accordingly, Wataghin's intention is not necessary.¹¹⁾

In conclusion, the authors wish to express our deep gratitude to Prof. S. Sakata for his kind interest to their work.

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We have recalculated the value β from Rossi and Greisen's Table (*Rev. Mod. Phys.* **13** (1941), 240).
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11) reference (5).

* T. Kameda has pointed out the burst of the size than 1000 at sea level is also caused by the nucleon component. Since his attention gives no change on our conclusion, we have neglected it.

** According to M. Koshiba's calculation (to be published), the energy loss of $\mu-\pi$ process is smaller than Sneddon-Taushek's one. Therefore, the energy loss taken in this note may be thought as an upper limit.

Note on the Statistical Theory of the Meson Shower

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December 4, 1951

Recent experiments¹⁾ by means of the photographic emulsions have shown that the pions are produced multiply in high energy nucleon-nucleon collisions and they are collimated into the forward and backward directions in the centre of mass system of two colliding nucleons. The energy dependence of the multiplicity, however, has not been determined empirically due to the lack of the knowledges on such a high energy nucleon collision. The difficulty of the identification of shower particles complicated this problem.

On the other hand, a satisfactory theory of meson shower has not been obtained owing to the difficulties for computing the multiple processes even if we stand on certain meson theories. Moreover, the theory of interaction between π -mesons and nucleons has not been established thoroughly.

Fermi²⁾ proposed an ingenious theory of meson shower basing on the fact that the pion interacts strongly with nucleon. His theory, however, seems to me a little unnatural on that point where he puts the

assumption that the space into which the energy is released is contracted with the initial velocity of colliding nucleon though the system attained at the statistical equilibrium. In this note we shall modify his theory and discuss the multiplicity and angular distribution of pions produced.

We suppose that when two nucleons with momentum P_0 and energy W_0 in their centre of mass system collide a part of their kinetic energies is released and two lumps of highly concentrated energies are generated. These systems are so to speak in high temperature and the pions are radiated from them according to the laws of radiation. If we assume the volume of the lump into which the energy is released to be the sphere with radius R in the reference frame where this lump is at rest, we have as the total number of emitted pions

$$N = 2(2R^3)^{1/4} (45/\pi^3)^{3/4} (a/3\pi) (W^*)^{3/4}$$

where W^* is the total energy of the radiated pions in this reference frame and $a = 2 \sum_{n=1}^{\infty} (1/n^3) = 2.413$.

Further we assume that the lump from which pions are radiated has momentum P_1 and energy W_1 in mean apart from these of pions and introduce the fictitious mass by means of the invariant relation in the reference frame of initial centre of mass system

$$W_1^2 = P_1^2 + M^{*2}.$$

(We put $\hbar=c=1$).

Putting $W_1 = \Gamma W_0$ and (the energies of radiated pions) $= \gamma W_0$, we have

$$N = 2(2R^3)^{1/4} (45/\pi^3)^{3/4} (a/3\pi) (\gamma M^*/\Gamma)^{3/4}$$

where of course the relation $\gamma + \Gamma \leq 1$ must hold.

Now, as pions are produced isotropically in the Lorentz frame attached to the system with momentum P_1 they are concentrated forward and backward in the initial c.m. system with mean angular spreads

$$\bar{\theta} = (2/\nu_3) M^*/\Gamma W_0,$$

The numerical values of N , $\bar{\theta}$ are listed for several $(\gamma M^*/\Gamma)$, $(\Gamma W_0/M^*)$ in the following table. γ , Γ , and M^* are the

$\gamma M^*/\Gamma$	1	2	5	7	8	10	20
N	3.2	5.4	10.8	13.9	15.3	18.1	30.8
$\bar{\theta}$	0.1	0.2	0.3	0.4	0.5		
$\Gamma W_0/M^*$	11.5	5.8	3.8	2.9	2.3		

functions of initial energy W_0 although their dependencies are not determined in this treatment. Putting $\gamma = \frac{1}{2}$, $\Gamma = 1/4$ taking account of the cosmic ray experiments we get $N \approx 15.3$, $\bar{\theta} = 0.2$ for $W_0 = 10^9$ Bev taking tentatively $M^* = 4M$, $R = \mu^{-1}$. The agreement with the experiments are rather good in such crude consideration as this note.

Thermodynamical treatment as above does not involve any detailed meson-theoretical consideration about the mechanism of energy exchange between nucleons and mesons except the fact that their interactions are very strong. It does not take into consideration any detailed transition of the system in the course of "cooling" by radiating pions since it takes the approximation method of statistical equilibrium. Thus the production of higher energy meson is rather unfavoured and consequently the multiplicity does not depend so much on the initial energy of nucleon. Above formulae

contain the initial energy implicitly in γ , Γ and M^* , but their exact forms would require meson-theoretical foundations.

On the other hand, when we treat the phenomena of meson shower meson theoretically, we get the factor $\epsilon^{2n}/(n!)^2$ from the density of final states where ϵ is the total energy of meson emitted and n is the number of radiated mesons. The Poisson's distribution law for the meson spectrum gives the further factor $1/n!$ and as the consequence we get the multiplicity of meson showers proportional to $\epsilon^{2/3}$ when n is large.³⁾ This treatment, however, takes the approximation that the nucleon does not change its state considerably before and after the emission of mesons. Further it takes the approximation that the matrix element for the transition into the final state does not essentially depend on the fluctuation of the energies of mesons produced. In the case of extremely high energy phenomena as the meson shower it is questionable that such a treatment is valid.

The two standpoints mentioned above lie, in a sense, on the different limits of approximation. The connection between these two treatments, which we may be able to make by developing the idea of Heisenberg⁴⁾ on the meson shower, will be discussed elsewhere.

* The main part of this work was made during the author's visit on Nagoya University. He is much indebted to the kind hospitality of the members of the Institute of Theoretical Physics of Nagoya University. Thanks are also expressed to the Press The Chubu-Nippon, the financial aid of which made his visit possible.

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Zero-Zero Transitions*

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December 10, 1951

In the decay of a nucleus from an excited level of zero spin to a lower-lying state of zero spin with no parity change, an orbital electron will be ejected from its path about the nucleus or, if the one excited level of the nucleus is higher than $2mc^2$ above the other level, an electron-positron pair may be created.¹⁾²⁾³⁾⁴⁾ There will be a complete conversion of the energy released in the nuclear decay into freeing a bound electron and giving it kinetic momentum, or into creating the pair with a certain total momentum. This is because the selection rules strictly forbid single gamma emission, and the only other processes which could take up the energy made available in the nuclear transition are less probable (for example, double gamma emission).³⁾

It is significant in these zero-zero, no-parity-change transitions that the perturbing fields which induce the internal conversion or pair formation exist only within the nucleus and vanish identically outside of the nuclear radius.¹⁾²⁾ The entire interaction causing the electron to be ejected, from the bound orbit in the conversion process and from a negative energy state in the internal pair formation, thus occurs within the charge-current distribution of the nucleus. This is to be contrasted with ordinary radiative internal conversion and pair formation.

Because of this unique property of zero-zero transitions it is of particular interest to consider the effect of finite nuclear radius on the rates of internal conversion and of pair formation for such processes.

From a calculational viewpoint the problem is to obtain stationary state solutions to the Dirac equation for an electron moving in the field of a charge distribution of finite radius. The field is Coulomb up to the edge of the nucleus and falls off inside according to the charge distribution that is assumed. We may limit our attention to electrons with but one-half unit of total angular momentum, since the solutions behave like $r^{j \mp 1/2}$ for small r , where r denotes the radial distance from the center of the nucleus to the electron coordinate. Thus for $j > 1/2$ the electrons will spend a considerably smaller fraction of their time within the nucleus, and will experience a correspondingly weaker interaction. For the electronic wave function we can develop a series solution in powers of r/R , where R is the nuclear radius.⁵⁾ The leading term in this expansion gives a transition rate for the conversion or pair production process that is calculated to be accurate within 10 percent correction terms containing $(Z/137)^2$ as a factor.⁶⁾ The rapid convergence of the series results directly from the fact that the entire interaction occurs within the nucleus.

In particular one can calculate the ratio of internal conversion in the K -shell to conversion in the L -shell. To leading order, this ratio is independent of nuclear matrix elements and may be written as the ratio of the squares of normalization constants for the initial and final state wave functions. This agrees within 10 percent with the value calculated for Dirac electrons in a coulomb field of a point nucleus, and is in

accord with recent experimental data.⁷⁾

There are in the literature three proposed cases of zero-zero transitions with no change of parity: a 1.42-Mev transition^{7,8)} in RaC' , a 6.04-Mev transition⁹⁾ in O ,¹⁶ and a 0.7-Mev transition¹⁰⁾ in Ge^{72} . The experimental lifetimes for these three transitions are 2.5×10^{-11} sec., $7. \times 10^{-11}$ sec., and 0.3×10^{-6} sec., respectively. Equating the experimental transition rates with the results of our calculation gives an estimate for the matrix element $\langle \sum_{\text{protons}} r_p^2 \rangle$ between initial and final states. For these three cases the reasonable result

$$\langle \sum r_p^2 \rangle \approx \left(\frac{1}{4} - \frac{1}{9} \right) \left(\frac{1}{2} \frac{e^2}{mc^2} A^{1/3} \right)^2$$

is obtained, indicating an element of similarity in the different excited states.¹¹⁾ Details of this calculation are contained in an unclassified report of the Oak Ridge National Laboratory (ORNL 792 issued September 6, 1950).

*) Work performed at the Oak Ridge National Laboratory under the auspices of the AEC.

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$$\langle \sum p^2 \rangle \approx (1.5 - 2.0) \left(\frac{1}{2} \frac{e^2}{mc^2} \right)^2$$

is too small by a factor three for the transition in RaC' . We do not feel that the data warrant a significant distinction between the two assignments.

On the Approximate Solutions of the Unified Field Theory of Einstein and Schrödinger

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December 31, 1951

A unified field theory has been put forward by Einstein¹⁾ and Schrödinger.²⁾ The fundamental field equations in the theory are

$$[E_1] \quad g_{\lambda\mu, \nu} - g_{\lambda\sigma} I'_{\nu\mu}^{\sigma} - g_{\sigma\mu} I'_{\lambda\nu}^{\sigma} = 0,$$

$$[E_2] \quad I'_{\nu\sigma}^{\sigma} = 0,$$

$$[E_3] \quad R_{\lambda\mu} + \lambda g_{\lambda\mu} = 0,$$

$$[E] \quad (R_{\lambda\mu, \nu} + \lambda g_{\lambda\mu, \nu}) + (R_{\mu\nu, \lambda} + \lambda g_{\mu\nu, \lambda}) + (R_{\nu\lambda, \mu} + \lambda g_{\nu\lambda, \mu}) = 0,$$

where

$$R_{\lambda\mu} = I'_{\lambda\mu, \sigma}^{\sigma} - I'_{\lambda\sigma, \mu}^{\sigma} - I'_{\lambda\sigma}^{\tau} I'_{\tau\mu}^{\sigma} + I'_{\lambda\mu}^{\sigma} I'_{\sigma\tau}^{\tau} = 0$$

and λ is a constant. λ is not zero for Schrödinger's case and zero for Einstein's one. Some exact solutions are obtained by Papapetrou³⁾ and Takeno and his co-workers,⁴⁾ but no one has to a definite conclusion as for the physical meanings of these solutions. On the other hand Kurşunoğlu,⁵⁾ Tonnelat⁶⁾ and Schrödinger⁷⁾ have studied what the field equations become in the case of weak field. In their papers it is assumed that $g_{\lambda\mu} = \eta_{\lambda\mu}$, ($\eta_{11} = \eta_{22} = \eta_{33} = -1$, $\eta_{44} = 1$), and $g_{\lambda\mu}^{\sigma}$ are of small quantities of different orders, which show that $g_{\lambda\mu}$ and $g_{\lambda\mu}^{\sigma}$ are treated as basic fields separately and consequently $g_{\lambda\mu}$ becomes merely a superposition of the two fields $g_{\lambda\mu}$ and $g_{\lambda\mu}^{\sigma}$. But, as is seen from the original construction of the theory, it is the full tensor $g_{\lambda\mu}$ but not $g_{\lambda\mu}$ and $g_{\lambda\mu}^{\sigma}$ separately that is fundamental.

And because of the non-linearity of the field equations, the procedure in which the full tensor $g_{\lambda\mu}$ is taken as a basic field will yield different results from the one in which $g_{\lambda\mu}$ and $g_{\lambda\mu}^{\sigma}$ are taken individually.

Thus it seems to us that their treatments are not natural. Therefore in the following we shall assume that the full tensor $g_{\lambda\mu} - \eta_{\lambda\mu}$ is of small quantity of the first order and examine what the field equations become.

Following the scheme of the general relativity, we assume

$$g_{\lambda\mu} \sim \eta_{\lambda\mu} + \epsilon g_{\lambda\mu}^{(1)} \quad (1)$$

where ϵ is an infinitesimal parameter of the first order and its square can be neglected in comparison with unity. If we substitute (1) into $[E_1]$ and solve them, we get

$$\Gamma_{\lambda\mu}^{\nu} \sim \varepsilon \eta^{\rho\sigma} (g_{\lambda\sigma,\mu}^{(1)} + g_{\sigma\mu,\lambda}^{(1)} - g_{\mu\lambda,\sigma}^{(1)})/2$$

where $\eta^{\lambda\alpha} \eta_{\mu\alpha} = \delta_{\mu}^{\lambda}$. (2)

Next we substitute this into $[E_3]$ and choose such a coordinate system as satisfy the condition :

$$\eta^{\sigma\tau} (g_{\sigma\mu}^{(1)} - \eta_{\sigma\mu} \eta^{\alpha\beta} g_{\alpha\beta}^{(1)}/2)_{,\tau} = 0. \quad (3)$$

Then in the same way as in the general relativity⁽⁸⁾ $[E_3]$ becomes

$$\square g_{\lambda\mu}^{(1)} = \eta_{\lambda\mu} \quad \text{in Schrödinger's case} \quad (4)$$

and

$$\square g_{\lambda\mu}^{(1)} = 0 \quad \text{in Einstein's case} \quad (4')$$

respectively, where in (4) we assumed $\lambda \sim \varepsilon$ in order that $[E_3]$ should be satisfied in the limit $\varepsilon \rightarrow 0$. Similarly, in any coordinate system, from $[E_2]$ and $[E_4]$ we get

$$g_{\nu,\sigma}^{(1)\mu\sigma} = 0 \quad \text{where} \quad g_{\nu}^{(1)\mu\sigma} = \eta^{\mu\alpha} \eta^{\sigma\beta} g_{\alpha\beta}^{(1)} \quad (5)$$

and

$$\square (g_{\lambda\mu,\nu}^{(1)} + g_{\mu\nu,\lambda}^{(1)} + g_{\nu\lambda,\mu}^{(1)}) = 0 \quad (6)$$

respectively.

Thus in the weak field of the first approximation we see that $g_{\lambda\mu}$ plays the same role as in the general relativity (gravitational field) and $g_{\lambda\mu}$ expresses a generalized electromagnetic field which contains the ordinary electromagnetic one as a special case, because (5) and

$$g_{\lambda\mu,\nu}^{(1)} + g_{\mu\nu,\lambda}^{(1)} + g_{\nu\lambda,\mu}^{(1)} = 0 \quad (7)$$

are Maxwell equations.

Lastly we shall define the equation of motion of a test particle by

$$\frac{d^2 x^\lambda}{ds^2} + \Gamma_{\mu\nu}^{\lambda} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

where $ds^2 = g_{\lambda\mu} dx^\lambda dx^\mu$. (8)

This reduces to

$$\frac{d^2 x^\lambda}{ds^2} + \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \quad (9)$$

by (2), where $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$ are the Christoffel symbols with respect to $g_{\lambda\mu}$ and

$$\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} \sim \varepsilon \eta^{\lambda\sigma} (g_{\sigma\nu,\mu}^{(1)} + g_{\sigma\mu,\nu}^{(1)} - g_{\mu\nu,\sigma}^{(1)})/2. \quad (10)$$

This shows that the motion of a test particle is not affected by $g_{\lambda\mu}$ in the first approximation. We can easily see that this is not the case in the second and higher ones.

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Group Theoretical Aspects in S-Matrix Theory

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According to the S-Matrix theory of Dyson,⁽¹⁾ the n -th order approximation S_n contains the sum over all n -th order permutation. When n is small, this procedure is easily achieved explicitly, but it is very complicated for a large value of n . The result is, however, only a sum of groups classified according to the type of the

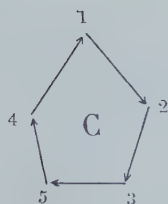


Fig. 1

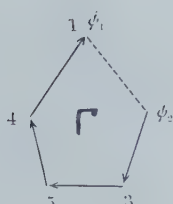


Fig. 2

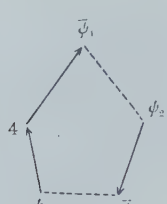


Fig. 3

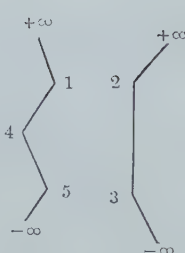


Fig. 4

Feynman diagrams. As the contributions from the terms corresponding to the same type of diagrams are equivalent, it is necessary only to find what kind of diagrams exists and to know the number of terms which belong to each diagram. As a matter of fact, we can meet these requirements with the elementary knowledge of the permutation group theory. There, Feynman diagrams are considered as the expressions of the Dyson permutation by the cycles, that is, S becomes a class function in the group theory.

For simplicity, we confine ourselves to the case of the quantum-electrodynamics. In calculating the $\langle S_n \rangle_{ap}$ it appears the term^{1)*}

$$\begin{aligned} & \langle P(\bar{\psi}_1 \psi_1, \bar{\psi}_2 \psi_2, \dots, \bar{\psi}_n \psi_n) \rangle_\alpha \\ &= \sum_{p, perm} \epsilon(p) \langle P(\bar{\psi}_1 \psi_{r1}) P(\bar{\psi}_2 \psi_{r2}) \dots \\ & \quad P(\bar{\psi}_n \psi_{rn}) \rangle_\alpha \end{aligned} \quad (1)$$

where \sum is taken over all possible permutations.

First, we consider the case $\alpha=0$. Then, Feynman diagrams become, as is well known, closed loops. For $\Gamma \equiv \begin{pmatrix} 12345 \\ 23514 \end{pmatrix}$ it is shown in Fig. 1 (for another P two or more closed loops may be present). This figure can also be considered as the expression of the cycle $C \equiv (12354) \equiv \begin{pmatrix} 12345 \\ 23514 \end{pmatrix}$. As such, the type of the diagrams depends only on

the structure of the Dyson permutations represented by cycles in the case of the vacuum expectation values.

Next, $\alpha=1$. We must take the one particle part for one of the pairs on the right hand side of (1). Now in the diagram, one of the cycles is cut off between neighbouring two points, say, 1 and 2, becoming an open-polygon (Fig. 2). For convenience, considering this still as a cycle we call it Γ -cycle and a normal type C -cycle. The operation of the expectation value, then transforms only one of the C -cycles to a Γ -cycle and the structure of the diagrams is essentially due to the permutations represented by cycles.

Lastly, $\alpha \geq 2$. In this case it is verified also that we can't cut off two or more lines in a C -cycle by taking the expectation values. (For example, Fig. 3 are forbidden).** Accordingly, some C -cycles change only to the corresponding Γ -cycles and the above conclusion on the structure of the diagrams is not affected at all.

Consequently, we obtain the following equation

$$\begin{aligned} S &= \sum_{n=0}^{\infty} S_n = \sum_{n=0}^{\infty} \sum_{perm} S'_n(perm.) \\ &= \sum_{n=0}^{\infty} \sum_{cycle} g_n(cycle) S''_n(cycle) \end{aligned} \quad (2)$$

where g_n is the number of the terms that have the same type of cycles, namely, that

belong to the same class grouptheoretically. After some elementary calculation we get

$$S_n = \left(\frac{-i}{\hbar c} \right)^n \sum_{\substack{\nu_1 \nu_2 \dots \nu_l \\ (\sum \nu_i = n)} \sum_{\substack{n_1 n_2 \dots n_k \\ (\sum n_k = n)}} \frac{(-1)^{n - \sum \nu_i - \sum n_k}}{(\nu_1!)(\nu_2!) \dots (\nu_l!)(n_1!)(n_2!) \dots}$$

$$\left(\frac{1}{1} \right)^{n_1} \left(\frac{1}{2} \right)^{n_2} \dots \int \dots \int dx_1^{(1)} \dots dy_1^{(1)} \dots dz_1 dz_2 \dots \times$$

$$\times \langle P(A_\mu(x_1^{(1)}) \dots A_\nu(y_1^{(1)}) \dots$$

$$A_{\lambda_1}(z_1) A_{\lambda_2}(z_2) \dots) \rangle,$$

$$\prod_{i,j=1}^{\nu_i} \{ \bar{\psi}(x_j^{(i)}) \Gamma_j^{(i)} \psi(y_j^{(i)}) \} \prod_{k,m=1}^{n_k} C_k^{(m)},$$

$$\{ \bar{\psi}(x_j^{(i)}) \Gamma_j^{(i)} \psi(y_j^{(i)}) \} \equiv \bar{\psi}(x_j^{(i)}) (-i\epsilon\gamma)$$

$$S(x_j^{(i)}, z_1) (-i\epsilon\gamma) S(z_1, z_2) \dots$$

$$S(z_{i-1}, y_j^{(i)}) (-i\epsilon\gamma) \psi(y_j^{(i)}),$$

$$C^{(k)} \equiv T_r (-i\epsilon\gamma) S(z_1, z_2) (-i\epsilon\gamma)$$

$$S(z_2 z_3) \dots (-i\epsilon\gamma) S(z_{k-1} z_k),$$

$$S(x, y) \equiv \frac{1}{2} S_F(x-y)$$

where ν_i and n_k are the numbers of the open-polygons and of the closed loops con-

stituted of i and k vertices respectively.

This equation is not so complicated as it seems, and is very convenient not only in practice but also for the theoretical point of view. For example, the diffusion equation of Feynman²⁾ and of Tomonaga-Fukuda³⁾ can be deduced easily and its physical meanings become clear. Furthermore, the relations with the path-integral of Morette,⁴⁾ the statistical method of Fermi⁵⁾ or with others are so interesting and instructive. The detailed and further research will soon appear in this journal.

* Boson lines can be considered only after the skeleton of the electron lines is constructed.

** Fig. 4 are obtained from $\langle (145)(23) \rangle_2 = \langle 145 \rangle_1 \langle 23 \rangle_1$.

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On Lagrangian and Hamiltonian Formalism

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The theory of transformation functions, as has been developed by Schwinger, Feynman, Dyson and others, is investigated systematically with mechanical models, keeping close analogy to classical theory; and a new light is thereby thrown on the relation between the Lagrangian and Hamiltonian formalism. In these investigations the idea of chronological ordering introduced by Dyson on the one hand, and a differential operation with respect to coupling constant (or some other parameters) on the other, play a decisive part. The formulation is extended to a point where dynamical systems with higher time derivatives as well as "non-local" systems (with integral equations of motion) are included.

§ 1. Introduction

The development of quantum electrodynamics in recent years seems to have brought us to the limit which current field theory can hope to reach without making any vital leap. Among the various achievements brought about by the theory of Tomonaga and Schwinger, we may summarize the main features in the following way.

In the first place, introduction of the interaction representation has enabled one to represent the equations of motion in a manifestly covariant way. It is true, that, as was shown by Dirac,¹⁾ interaction representation is not necessarily required to describe a state on a three-dimensional hypersurface, but the super-many time formalism was first discovered and developed in the form of interaction representation. For the interaction representation implies, apart from relativistic covariance, an important physical viewpoint, namely to investigate the development of a state when the free systems are taken as standards. The concept of free fields or particles has been the very basis of describing phenomena in classical as well as quantum theory, but underlying this is an essential fact that the interaction energy is small compared to the free energy. When this condition breaks down, the notion of "free individual particles" would lose its meaning; the particles which are observed could no longer be regarded as simple mass points.

The interaction representation is therefore based by nature on the assumption of weak coupling or validity of perturbation theory. But by preserving relativistic covariance throughout it has made easy the analysis of the various effects originated from the difficulties of self-action. Thus it was observed that the interaction between electron and electromagnetic field gives rise to a change or renormalization of mass and coupling constant (charge). The renormalization philosophy, though effective in quantum electrodynamics for the disposition of divergence difficulties, is in itself irrelevant to the physical origin of the

appearance of such divergences, and would be justified only by anticipating a correct, convergent theory. Yet it may be desirable to make a systematic investigation of the mathematical as well as physical structure underlying the Tomonaga-Schwinger theory, including the renormalization procedure. For this purpose we shall have to investigate not only the transformation functions, but also the nature of general Heisenberg operators²⁾ and their bearing on observation³⁾. But it may not be essential to work in the field theory; to a certain extent it may suffice to work in some convenient "models" with smaller dimensions which can be more easily handled.⁴⁾

Another new formulation of quantum mechanics has been brought forward by Feynman. Instead of using a correspondence between quantum and classical mechanics in Hamiltonian formalism, he bases his theory on a wave optical correspondence in the transformation function U . Here appears the Lagrangian rather than the Hamiltonian as the fundamental quantity. Since Lagrangian is a relativistic invariant in field theory while Hamiltonian is not, the Lagrangian formalism has a special advantage over the Hamiltonian formalism in relativistic quantum mechanics.

The relation between the conventional Hamiltonian formalism of quantum mechanics (to which the theory of Tomonaga and Schwinger belongs) and the Lagrangian formalism of Feynman has been clarified considerably by Dyson⁵⁾ and Feynman⁶⁾, but there remains much to be worked out along these lines. Since such situations are not necessarily characteristic of field theory, we may again be allowed at first to restrict ourselves to simpler dynamical systems.

Feynman's theory, though in most respects equivalent to ordinary quantum mechanics, has revealed us a much larger freedom and variety in the ways of attacking individual problems. Thus, for example, one can eliminate some of the dynamical variables ("field") and replace them by an equivalent action at a distance. After such a procedure, however, the ensuing equivalent dynamical system cannot be fitted into the frame of the ordinary Hamiltonian formalism, but has to be treated as a "non-local" system, i.e. a system having integral equations for its equations of motion. In view of this, it is desirable to extend our investigations to include systems with higher derivatives as well as non-local systems.

In compliance with the above mentioned considerations, we shall be engaged, in the following, in the analysis of the relation between Lagrangian and Hamiltonian formalism from a general point of view. For the time being field theory is taken out of account, and will be discussed separately. The parallelism between classical and quantum theory is carried as far as possible, which often serves us as a useful guide. Throughout the whole investigation the idea of chronological ordering of operators introduced by Dyson (symbolized by the notation P) on the one hand, and the differentiation operation of various quantities with respect to coupling constant or any other external parameters on the other, will play a decisive part. We can show in this way that the Lagrangian method and the Hamiltonian method can be related for general dynamical systems in the interaction representation. This is essentially a generalization of the arguments put forward by Koba,⁷⁾ Nishijima,⁸⁾ Matthews⁹⁾ and others. Feynman's original expression of the (total) transformation func-

tion by means of functional integral can be regarded as corresponding to a limiting case of the interaction representation. In succeeding papers we intend to investigate such problems as the elimination of variables, expression of the transformation functions by means of the "third quantization" introduced by the author,¹⁰⁾ and extension to field theory.

§ 2. Classical preliminaries

2. 1 Equations of motion

Equations of motion of a conservative system can be derived from a Lagrangian L by means of the variation principle. Let q be the dynamical variable of a system of freedom one*, and $L(\dot{q}, q)$ be the corresponding Lagrangian. The variation principle reads

$$\delta I = \delta \int_{t_0}^{t_1} L(\dot{q}, q) dt = 0, \quad (2.1)$$

where I is the action integral. From this follows first the Euler equation

$$L_q - DL_{\dot{q}} = 0; \quad L_q \equiv \partial L / \partial q, \quad L_{\dot{q}} \equiv \partial L / \partial \dot{q}, \quad D \equiv d/dt, \quad (2.2)$$

and the action integral becomes a function of initial and final values only. Thus for an arbitrary variation at the final point t_1 ,

$$\delta I = L \delta t + p \delta q = -H \Delta t + p \Delta q, \quad p \equiv L_{\dot{q}}, \quad (2.3)$$

Δt and Δq being the total (or substantial) variation:

$$\Delta t = \delta t, \quad \Delta q = \delta q + \dot{q} \delta t. \quad (2.4)$$

If we regard I as a function of t and the final conditions $q(t)$ and $p(t)$ (instead of $q(t_1)$ and $q(t_0)$), then the integrability of Eq. (2.3) yields the canonical equations of motion

$$dp/dt = -\partial H / \partial q, \quad dq/dt = \partial H / \partial p. ** \quad (2.5)$$

On the other hand, if we take $q(t_1)$ and $q(t_0)$ as independent variables, we obtain the Hamilton-Jacobi partial differential equation

$$\frac{\partial I}{\partial t} + H\left(\frac{\partial I}{\partial q}, q\right) = 0 \quad (2.6)$$

from Eq. (2.3).

Now let us assume that L contains, aside from the dynamical variable q , an external parameter η . The value of a dynamical variable at any instant then becomes a function of q for fixed initial conditions, and the generalized variation including $\Delta \eta$ leads to the expression

* This will be assumed for simplicity very frequently throughout the paper, but will not restrict the general validity of the results.

** We assume here that H can be expressed in terms of p and q by solving $L_{\dot{q}} = p$ for \dot{q} . Cases where this fails have been discussed in detail by Dirac.¹¹⁾

$$\begin{aligned}\delta I &= -H\Delta t + p\Delta q + \int_{t_0}^{t_1} (\partial L / \partial \eta) \Delta \eta dt \\ &\equiv -H\Delta t + p\Delta q - K\Delta \eta.\end{aligned}\quad (2.7)$$

Here K is evaluated by subjecting q to the equation of motion, so that it depends only on the initial and final conditions. Following the same arguments as were used in deriving Eq. (2.5), we readily obtain from Eq. (2.7) the "canonical equations of motion" in η :

$$dp/d\eta = -\partial K/\partial q, \quad dq/d\eta = \partial K/\partial p, \quad (2.8)$$

and the "Hamilton-Jacobi equation" corresponding to Eq. (2.6):

$$\frac{\partial I}{\partial \eta} + K\left(\frac{\partial I}{\partial q}, q\right) = 0, \quad p = \frac{\partial I}{\partial q}. \quad (2.9)$$

Eq. (2.8) determines the variation of $p(t_1)$ and $q(t_1)$ caused by the variation of η when the initial values $p(t_0)$ and $q(t_0)$ are fixed. For any function which does not involve η explicitly, we have

$$\frac{df}{d\eta} = -[f, K] \quad \left(\equiv -\frac{\partial f}{\partial p} \frac{\partial K}{\partial q} + \frac{\partial f}{\partial q} \frac{\partial K}{\partial p} \right). \quad (2.10)$$

K itself, however, contains η in general since for its evaluation we must express $q(t)$, $t_0 \leq t \leq t_1$, explicitly by $q(t_1)$ and $p(t_1)$, so that

$$dK/d\eta = \partial K/\partial \eta \neq 0.$$

Hitherto a single parameter η has been considered. If η itself is an arbitrary function of time, the equations (2.8) are replaced by the functional differential equations

$$\frac{\delta p(t)}{\delta \eta(t')} = -\frac{\partial K(t')}{\partial q(t)}, \quad \frac{\delta q(t)}{\delta \eta(t')} = \frac{\partial K(t')}{\partial p(t)}, \quad (2.11)$$

where

$$\begin{aligned}K(t') &= -\partial L(t')/\partial \eta, \quad t \geq t' \geq t_0^*, \\ &= 0, \quad t' > t,\end{aligned}$$

and $\delta/\delta\eta(t)$ satisfies

$$(\delta/\delta\eta(t))\eta(t') - \eta(t')(\delta/\delta\eta(t)) = \delta(t-t').$$

The equation corresponding to (2.9) becomes

$$\frac{\delta I}{\delta \eta(t')} + K\left(t'; \frac{\delta I}{\delta q(t)}, q(t)\right) = 0. \quad (2.12)$$

Now the two sets of equations of motion (2.5) and (2.9) refer to two independent variables t and η , so that a condition of integrability (or compatibility) must hold for these equations. In other words, from

* Or more generally, $K = -\int_{t_0}^{t_1} \delta/\delta\eta(t') \cdot L dt$.

$$df/dt = -[f, H], \quad df/d\eta = -[f, K],$$

we have necessarily

$$-\frac{d}{d\eta} \frac{d}{dt} f = \frac{d}{d\eta} [f, H] = \frac{d}{dt} [f, K].$$

Noting that H and K involve η and t respectively, and denoting the resulting explicit differential by ∂ , we obtain

$$\partial H / \partial \eta - \partial K / \partial t - [H, K] = 0. \quad (2.13)$$

Because of the relation

$$dH/d\eta = \partial H / \partial \eta - [H, K], \quad dK/dt = \partial K / \partial t - [K, H],$$

we can also write (2.13) as

$$\frac{dH}{d\eta} = \frac{\partial K}{\partial t} = - \left. \frac{\partial}{\partial t} \int_{t_0}^t \frac{\partial L}{\partial \eta} dt' \right|_{\Delta q(t)=0} = -\partial L(t_0) / \partial \eta, \quad (2.14)$$

$$\frac{\partial H}{\partial \eta} = \frac{dK}{dt} = - \frac{d}{dt} \int_{t_0}^t \frac{\partial L}{\partial \eta} dt' = -\partial L(t) / \partial \eta. \quad (2.14')$$

The difference between d/dt and $\partial/\partial t$ arises according to whether $p(t_0)$ and $q(t_0)$, or $p(t)$ and $q(t)$, are fixed.

Example. We take a displaced harmonic oscillator with the Lagrangian

$$L = \frac{1}{2} (\dot{q}^2 - \nu^2 q^2) + \eta q \quad (2.15)$$

and regard the displacement η as an external parameter. The equation of motion is

$$\ddot{q} + \nu^2 q^2 = \eta, \quad (2.16)$$

and the solution is expressed in terms of initial values as

$$\begin{aligned} q(t) &= q^0(t) + \eta \int_{t_0}^t G(t-t') dt', \\ q^0(t) &= G(t-t_0) p(t_0) + \dot{G}(t-t_0) q(t_0), \\ G(t) &= \frac{1}{\nu} \sin \nu t, \quad p(t) = \dot{q}(t). \end{aligned} \quad (2.17)$$

Thus

$$\begin{aligned} K &= - \int_{t_0}^t (\partial L / \partial \eta) dt = - \int_{t_0}^t q dt \\ &= -p(t_0) \int_{t_0}^t G(t'-t_0) dt' - q(t_0) G(t-t_0) - \eta \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' G(t'-t''), \\ \text{or} \quad &= p(t) \int_{t_0}^t G(t-t') dt' - q(t) G(t-t_0) + \eta \int_{t_0}^t dt' \int_{t'}^t dt'' G(t'-t'') \end{aligned} \quad (2.18)$$

if expressed in terms of final values. From (2.17) and (2.18) we see clearly that the relations

$$\begin{aligned} dp(t)/d\eta &= -\partial K/\partial q(t) = G(t-t_0), \\ \frac{dq(t)}{d\eta} &= \frac{\partial K}{\partial p(t)} = \int_{t_0}^t G(t-t') dt' \end{aligned} \quad (2.19)$$

hold. Further, noting that

$$H = \frac{1}{2} (\dot{p}^2 + \nu^2 \dot{q}^2) - \eta q \Big|_t = \frac{1}{2} (\dot{p}^2 + \nu^2 \dot{q}^2) - \eta q \Big|_{t_0},$$

we get

$$\begin{aligned} dH/d\eta &= -q(t_0) = \partial K/\partial t, \\ \partial H/\partial \eta &= -q(t) = dK/dt. \end{aligned} \quad (2.20)$$

2. 2 Interaction representation

In the interaction representation we divide the original Lagrangian (which shall be called the total Lagrangian) conveniently in two parts :

$$L_{\text{tot}} = L_0 + L, \quad (2.21)$$

where L_0 is the "free" Lagrangian for which the solution is easily obtained, while L represents the "interaction" Lagrangian. Now we consider, in addition to the original system unfolding in accordance with L_{tot} , a hypothetical system which has L_0 for the Lagrangian, and compare the motion of the two systems at the same instant and at the same point of phase space. In other words, we observe the original system in a reference frame of phase space which is moving in accordance with L_0 . The action integral corresponding to this situation is provided by the difference of the two respective action integrals :

$$I = L_{\text{tot}} - I_{\text{free}}, \quad (2.22)$$

the variation of which yields

$$\delta I = (-H\Delta t + p\Delta x) \Big|_{\text{tot}} - (-H\Delta t + p\Delta q) \Big|_{\text{free}}. \quad (2.23)$$

Since we are comparing the two motions at the same point of phase space,

$$p_{\text{tot}} = p_{\text{free}}, \quad q_{\text{tot}} = q_{\text{free}} \quad (\text{numerically}); \quad (2.24)$$

but the velocities do not coincide in general :

$$\begin{aligned} \dot{q}_{\text{tot}} - \dot{q} &= -[q, H], \quad \dot{p}_{\text{tot}} - \dot{p}_{\text{free}} = -[p, H], \\ H &\equiv H_{\text{tot}}(p, q) - H_0(p, q). \end{aligned} \quad (2.25)$$

Expressed in terms of $\dot{q}_{\text{free}} \equiv \dot{q}$, we get

$$\begin{aligned} H(\dot{q}, q) &= H_{\text{tot}}(\dot{q} - [q, H], q) - H_0(\dot{q}, q) \\ &\quad - (\partial L_0/\partial q)[q, H] - L_{\text{tot}}(\dot{q} - [q, H], q) + L_0(\dot{q}, q); \end{aligned} \quad (2.26)$$

and the (Hamilton-Jacobi) equation derived from (2.23) :

$$\partial I / \partial t + H(p(t), q(t)) = 0 \quad (2.27)$$

is the classical counterpart of the equation of motion for the wave function ψ in interaction representation.

Now let us treat the above mentioned problem from the viewpoint of the parametric differential equation discussed in 2.2. We insert for this purpose a parameter η for the interaction Lagrangian and write

$$L_{\text{tot}} = L_0 + \eta L. \quad (2.28)$$

The corresponding Hamiltonian is also a function of η satisfying the differential equation (2.14') :

$$\partial H_{\text{tot}} / \partial \eta = \partial H / \partial \eta = -\partial L_{\text{tot}} / \partial \eta = -L. \quad (2.29)$$

Here we must express \dot{q}_{tot} which may be contained in L by means of \dot{q} , thus

$$\partial H / \partial \eta = -L(\dot{q} - [q, H], q). \quad (2.29')$$

This is considered as a (partial) differential equation for the unknown H , the solution of which with the initial condition $H=0$ at $\eta=0$ gives just the interaction Hamiltonian (2.26). In case L does not involve \dot{q} , we get at once

$$H = -\eta L; \quad (2.30)$$

but if L does involve \dot{q} , H takes on additional terms which are non-linear in η . As will be shown elsewhere, these additional terms just correspond to those terms in Tomonaga-Schwinger theory which are required for the integrability of the equation of motion.

2.3 Equations of motion with higher time derivatives

In the preceding analysis it was assumed that the Lagrangian contained only the coordinate q and the velocity \dot{q} , so that the Euler equation contained derivatives only up to the acceleration. If this restriction is dropped, Lagrangians with higher derivatives than the first come into question.

The variation principle and the Hamiltonian formalism for a Lagrangian containing $q, \dot{q}, \ddot{q}, \dots, q^{(n)}$ has been given by Ostrogradski.¹²⁾ Thus

$$\begin{aligned} \delta I &= \delta \int L(q^{(r)}) dt \\ &= \int \sum_{r=0}^n (-1)^r D^r L_q^{(r)} \delta q dt + L \delta t + \sum_{r=0}^{n-1} \left(\sum_{s=0}^{n-r-1} (-D)^s L^{(r+s+1)} \right) \delta q \Big|_t, \end{aligned} \quad (2.31)$$

from which follows the equation of motion

$$\sum_{r=0}^n (-D)^r L_q^{(r)} = 0 \quad (2.32)$$

and the relation

$$\delta I = -H\Delta t + \sum_{r=0}^{n-1} p_r \Delta q_r$$

with

$$\Delta q = \delta q + q^{(r+1)} \Delta t, \quad q_r \equiv q^{(r)}, \quad (2.32')$$

$$p_r \equiv \sum_{s=0}^{n-r-1} (-D)^s L^{(r+s+1)},$$

$$H \equiv \sum_{r=0}^{n+1} p_r q_{r+1} - L.$$

Expressing $q^{(r)}$ in H in terms of p_{n-1} , q_r through the relation defining p_{n-1} , we obtain the Hamiltonian equations of motion

$$dp_r/dt = -[p_r, H] \equiv -\partial H/\partial q_r, \quad dq_r/dt = -[q_r, H] \equiv \partial H/\partial p_r. \quad (2.33)$$

On this occasion we make a few remarks on the indeterminacy of Lagrangian by a total differential (or total divergence in field theory)*. As is well known, the Euler equation of motion is invariant against addition of an arbitrary total derivative dW/dt to the Lagrangian, so that the Lagrangian is not uniquely determined by the equation of motion. Here W is usually assumed to be a function of $q, \dot{q}, \dots, q^{(n-1)}$, and we have

$$\begin{aligned} \delta I &= L\delta t + \dot{W}\delta t + \sum_{r=0}^{n-1} (p_r + W_{q_r})\delta q_r \\ &= (L - \sum_{r=0}^{n-1} p_r q_{r+1})\delta t + \sum_{r=0}^{n-1} (p_r + W_{q_r})\delta q_r. \end{aligned} \quad (2.34)$$

The Hamiltonian is therefore uniquely determined in numerical values, but the canonical variables p_r, q_r undergo a canonical transformation

$$q_r \rightarrow q_r, \quad p_r \rightarrow p_r + W_{q_r}.$$

More generally, if W is expressed as a function of the p 's and q 's that obey the equation of motion, the canonical transformation is generated in accordance with the relation

$$\begin{aligned} \delta I &= -H\Delta t + \sum p_r \Delta q_r + \Delta W \\ &= -H'\Delta t + \sum p_r' \Delta q_r'. \end{aligned}$$

Though the p 's are in general functions of derivatives higher than $q^{(n)}$, the freedom is reduced by the restriction that they shall obey the equation of motion, thus admitting only n variations q_r to be independent. The same situation still prevails when W contains arbitrarily high derivatives of q since these derivatives are only of apparent nature.

When the order n of the highest derivatives in L increases indefinitely, we have essentially a "non-local" system, namely a system which obeys an integral equation of motion. In this case, $n = \infty$, and we cannot carry out the procedure of expressing $q^{(n)}$ by

* A more detailed discussion on this topic will be given later (Section 3.6).

p_r and q_r ($r=0, 1, \dots, n-1$). The variation principle, however, can be treated analogously to the case of finite n . Let us consider for example the Lagrangian

$$L(t) = \frac{1}{2} \int_{-\infty}^{\infty} \varphi(t) K(t-t') \varphi(t') dt' \quad (2.35)$$

which gives a linear equation of motion. Here $K(t-t')$ is an even function characterizing the dynamical property of the system, and $\varphi(t)$'s are the dynamical variables. As a generalization of the differential operator, it may be assumed that $K(t-t')$ takes large values only when $t-t'$ is small. The indeterminacy of Lagrangian mentioned above is reflected in this case in the existence of a series of equivalent Lagrangians

$$L(t) = \frac{1}{2} \int_{-\infty}^{\infty} \varphi(t+a\tau) K(\tau) \varphi(t-b\tau) d\tau \quad (2.35')$$

with $a+b=1$. Indeed they lead to one and the same equation of motion. Let us integrate $L(t)$, Eq. (2.35) over a finite interval $t_0 t_1$:

$$I = \int_{t_0}^{t_1} L(t) dt = \frac{1}{2} \int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} dt' \varphi(t) K(t-t') \varphi(t') \quad (2.36)$$

and take its variation

$$\delta I = \frac{1}{2} \int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} dt' (\delta \varphi(t) \varphi(t') + \varphi(t) \delta \varphi(t')) K(t-t').$$

Since we demand such an equation of motion that is independent of the limits of integration t_0 and t_1 , we rewrite this as

$$\begin{aligned} \delta I = & \int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} dt' \delta \varphi(t) \varphi(t') K(t-t') - \frac{1}{2} \int_{t_0}^{t_1} dt \left(\int_{-\infty}^{t_0} + \int_{t_1}^{\infty} \right) dt' \delta \varphi(t) \varphi(t') K(t-t') \\ & + \frac{1}{2} \left(\int_{-\infty}^{t_0} + \int_{t_1}^{\infty} \right) dt \int_{t_0}^{t_1} dt' \delta \varphi(t) \varphi(t') K(t-t'). \end{aligned} \quad (2.37)$$

Considering the nature of the kernel K , contributions to the second and third term will come mainly from those $\varphi(t)$ which lie in the neighborhood of $t=t_0$ and t_1 . Thus we obtain for the equation of motion the first term

$$\int_{-\infty}^{\infty} K(t-t') \varphi(t') dt' = 0 \quad (t_1 > t > t_0). \quad (2.38)$$

If we let further $t_0 \rightarrow -\infty$ for simpleness, the interference between the two limits of integration vanishes, giving

$$\begin{aligned} \delta I = & L(t_1) \delta t + \frac{1}{2} \int_{-\infty}^{t_1} \pi_{t_1}^-(t) \delta \varphi(t) dt + \frac{1}{2} \int_{t_1}^{\infty} \pi_{t_1}^+(t) \delta \varphi(t) dt \\ = & -H(t_1) \delta t + \frac{1}{2} \int_{-\infty}^{\infty} \pi_{t_1}(t) \delta \varphi(t) dt, \end{aligned}$$

with

$$\pi_{t_1}(t) = \begin{cases} \pi_{t_1}^-(t) = - \int_{t_1}^{\infty} K(t-t') \varphi(t') dt' & \text{for } t \leq t_1, \\ \pi_{t_1}^+(t) = + \int_{-\infty}^{t_1} K(t-t') \varphi(t') dt' & \text{for } t \geq t_1, \end{cases} \quad (2.39)$$

$$H(t_1) = -L(t_1) + \frac{1}{2} \int_{-\infty}^{\infty} \pi_{t_1}(t) \dot{\varphi}(t) dt.$$

Here t serves as an internal coordinate representing the dynamical freedom of φ .

In order to go over to the Hamiltonian formalism we must first find out a set of canonical variables. In the present case it is clear that we cannot take all the $\varphi(t)$'s as independent because of the equation of motion (2.37). The same equation also requires that

$$\pi_{t_1}^-(t_1) = \pi_{t_1}^+(t_1),$$

with which the Poisson bracket must be compatible.

For this purpose, we assume that

$$[\pi_{t_1}^{\pm}(t), \varphi(t_1)] = \delta(t_1 - t), \quad (2.40)$$

which means

$$[\varphi(t'), \varphi(t'')] = D(t' - t''). \quad (2.40')$$

D must then satisfy, as a consequence of (2.39), (2.40) and (2.40'), the following requirements :

$$\begin{aligned} - \int \frac{1}{2} K(t' - t'') \varepsilon(t - t'') D(t - t'') dt'' &= \delta(t - t'), \\ \int K(t - t') D(t' - t'') dt' &= 0, \quad D(t) = -D(-t). \end{aligned} \quad (2.41)$$

In general cases we must first demand the equation

$$df/dt = -[f, H]$$

to hold for an arbitrary dynamical variable f , and determine the canonical variables or Poisson brackets in such a way as to meet this requirement. Especially the relation

$$dH/dt = -[H, H] = 0$$

leads to the Euler equation, thereby closing the translational operation $f(t) \rightarrow f(t+dt)$ generated by Eq. (2.41) on the set of dynamical variables with the parameter t .

2. 4 Interaction representation for systems with higher derivatives

Since a dynamical system with higher derivatives is equivalent to a system with higher degrees of freedom, it is expected that the interaction representation holds for this case in much the same way as before. But the degree of freedom of the system should naturally not increase by the introduction of interaction. In other words, the "interaction" Lagrangian should not contain higher derivatives than the "free" Lagrangian does.

The parametric differential equation which was discussed previously can directly be extended to this case, so that again

$$\partial H / \partial \eta = -L$$

holds in general. L contains, by assumption, only $q, \dot{q}, \dots \overset{(n)}{q}$. Of these, $q, \dots \overset{(n-1)}{q}$ are the independent coordinates q_0, \dots, q_{n-1} in the Hamiltonian formalism, and are common to both the free and the interacting system. On the other hand, $\overset{(n)}{q}$ is a function of q_0, \dots, q_{n-1} and p_{n-1} , so that it varies as the definition of p_{n-1} does when going over from the free to the interacting system. This situation can be expressed by the formula

$$\begin{aligned} \overset{(\tilde{r})}{q} &= \overset{(r)}{q} \quad r=1, 2, 1 \dots n-1, \\ \overset{(\tilde{n})}{q} &= D \overset{(n-1)}{q} = -[\overset{(n-1)}{q}, H_0 + H] = \overset{(n)}{q} - [\overset{(n-1)}{q}, H], \end{aligned} \quad (2.42)$$

where $\overset{(r)}{q}$ and $\overset{(\tilde{r})}{q}$ refer to the free and the interacting system respectively. Thus we get the parametric differential equation for H :

$$\partial H / \partial \eta = -L(\overset{(r)}{q}, \overset{(n)}{q} - [\overset{(n-1)}{q}, H]). \quad (2.43)$$

In particular, if L contains derivatives of q at most to the $(n-1)$ th degree, we have at once

$$H = -\eta L.$$

Additional terms arise only when L contains the highest derivative $\overset{(n)}{q}$.

When we go over to non-local systems, the above considerations do not apply directly, but the relation $\partial H / \partial \eta = -L$ can still be shown to hold.* Thus if the degree of freedom does not essentially increase by the introduction of interaction terms, we can get a differential equation for H by expressing the above relation in terms of "free" quantities.

We summarise the results of the preceding analysis by saying that the interaction representation exists in general if the "interaction" Lagrangian does not include higher derivatives (or degree of freedom) than the "free" Lagrangian. But there occur sometimes singular cases which merit our attention. Let us, for example, take a Lagrangian of coupled oscillator

$$L = (\dot{q}_1^2 - \nu_1^2 q_1^2) / 2 + (\dot{q}_2^2 - \nu_2^2 q_2^2) / 2 + \eta \dot{q}_1 \dot{q}_2. \quad (2.44)$$

The Euler equations read

$$\ddot{q}_1 + \nu_1^2 q_1 = -\eta \ddot{q}_2, \quad \ddot{q}_2 + \nu_2^2 q_2 = -\eta \ddot{q}_1.$$

Thus, if $\eta = \pm 1$ and $\nu_1^2 \neq \nu_2^2$, we have no solution except $q_1 = q_2 = 0$; moreover, $p_1 = p_2$, and the \dot{q} 's cannot be expressed by the p 's.** In such cases the parametric differential equation for H still holds formally, but $\eta = \pm 1$ just corresponds to a singular point of this

* In this case take for K (Eq. 2.7), $K = \int_{-\infty}^t \partial L / \partial \eta dt$.

** This assumption corresponds to the Case IV of Dirac⁽¹⁾; $p_1 = p_2$ is a ϕ -equation.

equation. Indeed, assuming $\eta \neq \pm 1$, we obtain H as

$$\begin{aligned} H &= \frac{1}{2} \frac{1}{1-\eta^2} (\eta^2 \dot{q}_1^2 - 2\eta \dot{q}_1 \dot{q}_2 + \dot{q}_2^2) \\ &= \frac{1}{2} \frac{1}{1-\eta^2} (\eta \dot{q}_1 - \dot{q}_2)^2 - \frac{1}{2} \dot{q}_2^2 \\ &= \frac{1}{2} \frac{1}{1-\eta^2} (\dot{q}_1 - \eta \dot{q}_2)^2 - \frac{1}{2} \dot{q}_1^2. \end{aligned} \quad (2.45)$$

This H will lose meaning for $\eta \rightarrow \pm 1$ unless $\dot{q}_1 \rightarrow \pm \dot{q}_2$ holds simultaneously.

§ 3. Quantum mechanics

3. 1 *Introductory remarks*

At present we have two ways of transition from classical to quantum mechanics. One is the well-known procedure due to Heisenberg, Schrödinger and Dirac, which essentially consists in the replacement of the classical Poisson bracket by a quantum mechanical commutation relation. From this follows the Heisenberg equation of motion for dynamical variables, and the Schrödinger equation for the wave function ψ which is related to the classical action integral I by the correspondence $\psi \sim (iI/\hbar)$.

The other method, which is due to Feynman, starts from a very intuitive picture having a close analogy to wave optics. Thus one assumes that a transformation function $S(q'(t) | q''(t_0))$ from time t_0 to t is given by the sum of elementary amplitudes along all possible paths connecting $q''(t_0)$ and $q'(t)$ in the $q-t$ space:

$$S = \exp \left[i/\hbar \int_{t_0}^t L(q, \dot{q}) dt \right] \mathcal{D}(\text{path}).$$

The former method therefore gives a differential description of motion, while the latter gives the solution directly in the integral form. Another difference of the two methods lies in that the former is a Hamiltonian, whereas the latter a Lagrangian formalism.

The equivalence of the two alternative formalisms have been shown by Dyson⁽⁵⁾ and Feynman⁽⁶⁾ to a considerable extent. Dyson has introduced in his analysis of the S -matrix the concept of chronological ordering of operators, according to which S (in the interaction representation) is expressed as

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} (-i/\hbar c)^n \int P(H(x_1), \dots, H(x_2)) dx_1 \dots dx_n.$$

Here $P(\dots)$ denotes the ordering of the arguments with respect to time.* In our present analysis of the relation between the Hamiltonian and Lagrangian formalism, we also start, like Dyson, from the transformation function (or S -matrix) in interaction representa-

* Feynman⁽⁶⁾ has also made use of this idea in the exposition of his theory.

tion. For this purpose, the properties of the chronological ordering operator P will first be examined.⁷⁾

Let us take a set of q -numbers $q(t)$ furnished with a parameter t . We define the ordered product (P -product) of two elements $q(t)$ and $q(t')$ as follows:

$$\begin{aligned} P(q(t), q(t')) &= q(t)q(t'), \quad t \geq t', \\ &= q(t')q(t), \quad t' \geq t. \end{aligned} \quad (3.1)$$

For products of more than two elements, we apply successively the above definition. When the ordering is degenerate, i.e. there are more than one operators belonging to the same t , we must either assume that they are commutable, or else redefine their ordering in an unambiguous way. From (3.1) follows at once the fundamental properties of the P -operator:

$$\begin{aligned} P(a, b) &= P(b, a), \\ P(a, P(b, c)) &= P(a, b, c), \\ P(a, b+c) &= P(a, b) + P(a, c). \end{aligned} \quad (3.2)$$

Here it is required that each of the operators a, b, c shall belong to some definite parameter t so as to determine unambiguously its position in the ordered product. In other words, operators belonging to different t 's must be regarded as independent, so that no functional relations such as

$$\int c(t)q(t)dt = 0 \quad (c = c\text{-number}) \quad (3.3)$$

should be used as an operator equation in the P -product. It is clear that an equation of motion for q just belongs to this type of relation.

In quantum theory, however, we often come across an operator like

$$\dot{q}(t) \equiv Dq = \lim_{\varepsilon \rightarrow 0} \frac{q(t+\varepsilon) - q(t)}{\varepsilon} \quad (3.4)$$

which is really a "dipole" quantity related to two parameters, and we find it necessary to regard as if it were a new independent operator belonging to a single parameter t . Since Eq. (3.4) is of the form (3.3), we cannot use this equation in the P -product. This means that the differential operation D and the ordering operation P do not commute. To see the commutation relation of D and P , we write

$$\begin{aligned} P(q(t), q(t')) &= \frac{1}{2} \{q(t), q(t')\} + \frac{\varepsilon(t, t')}{2} [q(t), q(t')], \\ \{a, b\} &\equiv ab + ba, \quad [a, b] \equiv ab - ba, \\ \varepsilon(t, t') &= \begin{cases} 1 & t > t', \\ 0 & t = t', \\ -1 & t < t'. \end{cases} \end{aligned} \quad (3.5)$$

Then it is easy to see that*

$$DP(q, q') - PD(q, q') = \delta(t - t')[q, q']$$

or in general

$$DP(q, f') = PD(q, f') + \delta(t - t')[q, f']. \quad (3.6)$$

This relation will be used later when we go from the Lagrangian to the Hamiltonian formalism.

Eq. (3.6) can readily be extended to cases of higher derivatives. We get from (3.5)

$$D^n P(q, f') = PD^n(q, f') + \sum_{k=1}^n \binom{n}{k} (D^{k-1} \delta(t - t')) D^{n-k}[q, f'] \quad (3.7)$$

which involves derivatives of the delta function.

3. 2 Dyson's *S*-matrix

In quantum theory, the wave equation (in an arbitrary representation)

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

can be integrated according to

$$\psi(t) = U(tt_0)\psi(t_0),$$

$$U(tt_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar} \right)^n \int_{t_0}^t P(H(t_1), \dots, H(t_n)) dt_1 \dots dt_n. \quad (3.8)$$

In this expression of the transformation function U , the Hamiltonian $H(t)$ is regarded as a function of momenta $p(t)$'s and coordinates $q(t)$'s, which belong to one parameter t , though p and q are not commutable. It is for this reason that the symbol P has here a definite meaning, and Eq. (3.8) is equivalent to the more natural expression

$$U(tt_0) = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar} \right)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n H(t_1) H(t_2) \dots H(t_n).$$

If often occurs, however, that the Hamiltonian $H(t)$ is conveniently expressed in terms of coordinates q and velocities \dot{q} . Since in this case the velocities are substitutes of the momenta p , they also belong to a single parameter t , so that the differentiation $Dq = \dot{q}$ should be understood to be carried out *before* the ordering P . This is the characteristic feature of the Hamiltonian formalism.

Next let us go over to the Lagrangian formalism. For the time being we restrict ourselves to the interaction representation. We shall immediately prove the following relation

$$U(tt_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar} \right)^n \int_{t_0}^t P^*(L(t_1), L(t_2), \dots, L(t_n)) dt_1 \dots dt_n, \quad (3.9)$$

* D operates on the first argument q .

or

$$i\hbar \frac{\partial}{\partial t} U = P^*(-L, U),$$

where

$$P^*(\dot{q}(t), f(t')) \equiv D_t P^*(q(t), f(t')) \quad \text{etc.,}^* \quad (3.10)$$

and L is the interaction part of the Lagrangian. In words, the transformation function U is obtained by means of the Lagrangian L and the operator P^* ; the latter demands that the velocities \dot{q} which may appear in L should be regarded as genuine "dipole" quantities, and thus every differentiation in t should be carried out *after* the ordering of operators. (The second equation of (3.9) has only a vague sense, being a symbolical differential form of the first equation.) In Lagrangian formalism, coordinates and velocities are the fundamental quantities, the latter of which belongs to two adjacent parameters t and $t+dt$. In Hamiltonian formalism, coordinates and momenta are the fundamental quantities, both of which belong to a single parameter t .

When the interaction Lagrangian does not contain velocities at all, we have

$$H(t) = -L(t), \quad (3.11)$$

and it is clear that (3.8) and (3.9) amounts to the same expression, since there is no distinction between P and P^* . What we have to examine is the case in which L really contains a velocity \dot{q} . We here assume that $L(q, \dot{q})$ forms a power series in q and \dot{q} . Since q and \dot{q} do not commute, we must define their order in L . For the present purpose it is convenient to assume that L itself is an ordered operator. In a product of several q 's and \dot{q} 's, their order will be arbitrarily defined, each factor being regarded as belonging to distinct but adjacent times. Thus for example

$$\dot{q}^0 \rightarrow \lim_{t' \rightarrow t+0} P(\dot{q}(t'), \dot{q}(t)) = \lim_{t' \rightarrow t+0} P^*(\dot{q}(t'), \dot{q}(t)).$$

we often let t and t' be finitely different, and reserve the limiting procedure to the last. $L(t)$ will then be an ordered operator with finite time width.

Now our aim is to convert the Eq. (3.9), expressed by means of P^* , into an equation expressed by means of P . Take a representative term in the expansion (3.9):

$$\frac{1}{n!} \left(\frac{i}{\hbar} \right)^n \int P^*(L_1, L_2, \dots, L_n) dt_1 dt_2 \dots dt_n, \quad (3.12)$$

As long as all of t_1, t_2, \dots, t_n to which belong L_1, L_2, \dots, L_n are different by finite amounts, we may effectively regard the L 's as "monopoles", and there is no distinction between P and P^* . Next let us suppose that L_1 and L_2 come nearer to an infinitesimal distance so that the time interval occupied by L_1 overlaps with that of L_2 . By moving L_2 relative to L_1 , a particular factor q or \dot{q} in L_1 and a corresponding one in L_2 will cross each other at a certain time. At this instant, we have according to (3.6)

$$P^*(\dot{q}, \dot{q}') = P(\dot{q}, \dot{q}') + \delta(t-t')[q, \dot{q}']; \quad (3.13)$$

there arise no differences between P^* and P when q and q' , or q and \dot{q}' cross. Thus

* This P^* differs in definition from Koba's P^* .⁷⁾

on replacing the symbol P^* by P , we get an additional term at each crossing of a pair of \dot{q} 's. If we take for simplicity the time interval between individual q 's in L_1 larger than the total interval occupied by L_2 , the additional terms obtained by letting L_2 move through L_1 from past to future are a sum of contributions which are given by the substitution

$$\dot{q} \rightarrow (i/\hbar)[q, \dot{q}] \cdot L_{2\dot{q}} = (i/\hbar)[q, L_2], \tag{3.14}$$

applied successively to each \dot{q} in L_1 . Here the product supplied with \cdot is a symbolical one in the sense that the order of operators is not rigorously taken into account.* In the same sense, we can therefore write this sum as

$$L_{1\dot{q}} \cdot (i/\hbar)[q, L_2] = -(i/\hbar)[L_1, q] \cdot L_{2\dot{q}}. \tag{3.15}$$

Further there occurs a circumstance in which a third L_3 approaches the complex formed by the overlapping of L_1 and L_2 . In this case, we get new additional terms which should be supplemented to the pair (L_1L_2) , (L_1L_3) as well as to the already obtained additional term (3.15). This situation can be described symbolically as :

From (L_1L_2) arises $\mathcal{A}(L_1L_2) (= (i/\hbar)L_{1\dot{q}} \cdot [q, L_2])$,
From $(L_1L_2L_3)$ arises $\mathcal{A}(L_1L_2) + \mathcal{A}(L_2L_3) + \mathcal{A}(L_3L_1) + \mathcal{F}(L_1L_2L_3)$.
.....

We denote by S_r the r th order additional term $\mathcal{A}^{r-1}(L_1, \dots, L_r)$ arising from simultaneous overlapping of r L 's. Then (3.12) becomes

$$\begin{aligned} & \frac{1}{n!} \left(\frac{i}{\hbar} \right)^n \int P^*(L_1, \dots, L_n) dt_1 \dots dt_n \\ &= P \left(\frac{1}{n!} \sum_{r_1+r_2+\dots+r_k=n} \left(\frac{i}{\hbar} \right)^k \frac{n!}{r_1! r_2! \dots r_k!} \int S_{r_1} S_{r_2} \dots S_{r_k} dt_1 \dots dt_k \right), \\ & S_1 \equiv L. \end{aligned}$$

Adding up with respect to n ,

$$\begin{aligned} L &= \sum_{n=0}^\infty \frac{1}{n!} \left(\frac{i}{\hbar} \right)^n \int P \left(\left(\sum \frac{S_r}{r!} \right)_1, \dots, \left(\sum \frac{S_r}{r!} \right)_n \right) dt_1 \dots dt_n \\ &= \sum_{n=0}^\infty \frac{1}{n!} \left(\frac{-i}{\hbar} \right)^n \int P(\mathcal{H}_1, \dots, \mathcal{H}_n) dt_1 \dots dt_n, \\ \mathcal{H} &\equiv - \sum_{r=1}^\infty \frac{S_r}{r!}. \end{aligned} \tag{3.16}$$

* In general we may get different results by adopting different conventions on the ordering of operators in L and the application of the procedure (3.13). In ordinary cases, however, essential differences do not arise.

** In classical theory, (3.13) and (3.14) correspond respectively to

$$-\partial \dot{q} / \partial \dot{p} \text{ and } -L_{1\dot{q}} \cdot \partial L_2 / \partial \dot{p} = -\dot{p}_1 \partial L_2 / \partial \dot{p}.$$

Thus we have obtained the "effective Lagrangian" \mathcal{H} which is to be used to express U by means of the ordering P . Our proof will be complete if we can show that \mathcal{H} agrees with the Hamiltonian H . For this purpose we supply $L(t)$ with an external parameter $\eta(t)$ as $\eta(t)L(t)$. Putting $\eta = \text{const.}$, S_r changes to $\eta^* S_r$, and \mathcal{H} becomes a power series in η . Now from (3.16) we get

$$\frac{\partial U(t t_0)}{\partial \eta(t')} = P \left(\frac{-i}{\hbar} \frac{\partial}{\partial \eta(t')} \mathcal{H}(t'), U(t t_0) \right), \quad t > t' > t_0^*. \quad (3.17)$$

On the other hand, from (3.9),

$$\frac{\partial U(t t_0)}{\partial \eta(t')} = P^* \left(\frac{i}{\hbar} L(t'), U(t t_0) \right). \quad (3.18)$$

But we have according to (3.6) and (3.16)

$$P^*(\dot{q}(t'), U) = P(\dot{q}(t') - (i/\hbar)[q, \mathcal{H}(t')], U),$$

so that

$$\frac{\partial U}{\partial \eta(t')} = P \left(\frac{i}{\hbar} L \left(\dot{q} - \frac{i}{\hbar} [q, \mathcal{H}], q \right), U \right). \quad (3.19)$$

By taking the limit $t \rightarrow t_0$ we finally get from (3.17) and (3.19)

$$\frac{\partial \mathcal{H}}{\partial \eta} = -L \left(\dot{q} - \frac{i}{\hbar} [q, \mathcal{H}], q \right). \quad (3.20)$$

This is just the same relation as the classical one (2.30). Since the usual formalism tells that transition from classical to quantum mechanics is achieved by replacing the Poisson bracket by a commutator, we have equally in quantum mechanics the relation

$$\frac{\partial H}{\partial \eta} = -L \left(\dot{q} - \frac{i}{\hbar} [q, H], q \right). \quad (3.20')$$

Both \mathcal{H} and H thus satisfy the same differential equation, and in addition the same initial condition: $H = \mathcal{H} = 0$ for $\eta = 0$. Therefore we have $\mathcal{H} = H$ in general, q.e.d.

In the above discussion, L has been assumed to be a power series in q and \dot{q} . Otherwise the meaning of Eq. (3.9) will be obscured. Further it will often occur that L is a power series in q and \dot{q} , but that H as a function of p and q takes on a complicated form and their ordering becomes ambiguous. In such cases the present formulation may be used as a possible prescription.

Example 1. We take

$$L_{\text{tot}} = L_0 + \eta f(q) \dot{q}, \quad L_0 = \frac{1}{2} (\dot{q}^2 - v^2 q^2). \quad (3.21)$$

* This is true only when \mathcal{H} is a local operator, which is the case for systems with finite order of derivatives.

Then

$$\begin{aligned} S_1 &= L = \eta f(q) \dot{q}, \\ S_2 &= \Delta(L_1 L_2) = \eta f(q) (i/\hbar) [q, \eta f(q) \dot{q}] = -\eta^2 f(q)^2. \end{aligned}$$

The series terminates here since S_2 no more contains \dot{q} . Thus

$$H = -S_1 - \frac{1}{2!} S_2 = -\eta f(q) \dot{q} + \frac{1}{2} \eta^2 f(q)^2. \quad (3.22)$$

Example 2. Put

$$L_{\text{tot}} = (1 + \eta) \frac{1}{2} \dot{q}^2 - \frac{1}{2} \nu^2 q^2 = L_0 + \frac{1}{2} \eta \dot{q}^2. \quad (3.23)$$

The interaction Hamiltonian obtained from (2.26) is

$$H = -\frac{1}{2} \frac{\eta}{1 + \eta} \dot{p}^2 = -\frac{1}{2} \frac{\eta}{1 + \eta} \dot{q}^2. \quad (3.24)$$

To derive this result from (3.16), we first consider $S_2 = \Delta(L_1 L_2) = \frac{1}{4} \eta^2 \Delta(\dot{q}_1^2, \dot{q}_2^2)$. From each pair of \dot{q}_1 and \dot{q}_2 there arises a contribution $(i/\hbar) [q, \dot{q}] = -1$, so that we get

$$S_2 = -2^2 \frac{1}{4} \eta^2 \dot{q}_1 \dot{q}_2 \rightarrow -\eta^2 \dot{q}^2.$$

In calculating $S_n = \Delta^{n-1}(L_1 L_2 \dots L_n)$, we note that for each permutation of $L_1 L_2 \dots L_n$ there correspond 2^n different chains of pairs $(\dot{q}_r, \dot{q}_{r+1})$ taken out of L_r and L_{r+1} , each pair contributing a factor $(i/\hbar) [q, \dot{q}] = -1$ and leaving \dot{q}_1 and \dot{q}_n as operator. Dividing by 2 since a permutation and its inverse give identical contributions, we get

$$S_n = -(-\eta/2)^n (n!/2) 2^n \dot{q}^2 = -n! \frac{1}{2} (-\eta)^n \dot{q}^2.$$

Hence

$$\mathcal{H} = -\sum_{n=1}^{\infty} \frac{S_n}{n!} = \frac{1}{2} \dot{q}^2 \sum_{n=1}^{\infty} (-\eta)^n = -\frac{1}{2} \frac{\eta}{1 + \eta} \dot{q}^2.$$

The validity of Eq. (3.20) can readily be verified. It should be noted, however, that the \mathcal{H} as defined by the power series (3.16) has a meaning only if the series converges. In the present example, this corresponds to $|\eta| < 1$. Otherwise (3.16) must be looked upon as a formal equation giving \mathcal{H} through analytic continuation.

3.3 Cases of higher order equations

The foregoing discussions can be easily extended to the case of dynamical systems with higher order time derivatives. The fundamental equation (3.9) still holds if the interaction Lagrangian does not involve higher derivatives than the free Lagrangian. In this case the operator P^* is naturally defined by

$$P^*(q, f') = D^n P(q, f').$$

Let us consider, as before, the expression

$$\frac{1}{n!} \left(\frac{i}{\hbar} \right)^n \int P^*(L_1, L_2) dt_1 dt_2.$$

Making use of the equation (3.7), the additional terms arising from the substitution of P^* by P are obtained as

$$\Delta(L_1 L_2) = \sum_{r=1}^n L_{1(r)} \cdot \sum_{k=1}^r \binom{r}{k} (D^{k-1} \delta(t_1 - t_2)) \frac{i}{\hbar} [D^{r-k} q, L_2] dt_1, \quad (3.25)$$

in which $L_2(t_2)$ is kept fixed and $L_1(t_1)$ is integrated through the region occupied by L_0 . The derivatives of the δ -function can be removed by partial integration; with the aid of the relation

$$\sum_{k=1+l}^r \binom{r}{k} \binom{k-1}{l} (-1)^{k-1} = (-1)^l,$$

we get

$$\Delta(L_1 L_2) = \sum_{r=1}^n \sum_{l=0}^{r-1} (-1)^l (D^l L_{1(r)}) \cdot \frac{i}{\hbar} [q^{(r-l-1)}, L_2] \quad (3.26)$$

or, by Eq. (2.32)

$$= \sum_{r=0}^{n-1} p'_r \cdot \frac{i}{\hbar} [q^{(r)}, L_2], \quad (3.26')$$

where p'_r is that part of the momentum p_r which is due to the interaction Lagrangian, and \cdot signifies, as in Eq. (3.14), an appropriate ordered product.

Continuing this procedure, we can calculate the successive correction terms S_r and finally obtain

$$\mathcal{K} = - \sum_{r=1}^{\infty} \frac{S_r}{r!}.$$

Actually, the relations*

$$[q^{(r)}, q^{(n)}] = 0, \quad [q^{(n-1)}, q^{(n)}] \neq 0, \quad r+s \neq 2n-1, \quad r, s \leq n \quad (3.27)$$

hold (both in classical and quantum theory), so that only the term with $r=n-1$ survives in (3.26') :

* To prove these relations, note that

$$\begin{aligned} [q, q]^{(n)} &= [q_0, f(q_r, p_{n-1})] = 0, \\ [q, q]^{(n+1)} &= -[q, [q, H]] = -[[q, H], q] = -[q^{(1)}, q^{(n)}] = 0, \end{aligned}$$

and in general

$$[q, q]^{(n+k)} = (-1)^k [q^{(k)}, q^{(n)}] = 0 \quad \text{until} \quad k=n-1.$$

$$\Delta(L_1 L_2) = p'_{n-1} \cdot [{}^{(n-1)}_q, L_2] = L^{(n)}_q \cdot [{}^{(n-1)}_q, L_2]. \quad (3.26'')$$

Then we can repeat the same argument as after Eq. (3.12) and obtain the result

$$\frac{\partial \mathcal{H}}{\partial \eta} = -L\left({}^{(r)}_q, {}^{(n)}_q - \frac{i}{\hbar} [{}^{(n-1)}_q, \mathcal{H}]\right), \quad (3.28)$$

which agrees with Eq. (2.42).

In Eq. (3.25) we have fixed t_2 and the partial integration was performed with respect to t_1 . But this was not the only way, and we could also have integrated with respect to t_2 . Moreover, the symbol P^* means by nature that all the constituents of L_1 and L_2 are analysed into monopoles, so that we should have dealt with expressions such as

$$P^*({}^{(r)}_q, {}^{(s)}_{q'}) = D_t^r D_{t'}^s P(q, q').$$

The results, however, do not depend on these minute details of calculation, leading always to the same expression for \mathcal{H} . This is due to the special situation (3.27). Indeed, we could equally write (3.28) as

$$\partial H / \partial \eta = -\tilde{L}, \quad \tilde{L} = L({}^{(r)}_q), \quad (3.28')$$

with
$${}^{(r)}_q = \left(D + \frac{i}{\hbar} [H]\right)^r q, \quad ([H]q \equiv [H, q]).$$

When in the interaction Lagrangian appear higher derivatives than in the free Lagrangian, the interaction representation does not exist and Eq. (3.28) fails. As will be shown later, however, the expression (3.9) for the transformation function can still be applied in this case with the correct definition of P^* , namely that every quantity should be analyzed down to the monopole $q(t)$ and then arranged according to the chronological order. But in deriving the effective Hamiltonian \mathcal{H} according to the above mentioned procedure, there will arise ambiguities related to the partial integration. The transformation function thus obtained has a restricted freedom which corresponds to the original free Lagrangian.

As a means of handling interactions with such exceptional high derivatives we may also use the trick of attaching an auxiliary term with sufficiently high derivatives to the original free Lagrangian, thus augmenting formally the degree of freedom of the free system.* If after deriving the transformation function we let this added term vanish, some special part of initial and final states will thereby be selected out.

These considerations become important in the case of non-local Lagrangians. The formula (3.9) can still be used to get the transformation function. This seems clear if the free Lagrangian is also a non-local one, and the degree of freedom is not changed by the interaction term. But even if the non-locality appears only in the interaction, the expression (3.9) can give a meaningful transformation function, as will be discussed elsewhere when dealing with the elimination of variables. Thus if we take for example

* This is analogous to the regulator method in field theory.

$$L(t) = P^* \int_{-\infty}^{\infty} q_1(t) f(t-t') q_2(t') dt' \quad (3.29)$$

we have

$$i\hbar \partial U / \partial t = P^* (-L, U). \quad (3.29')$$

In this case, however, L contains $q(t)$'s at all times t , so that quantities such as $U(t, t_0)$ does not have a precise meaning as a transformation function. The only well defined quantity is the S -matrix $S = U(\infty, -\infty)$, to which the indeterminacy of L itself discussed in section 2.3 does not give influence. The operation P^* in (3.28) must be carried out by regarding every $q(t)$ with different t as independent quantity, and any equation of motion connecting different $q(t)$'s is not allowed to work prior to the operation P^* .

3. 4 Extention to general representations

We have seen that in interaction representation we may use either the interaction Hamiltonian together with the operator P , or the interaction Lagrangian together with the operator P^* , to obtain the transformation function U . Now we attempt to extend this result to more general cases.

What we have called the interaction representation does not necessarily correspond to a physical reality in which "free" systems are influenced by each other through weak "interactions". We have merely divided the total Lagrangian conveniently and arbitrarily into two parts

$$L_{\text{tot}} = L_0 + L,$$

and described the motion with L_{tot} relative to the motion with L_0 . Thus it is possible even to make the division according to

$$L_{\text{tot}} = aL + bL, \quad a + b = 1. \quad (3.30)$$

The transformation is expressed by means of bL and P^* , which formally holds for any a and b . A limiting case is obtained if we let a tend to zero, and we have then the Schrödinger representation. Since the condition for the validity of P^*-L formalism breaks down for vanishing free Lagrangian, this must correspond to a singular case of the P^*-L formalism.

Another limiting case is attained for $b=0$, the Heisenberg representation. U is then equal to unity, and operators vary with time according to the total Hamiltonian. Let $A(t)$ be any "monopole" operator in a general interaction representation, and $\tilde{A}(t)$ be the corresponding one in the Heisenberg representation. They are connected by the relation

$$\tilde{A}(t) = U(t, t_0) A(t) U(t_0), \quad (3.31)$$

both representations being assumed to coincide for $t=t_0$. For \tilde{A} holds the Heisenberg equation of motion

$$\begin{aligned} -i\hbar d\tilde{A}/dt &= [H_{\text{tot}}, \tilde{A}], \\ i\hbar d\tilde{A}/dt_0 &= [H(t_0), \tilde{A}] = [\tilde{H}(t_0), \tilde{A}]. \end{aligned} \quad (3.32)$$

In addition to the Heisenberg representation, we introduce the mixed representation by means of

$$\begin{aligned}\underline{A}(t') &= U(t't) A(t') U(t't_0) \\ &= P^*(A(t'), U(tt_0)),\end{aligned}\quad t > t' > t_0 \quad (3.33)$$

assuming that the multiplication law

$$U(tt_0) = U(t't) U(t't_0)$$

holds, which is the case for "local" systems. \tilde{A} and A are related by

$$\begin{aligned}\tilde{A}(t') &= U(t_0t) \underline{A}(t') = \underline{A}(t')^* U(tt_0), \\ \underline{A}(t') &= U(tt_0) \tilde{A}(t') = \tilde{A}^{(t)}(t') U(tt_0),\end{aligned}\quad (3.34)$$

with

$$\tilde{A}^{(t)}(t') = U(t't) A(t') U(t't).$$

Here \underline{A}^* is the Hermite conjugate of \underline{A} , and arranged according to the anti-chronological order (denoted by P^{*-1}). That the definition (3.32) using P^* is reasonable will be demonstrated by observing that

$$\begin{aligned}d\underline{A}(t')/dt' &= D_t P^*(A(t'), \sum \frac{1}{n!} (-i/\hbar)^n \int H_1 \dots H_n dt_1 \dots dt_n) \\ &= P^*(\dot{A}(t') - i/\hbar [A(t'), H(t')], U) \\ &= \underline{\dot{A}} - i/\hbar [\underline{A}, H] \\ &= -i/\hbar [\underline{A}, H_{\text{tot}}], \quad (\text{Note that } \underline{AB} \neq \underline{AB}.)\end{aligned}\quad (3.35)$$

thus yielding the total derivative. By multiplying both members of (3.35) by $U(t_0t)$ we get

$$d\tilde{A}(t')/dt' = U(t_0t) d\underline{A}(t')/dt' = -i/\hbar [\tilde{A}(t'), \tilde{H}_{\text{tot}}].$$

In the limit $t \rightarrow t_0$, $U(t_0t)$ tends to 1, so that the Heisenberg representation coincides with the mixed representation. Then Eq. (3.35) tells that the "instantaneous Heisenberg representation"

$$d\tilde{A}/dt = P^*(\dot{A}, U)|_{U=1}.$$

differs from the interaction representation

$$\dot{A} = P(\dot{A}, U)|_{U=1}$$

by a term $(-i/\hbar)[A, H]$. Putting $A=q$, we find here an interpretation of the equation (3.20).

With regard to the transformation function U , we note that

$$\begin{aligned}i\hbar \partial U(tt_0)/\partial t &= HU(tt_0) = U(tt_0) U(t_0t) HU(tt_0) \\ &= U(tt_0) \tilde{H}^{(t_0)}(t).\end{aligned}\quad (3.36)$$

Thus there arise two alternative expressions for U :

$$U(tt_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar} \right)^n \int_{t_0}^t P(H_1, \dots, H_n) dt_1 \dots dt_n$$

$$= \tilde{U}(tt_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar} \right)^n \int_{t_0}^t P^{-1}(\tilde{H}_1, \dots, \tilde{H}_n) dt_1 \dots dt_n, \quad (3.37)$$

$$\tilde{H}(t) \equiv \tilde{H}^{(t_0)}(t).$$

Since the two expressions differ only by the ordering and representation of the operators, we can also write U , following the previous results, in terms of the Lagrangian L :

$$\tilde{U}(tt_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar} \right)^n \int_{t_0}^t P^{*-1}(\tilde{L}_1, \dots, \tilde{L}_n) dt_1 \dots dt_n. \quad (3.37')$$

For U holds evidently the multiplication law

$$\tilde{U}(tt_0) = \tilde{U}(t't_0) \tilde{U}(tt').$$

Now let us consider, as in classical theory, the action integral \tilde{I} (putting here $L_{\text{tot}} \equiv L$):

$$\begin{aligned} \tilde{I} &= \int_{t_0}^t \tilde{L}^{(t_0)} dt = \int_{t_0}^t U(t_0 t') L(t') U(t' t_0) dt, \\ \underline{I} &= \int_{t_0}^t L dt = P^* \int_{t_0}^t U(tt') L(t') U(t' t_0) dt' \\ &= P^*(I, U(tt_0)), \\ I &= \int_{t_0}^t L dt. \end{aligned} \quad (3.38)$$

In view of (3.37'), \underline{I} can also be expressed as

$$\underline{I} = P^{*-1} \int_{t_0}^t \tilde{U}(t' t_0) \tilde{L}(t') \tilde{U}(tt') dt' = P^{*-1}(\tilde{I}, \tilde{U}(tt_0)). \quad (3.38')$$

Since operators in Heisenberg representation correspond to classical quantities, the variation principle applied formally to \tilde{I} yields the Heisenberg equation of motion together with the relation

$$\delta \tilde{I} = \tilde{L} \delta t + \tilde{L}_q \delta q + \dots = -\tilde{H} \delta t + \sum \tilde{p}_r \delta q_r, \quad (3.39)$$

$$\tilde{H} = \sum \tilde{p}_r^{(r+1)} \tilde{q} - \tilde{L} \quad (\equiv H_{\text{tot}}).$$

Now we can express the relation between \tilde{L} and \tilde{H} in a somewhat different way. Thus, we can write

$$\tilde{H} = -\tilde{L} + \mathcal{A}^{-1}(\tilde{L}, \tilde{H}), \quad (\mathcal{A} = -\tilde{L} - (i/\hbar) \tilde{L}_q \cdot [\tilde{q}, \tilde{H}] + \dots). \quad (3.40)$$

The symbol \mathcal{A}^{-1} bears an analogous meaning to \mathcal{A} in (3.25)–(3.26'), but with the operator P^{*-1} standing in place of P^* . Then using Ep. (3.26') ($L_1 \rightarrow L$, $L_2 \rightarrow H$, and a sign change because of P^{*-1}) and the nature of the (total) Hamiltonian:

$$-i/\hbar [\tilde{q}^{(r)}, \tilde{H}] = \tilde{q}^{(r+1)},$$

Eq. (3.39) at once leads to the familiar form (3.39).*

In (3.40) only the nature of the Hamiltonian (Eq. (3.32)) has been used, without regard to the precise form of the commutation relations. If we next fix the commutation relations between p_r and q_r , the Hamiltonian is thereby completely defined. But this circumstance can also be reversed. Thus if \tilde{H} is given as a function of the velocities $\dot{q}^{(r)}$, the equation $\dot{\tilde{H}}=0$ leads to the Euler equations of motion, which in turn requires that the canonical commutation relations be determined so as not to contradict these equations. (Uniqueness, however, is another question.)

It is also to be noted that Eq. (3.40) holds only for the total Hamiltonian, so that \tilde{H} and \tilde{L} in this equation are supposed to consist of such quantities whose motion is determined by this same H . On the other hand, Eq. (3.20) or (3.26) does hold for any (interaction) Lagrangian or Hamiltonian, irrespective of the free motion to which the q 's are subject.

Eq. (3.40) has a wide range of applicability, including even the non-local system. In fact let us take up the previous example (2.35) :

$$L(t) = \frac{1}{2} \int_{-\infty}^{\infty} P^*(\varphi(t) K(t-t') \varphi(t')) dt'. \quad (3.41)$$

Here the Lagrangian is adapted to quantum theory by assuming it to be an ordered function, but the Euler equation turns out to be the same as before. The generalization of Eq. (3.40), which is to be used here, is then given in the form

$$\tilde{H} = -\tilde{L} + i/\hbar \int_{-\infty}^{\infty} (P^{*-1} - P_{\nu}^{-1}) (\tilde{L}(t'), \tilde{H}(t)) dt'. \quad (3.42)$$

The meaning of this is as follows. $\tilde{L}(t)$ is really not a point function, but a "multipole"-like quantity consisting of the φ 's at all times. P^* demands that these quantities be arranged in chronological order, whereas P_{ν} represents an operation in which $\tilde{L}(t')$ as a whole is registered at time t' . The difference $P^{*-1} - P_{\nu}^{-1}$ is then calculated as

$$\begin{aligned} \int (P^{*-1} - P_{\nu}^{-1}) (\tilde{L}', \tilde{H}) dt' = & + \frac{1}{2} \int_{-\infty}^t dt'' \int_{-\infty}^{\infty} dt' \frac{1}{2} (\varepsilon(t'-t) - \varepsilon(t''-t)) \\ & \times [\tilde{\varphi}(t''), \tilde{H}(t)] \tilde{\varphi}(t') K(t'-t') \\ & + \frac{1}{2} \int_t^{\infty} dt'' \int_{-\infty}^{\infty} dt' \frac{1}{2} (\varepsilon(t'-t) - \varepsilon(t''-t)) \\ & \times \varphi(t') [\varphi(t''), \tilde{H}(t)] K(t'-t''). \end{aligned} \quad (3.43)$$

On the other hand, the quantities $\pi^{\pm}(t)$ in (2.39) is, in our present notation,

$$\pi_t(t') = \int_{-\infty}^{\infty} \frac{1}{2} (\varepsilon(t'-t) - \varepsilon(t''-t)) K(t'-t'') \varphi(t'') dt'', \quad (3.44)$$

* This relation (3.40) suggests that it could be derived from some basic nature of the transformation function U as expressed in terms of L and P^* , from which the usual relation (3.39) could conversely be inferred. But it is not yet successfully done.

so that we get the result

$$(P^{*-1} - P^{-1}) \int (L, H) dt' = -\frac{1}{2} \int_{-\infty}^t [\tilde{\varphi}(t'), \tilde{H}(t)] \tilde{\pi}_t(t') dt' \\ - \frac{1}{2} \int_t^{\infty} \tilde{\pi}_t(t') [\tilde{\varphi}(t'), \tilde{H}(t)] dt',$$

and

$$\tilde{H} = -\tilde{L} + \frac{1}{2} \int_{-\infty}^t \dot{\tilde{\varphi}}(t') \tilde{\pi}(t') dt' + \frac{1}{2} \int_t^{\infty} \tilde{\pi}(t') \dot{\tilde{\varphi}}(t') dt', \quad (3.45)$$

which is just the Eq. (2.39) or (2.40), except for the ordering of operators (in this case, antichronological).

In this way the significance of the operator P^* has been exhibited, not only in the transformation function U , but also in various other aspects of quantum theory. Now we take advantage of Eq. (3.37') in going to the limit of the total Lagrangian discussed at the beginning of this section ($\alpha \rightarrow 0$ in (3.30)). We decompose the matrix elements of $\tilde{U}(tt_0)$ into a product of infinitesimal factors:

$$(q' | \tilde{U}(tt_0) | q_0') = (q' | \Pi_n \tilde{U}(t_{n+1} t_n) | q_0') \\ = \Pi_n \int (q'_{n+1} | \tilde{U}(t_{n+1} t_n) | q_n') dq'. \quad (3.46)$$

Since every quantity that enters the expression L_{tot} is itself an operator in Heisenberg representation, we can naturally express the matrix elements $(q_n' | U(t_{n+1} t_n) | q_n')$ in a (mixed) representation in which all $q(t')$'s ($t \geq t' \geq t_0$) are taken to be diagonal:

$$(q'_{n+1} | U(t_{n+1} t_n) | q_n') = \exp \left[\frac{i}{\hbar} \int_{t_n}^{t_{n+1}} L_{\text{tot}}(q'_{n+1}, q'_n) dt \right], \quad (3.47)$$

$$(q' | U(tt_0) | q_0') = \int_{-\infty}^{\infty} \exp \left[\frac{i}{\hbar} \int_{t_0}^t L(q^{(r)}(t'')) dt'' \right] \mathcal{D}q(t''), \quad (3.48)$$

which is nothing but the starting point of Feynman's formulation.

In concluding this section, we add a few remarks concerning the parametric differential equation. Putting $L_{\text{tot}} = L_0 + \eta L$, we have

$$U(tt_0) = \sum \frac{1}{n!} \left(\frac{i}{\hbar} \right)^n \int_{t_0}^t P^*(\eta L_1, \dots, \eta L_n) dt_1 \dots dt_n \\ = P^* \exp \left[\frac{i}{\hbar} \int_{t_0}^t \eta L dt \right].$$

Differentiating with respect to η ,

$$i \hbar \frac{\partial U}{\partial \eta} = -P^* \left(\int_{t_0}^t L dt, U \right) = -P^*(I, U) \\ = -\tilde{I}^{(t)} U = -U \tilde{I}^{(t_0)}, \quad (3.49)$$

with

$$\tilde{q}^{(t)}(t') = U(tt') q(t') U(t't_0),$$

$$\tilde{q}^{(t_0)}(t) = U(t_0 t') q(t') U(t't_0).$$

This equation shows that the η -dependence of U is governed by the generating operator I , which corresponds to the quantity K (2.7) in classical theory. Exactly in an analogous manner as there, the two equations of motion

$$i\hbar\partial U/\partial t = HU \quad \text{and} \quad i\hbar\partial U/\partial \eta = -\tilde{I}^{(t)}U$$

entail the integrability condition

$$\frac{\partial I^{(t)}}{\partial t} + \frac{\partial H}{\partial \eta} + \frac{i}{\hbar}[H, \tilde{I}^{(t)}] = 0, \quad (3.50)$$

which, however, is fulfilled as a consequence of

$$\frac{\partial I^{(t)}}{\partial t} = L - \frac{i}{\hbar}[H, I] \quad \text{and} \quad \frac{\partial H}{\partial \eta} = -L.$$

Finally, Eq. (3.49) gives rise to the expansion

$$U(\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar} \right)^n \int_0^{\eta} \Pi(\tilde{I}_1^{(t)}, \dots, \tilde{I}_n^{(t)}) d\eta_1 \dots d\eta_n, \quad (3.51)$$

where Π denotes an ordering with respect to η .

3.5 Fermi statistics

Hitherto we have exclusively treated the "Bose systems", which has a close correspondence to classical systems. Our next investigations now concern the "Fermi systems", an analog of the Fermi fields. Since the functional relation between Lagrangian and Hamiltonian for a Fermi system is not essentially different from that for a Bose system, it is expected that the parametric differential equation for H and the Lagrangian expression of the transformation function U are still valid. But we must re-examine the definition of P^* so as to fit the present purpose.

The simplest example of a free Fermi system is the "Fermi oscillator", with the Lagrangian

$$L_0 = i\hbar\dot{\phi}^*\dot{\phi} - m\phi^*\phi, \quad (3.52)$$

where the amplitudes ϕ and ϕ^* satisfy

$$i\hbar\dot{\phi} - m\phi = 0, \quad -i\hbar\dot{\phi}^* - m\phi^* = 0,$$

$$\{\phi, \phi^*\} \equiv \phi\phi^* + \phi^*\phi = 1.$$

We may also supply the amplitudes with spin indices, and then the system acquires spin degrees of freedom.

When an interaction term is introduced which does not contain the derivatives of ϕ and ϕ^* , as is usually the case, ϕ^* and ϕ belonging to the same instant appear always in pairs in the interaction Lagrangian, so that we can treat the product $\phi^*\phi$ (as well as $\phi\phi^*$) as a point (monopole) function. Under such a convention, there arises no difference between P and P^* when applied to a Lagrangian like

$$\eta L = \eta\phi^*\phi,$$

and we have evidently

$$H = -\eta L, \quad \partial H / \partial \eta = -L.$$

Not trivial is the case where the interaction part contains also derivatives of the amplitudes. Thus we consider for example

$$P^*(\phi^*\dot{\phi}, \phi^{*'}\dot{\phi}'). \quad (3.53)$$

To adapt our notation to this circumstance, we define P^* as follows. Let a sign function be introduced which changes sign when in a product of ϕ 's and ϕ^* 's a pair of neighboring amplitudes are interchanged. P^* shall then imply an operation of arranging such a product in chronological order, taking due account of the sign function. When ϕ^* and ϕ belong to the same time, we suitably define their order and sign, e.g. $+\phi^*\phi$ or $+\frac{1}{2}(\phi^*\phi - \phi\phi^*)$. The operator P of course bears the meaning that derivatives are regarded as a whole to belong to one and single t .

According to this definition, (3.53) leads to

$$\begin{aligned} P^*(\phi^*\dot{\phi}, \phi^{*'}\dot{\phi}') &= P(\phi^*\dot{\phi}, \phi^{*'}\dot{\phi}') \\ &+ \delta(t-t')\phi^*\{\phi, \phi^{*'}\}\dot{\phi}' + \delta(t'-t)\phi^{*'}\{\phi', \phi^*\}\dot{\phi} \\ &= P(\phi^*\dot{\phi}, \phi^{*'}\dot{\phi}') + 2\delta(t-t')\phi^*\dot{\phi}. \end{aligned} \quad (3.54)$$

As an example, let us take

$$L_{\text{tot}} = i\hbar(1+\eta)\phi^*\dot{\phi} - m\phi^*\phi = L_0 + i\hbar\eta\phi^*\dot{\phi}. \quad (3.55)$$

The conjugate momenta to ϕ and ϕ^* are

$$\pi = \delta L / \delta \dot{\phi} = i\hbar(1+\eta)\phi^*, \quad \pi^* = \delta L / \delta \dot{\phi}^* = 0,$$

with the commutation relation

$$\{\phi, \pi\} = i\hbar, \quad \text{or} \quad \{\phi, \phi^*\} = 1/(1+\eta),$$

and the Hamiltonian

$$H_{\text{tot}} = m\phi^*\phi.$$

If we introduce the "free" amplitudes ϕ_0 and ϕ_0^* which obey the familiar commutation relation $\{\phi_0, \phi_0^*\} = 1$, we can write

$$H_{\text{tot}} = (m/1+\eta)\phi_0^*\phi_0, \quad H = (-m\eta/1+\eta)\phi_0^*\phi_0. \quad (3.56)$$

Next we apply the P^*-L formalism (3.9) to this case. To begin with,

$$\begin{aligned} S_1 &= L = i\hbar\eta\phi^*\dot{\phi}, \\ S_2 &= \Delta(L_1L_2) = 2(i/\hbar)\eta^2(i\hbar)^2\phi^*\dot{\phi} = -2i\hbar\eta^2\phi^*\dot{\phi}. \end{aligned}$$

General terms can be obtained by the same argument as in *Example 1* of Section 3.2, with the result

$$S_r = -i\hbar r! (-\eta)^r \phi^*\dot{\phi}. \quad (3.57)$$

Thus

$$\mathcal{H} = - \sum_{r=1}^{\infty} \frac{S_r}{r!} = -i\hbar(\eta/1+\eta)\dot{\phi}^*\dot{\phi} \quad (3.57)$$

$$= -m\eta/(1+\eta)\dot{\phi}^*\dot{\phi} = H. \quad (3.57')$$

Note that the "free" equation $i\hbar\dot{\phi} = m\dot{\phi}$ can be used only at the final stage; otherwise we are led to erroneous results. The same remark applies to the parametric differential equation. In the relation

$$P(\delta\mathcal{H}/\delta\eta(t), U) = P^*(-L, U)$$

(which is obtained by differentiating U by $\eta(t)$), we should use \mathcal{H} (3.57), and not H (3.57'), until P and P^* on both sides are taken off. Taking this into account, we can verify the relation

$$\begin{aligned} \partial\mathcal{H}/\partial\eta &= -L = -i\hbar\dot{\phi}^*\dot{\phi} = -i\hbar\left(\dot{\phi}^* - \frac{i}{\hbar}\frac{\partial\mathcal{H}}{\partial\dot{\phi}^*}\{\phi, \phi^*\}\right)\left(\dot{\phi} - \frac{i}{\hbar}\{\phi, \phi^*\}\frac{\partial\mathcal{H}}{\partial\dot{\phi}}\right) \\ &= -i\hbar(\dot{\phi}^* - \eta/(1+\eta)\dot{\phi}^*)(\dot{\phi} - \eta/(1+\eta)\dot{\phi}) \\ &= -i\hbar\frac{1}{(1+\eta)^2}\dot{\phi}^*\dot{\phi}. \end{aligned}$$

Extension to higher order equations meets no new difficulties. If the Lagrangian is bilinear in ϕ and ϕ^* , we can quantize the system according to Fermi statistics. If the interaction term does not involve the highest derivative, we have $H = -L$; if it does, U can still be obtained by means of P^* and L . As for the Hamiltonian, we must distinguish between H and \mathcal{H} . \mathcal{H} is defined by $-\sum S_r/r!$, contains in general the highest derivatives, and satisfies the parametric differential equation. H , on the other hand, is derived from \mathcal{H} by eliminating the highest derivatives with the aid of the (free) equations of motion, and can be used as Hamiltonian in the ordinary sense. Both H and \mathcal{H} amount to the same thing whenever P instead of P^* stands in the expression. Peculiar to the Fermi system is also the situation that the amplitudes ϕ^* (or ϕ) itself differs in the interaction and the instantaneous Heisenberg representation. These two representations are related again by

$$\dot{A} = P^*(\dot{A}, U)|_{U=1} = A + \Delta(A, \mathcal{H}). \quad (3.58)$$

Note that \mathcal{H} cannot be replaced by H .

It would be interesting if we could generalize the $P^* - L$ formalism in Fermi systems, as in Bose systems, to include the extreme case of vanishing free Lagrangian, and write down the transformation function as something like Feynman's path integral. Instead of allowing the coordinates q to take on all real values from $-\infty$ to $+\infty$, we could assign here two values 1 and 0 (or discrete values in general) to the quantity $\phi^*\dot{\phi}$, but the formulation does not seem to be so simple and advantageous as in Bose systems.

3.6 Canonical transformation

The purpose of this section is to investigate how the canonical transformation in Hamiltonian formalism is interpreted in our Lagrangian formalism. As is well known, a canonical transformation between two pairs of canonical variables p, q and p', q' is described by the relation

$$\begin{aligned} p\delta q - p'\delta q' &= \delta W(q, q') \\ (\text{or } p\delta q + q'\delta p' &= \delta W(q, p'), \text{ etc.}) \end{aligned} \quad (3.59)$$

from which follows

$$\begin{aligned} p &= \partial W / \partial q, \quad p' = -\partial W / \partial q', \\ (\text{or } p &= \partial W / \partial q, \quad q' = \partial W / \partial p', \text{ etc.}). \end{aligned}$$

In this formulation, the generating function W depends on the new as well as old variables. On the other hand, an infinitesimal canonical transformation is generated according to the relation

$$dp/d\eta = -\partial w / \partial q, \quad dq/d\eta = \partial w / \partial p, \quad (3.60)$$

with the generating function w that depends on p, q and the parameter η . Integrating (3.60) we get a finite transformation:

$$p' = S[\eta] p, \quad q' = S[\eta] q, \quad (3.61)$$

where

$$\begin{aligned} S[\eta] &\equiv P_\eta \exp\left(\int [\eta] d\eta\right), \\ [\eta] f &\equiv \frac{\partial w}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial w}{\partial q} \frac{\partial f}{\partial p}, \end{aligned}$$

and P_η symbolizes an ordering of the operation $[\eta]$ with respect to η .

Now let us suppose that the variation principle

$$\delta I = L\delta t + p\delta q = -H\delta t + p\delta q \quad (3.62)$$

is modified by adding to the Lagrangian L a total differential term dW/dt :

$$\delta I' = L\delta t + p\delta q + \delta W. \quad (3.63)$$

In order not to affect the equations of motion and the dynamical degrees of freedom, we first assume $W(t)$ to depend only on $q(t)$, $\dot{q}(t)$ and t (or better p, q and t). Eq. (3.63) suggests that a canonical transformation

$$p\delta q + \delta W = p'\delta q'$$

restores the original relation (3.62). The function W , however, is given explicitly only as a function of the older variables, so that Eq. (3.59) cannot be applied directly. To overcome this difficulty, we again make use of the parametric differential equation. Assuming W to be a function of p and q , the term dW/dt can be treated as an interaction

term which, however, does not change the dynamical nature of the system. Thus, putting $W = \eta w$ and writing H for H_{tot} , we get the equation

$$\partial H / \partial \eta = -dw/dt = -[H, w] - \partial w / \partial t. \quad (3.64)$$

When W does not depend explicitly on t , Eq. (3.64) is just of the form of an infinitesimal canonical transformation, the integral of which yields

$$H = \exp([w]\eta) H_0 = \exp([W]) H_0, \quad (3.65)$$

where H_0 is the original Hamiltonian. This shows clearly that the old and new Hamiltonians are related by a canonical transformation generated by W . When W depends explicitly on t , the solution of (3.64) is given by

$$H = \exp([w]\eta) H_0 - \exp([w]\eta) \int_0^\eta d\eta' \exp(-[w]\eta') \partial w / \partial t. \quad (3.65')$$

The second term on the right-hand side becomes

$$-\frac{1 - \exp([w]\eta)}{[w]} \frac{\partial w}{\partial t} = \sum_{n=1}^{\infty} \frac{1}{n!} [W]^{n-1} \frac{\partial W}{\partial t}.$$

It is evident that (3.65) and (3.65') hold equally in classical and quantum theory. In particular, the latter equation corresponds term to term to the transformation

$$H = S^{-1} H_0 S - S^{-1} \dot{S},$$

which is induced in quantum theory on the Hamiltonian for a time-dependent transformation

$$\psi_0 = S\psi = \exp(-iW/\hbar)\psi.$$

If we drop the restriction that W be a function only of p and q , and assume its general dependence on the "multipoles" \dot{q}, \ddot{q}, \dots , the above arguments still hold provided that these derivatives are defined implicitly by the unknown Hamiltonian as $[q, H], [[q, H]H], \dots$ W then also involves H , but Eq. (3.64) remains unchanged.

In this way it is concluded that addition to the Lagrangian of an arbitrary function dW/dt , $W = W(q, \dot{q}, \ddot{q}, \dots)$, which does not depend on t explicitly, corresponds to a canonical transformation on the Hamiltonian:

$$\partial H / \partial \eta = [w, H] \quad (\eta w = W = W([H], q)). \quad (3.66)$$

Conversely, if any canonical transformation is generated by (3.66) with w given as a function of p, q and η , then it can be shown that this corresponds to an additional term in the Lagrangian:

$$\frac{d}{dt} \left(\int w d\eta \right) \equiv \frac{dW}{dt}, \quad (3.67)$$

where w is integrated by expressing p in terms of q, \dot{q} and η through the relation $\dot{q} = -[q, H]$. On the other hand, the case in which W explicitly depends on t still represents a canonical transformation, in the sense that it modifies the equation of motion that expresses nothing but an evolution of a canonical transformation.

For instance, put $L_{\text{tot}} = L_0 + L$. By choosing $W = -\int L_0 dt$ (where integration is performed by letting $q(t)$ obey the equation of motion due to L_{tot}), the transformed Lagrangian becomes simply L , since $dW/dt = -L_0$. This means a transition from the Heisenberg to the interaction representation. If we take instead $W = -\int L_{\text{tot}} dt$, then we have made a transition to the Schrödinger representation. In the latter case, the canonical transformation between the old variables $p(t_0)$, $q(t_0)$ and the new variables $p(t)$, $q(t)$ is given through the action integral W :

$$p(t_0) = \partial W / \partial q(t_0), \quad p(t) = -\partial W / \partial q(t),$$

where W is expressed in terms of $q(t_0)$ and $q(t)$ in a suitable manner.

In these investigations, it was primarily assumed that the original Lagrangian was of first order in the derivatives. But it will be obvious that no difficulties arise in extending the results to Lagrangians with higher order derivatives as long as they admit the Hamiltonian formalism.

Now we turn our attention exclusively to quantum theory, and attack the problem from the viewpoint of the $P^* - L$ formalism. In interaction representation, the transformation function

$$U(t_0) = P^* \exp(i/\hbar \int_{t_0}^t L dt) \quad (3.68)$$

is modified, by an additional term dW/dt attached to L , into

$$\begin{aligned} U' &= P^* \exp[i/\hbar \int (L + \dot{W}) dt] \\ &= P^* (\exp[i/\hbar \int L dt], \exp[i(W(t) - W(t_0))/\hbar]) \\ &= P^*(U, V) = \tilde{V}^{(t)} U, \\ V &\equiv P^* \exp[i(W(t) - W(t_0))/\hbar], \quad \tilde{V}^{(t)} = P^*(U, V) U^{-1}. \end{aligned} \quad (3.69)$$

Thus the wave function ψ' corresponding to the transformation function U' is connected with the original wave function ψ by

$$\psi'(t) = \tilde{V}(t, t_0) \psi(t), \quad \tilde{V}(t_0, t_0) = 1. \quad (3.70)$$

Since W is arbitrary, we may also write $\dot{W} \equiv L_1$, so that

$$V = P^* \exp(i/\hbar \int_{t_0}^t L_1 dt). \quad (3.71)$$

Now it can be shown that \tilde{V} is a unitary operator provided that the original Lagrangian L is of local nature. Let V be decomposed into its constituents $q(t)$, and consider a representative term in the expansion of $P^*(U, V)$:

$$P^*(\exp(i/\hbar \int L dt), q(t'), q(t''), \dots), \quad t' > t'' > \dots$$

Of course such quantities as \dot{q} , \ddot{q} , etc. in V are to be replaced here by finite differences. By the assumption of local nature of L , this expression can be factorized into

$$P^*(\exp(i/\hbar \int_{\tau'}^t L dt), q(t')) \cdot P^* \exp(i/\hbar \int_{\tau''}^{\tau'} L dt), q(t'')) \dots, (t' > \tau' > t'' > \tau'' \dots) \\ = U(t_0 t) \tilde{q}(t') \tilde{q}(t'') \dots$$

introducing

$$\tilde{q}(t') = P^{*-1} \exp(-i/\hbar \int_{t_0}^{t'} L dt) \cdot P^* (\exp(i/\hbar \int_{t_0}^{t'} L dt), q(t')) \\ = U(t_0 t') P^* (U(t t_0), q(t')) = U(t_0 t') q(t') U(t' t_0).$$

Thus

$$V = U(t_0 t) \cdot P^* \exp(i/\hbar \int \tilde{L}_1 dt), \\ \tilde{V} = P^* \exp(i/\hbar \int \tilde{L}_1 dt) = P \exp(-i/\hbar \int \tilde{H}_1 dt), \quad (3.72)$$

which shows that \tilde{V} is of the form of a transformation function.

When L is of non-local nature, the factorization of V fails so that the above argument cannot be applied directly. But if such a non-local Lagrangian is described by means of an infinite set of canonical variables, and V can be given in terms of them as a Hermitic operator, the problem goes back to the local case.

The transformed Hamiltonian arising from $L' = L + d\Pi'/dt$ can be obtained again along the ready made procedure. One may either evaluate the correction terms $\Delta(L_1' L_2') = S_2'$, $\Delta^2(L_1' L_2' L_3') = S_3', \dots$, and put

$$H' = - \sum_{r=1}^{\infty} \frac{S_r'}{r!}; \quad (3.73)$$

or one may use the relation

$$\partial H' / \partial \eta = - d\tilde{w} / dt = - \partial \tilde{w} / \partial t - i/\hbar [H', \tilde{w}], \quad (IV = \eta \tilde{w}) \quad (3.73')$$

which is obtained by differentiating

$$U' = P^* \exp(i/\hbar \int L' dt).$$

At this place we must investigate more closely the case where Π' involves higher derivatives than are contained in the original Lagrangian. Strictly speaking, such a Π' does not modify the nature of the Euler equation of motion, but imposes additional restriction on the boundary condition in order that the effective degree of freedom be reduced to the one corresponding to this equation of motion. With this understood, Eq. (3.73') uniquely determines the canonical transformation for a given Π' , since the derivatives $\tilde{q}^{(r)}$ that may appear in \tilde{w} are expressed by means of the Hamiltonian H' .

On the other hand, there arise ambiguities if one will use Eq. (3.73) to obtain H' . This is due to the fact that in calculating S_r in accordance with the formulas (3.7) and (3.26), derivatives of the δ -function $\delta(t-t')$ lead to different results according as whether one integrates with respect to t or t' . This is indeed the very reason why the interaction representation fails in general if the interaction Lagrangian is of higher order than the free part. In the present case, however, such ambiguities again amount to a total differential

because of the special form dW/dt , so that it only means that the canonical transformation is not uniquely determined by giving dW/dt if W comprises higher derivatives. Eq. (3.73') corresponds to a particular specification on this point.

Example 1. Let us illustrate the results of this section with a few examples which are often called equivalence theorems in field theory.

First let us take two harmonic oscillators coupled by the interaction

$$L = \eta q_1 \dot{q}_2. \quad (3.74)$$

An "equivalent" Lagrangian is at once obtained by adding to this a total derivative:

$$L' = L - d(\eta q_1 q_2)/dt = -\eta \dot{q}_1 q_2. \quad (3.74')$$

The corresponding Hamiltonians, obtained by our method, read

$$H = -\eta q_1 \dot{q}_2 + \frac{1}{2} \eta^2 q_1^2, \quad (3.75)$$

$$H' = +\eta \dot{q}_1 q_2 + \frac{1}{2} \eta^2 q_2^2, \quad (3.75')$$

$$H_{\text{tot}} = \frac{1}{2} (\dot{p}_1^2 + \nu_1^2 q_1^2) + \frac{1}{2} (\dot{p}_2^2 - \eta q_1)^2 + \nu_2^2 q_2^2, \quad (3.76)$$

$$H'_{\text{tot}} = \frac{1}{2} (\dot{p}_1 + \eta q_2)^2 + \nu_1^2 q_1^2 + \frac{1}{2} (\dot{p}_2 + \nu_2^2 q_2^2). \quad (3.76')$$

These two expressions are obviously connected by the canonical transformation

$$\begin{aligned} p_1' &= p_1 - \eta q_2, & p_2' &= p_2 - \eta q_1, \\ q_1' &= q_1, & q_2' &= q_2, \end{aligned}$$

which is generated in accordance with

$$H' = \exp([w]\eta) H$$

$$\text{or} \quad = \exp(-i\eta q_1 q_2/\hbar) H \exp(i\eta q_1 q_2/\hbar).$$

Example 2. Next take (see Eq. (2.44))

$$L = \eta \dot{q}_1 \dot{q}_2. \quad (3.77)$$

The corresponding Hamiltonian (2.45) can easily be obtained by successive "hamiltonianization" (see Appendix C) of L with respect to q_1 and q_2 :

$$H = \frac{1}{2} \frac{1}{1-\eta^2} (\eta^2 \dot{q}_1^2 - \eta \dot{q}_1 \dot{q}_2 + \eta^2 \dot{q}_2^2), \quad (3.78)$$

$$H_{\text{tot}} = \frac{1}{2} \frac{1}{1-\eta^2} (\dot{p}_1^2 + \dot{p}_2^2 - \eta \dot{p}_1 \dot{p}_2) + \frac{1}{2} \nu_1^2 q_1^2 + \frac{1}{2} \nu_2^2 q_2^2.$$

An equivalent Lagrangian is supplied by

$$L' = -\eta \ddot{q}_1 q_2. \quad (3.79)$$

It is expected that the Hamiltonian H' becomes $H' = -H' + \dots = -\eta \nu_1^2 q_1 q_2 + \dots$ Indeed the transformation

$$\begin{aligned}q_1' &= q_1 + \eta q_2, & p_2' &= p_2 - \eta p_1, \\p_1' &= p_1, & q_2' &= q_2\end{aligned}$$

brings H into

$$\begin{aligned}H_{\text{tot}}' &= \frac{1}{2} p_2'^2 / (1 - \eta^2) + \frac{1}{2} p_1'^2 + \frac{1}{2} \nu_1^2 (q_1 - \eta q_2)^2 + \frac{1}{2} \nu_2^2 q_2'^2, \\H' &= \frac{1}{2} \eta^2 \dot{q}^2 / (1 - \eta^2) - \nu_1^2 \eta q_1 q_2 + \frac{1}{2} \nu_1^2 \eta^2 q_2'^2.\end{aligned}\quad (3.78')$$

In calculating H' by Eq. (3.73), we meet with an ambiguity in S_2 :

$$\Delta(\ddot{q}_1 q_2, \ddot{q}_1' q_2') = q_2 q_2' D_t^2 D_{t'}^2 \left(\frac{1}{2} \varepsilon(t - t') [\dot{q}_1, \dot{q}_1'] \right),$$

which yields different results according to whether we integrate simply by t :

$$\begin{aligned}S_2 &= \eta^2 (\dot{q}_2 q_2 [\dot{q}_1, q_1] + q_2^2 [\dot{q}_1, \ddot{q}_1]) = -\eta^2 (\nu_1^2 + \nu_2^2) q_2^2, \\H' &= -\nu^2 \eta q_1 q_2 + \frac{1}{2} \eta^2 (\nu_1^2 + \nu_2^2) q_2^2 + \dots,\end{aligned}\quad (3.80)$$

or proceed as

$$\begin{aligned}\Delta(\ddot{q}_1 q_2, \ddot{q}_1' q_2') &= \Delta(D(\dot{q}_1 q_2) - \dot{q}_1 \dot{q}_2, \ddot{q}_1' q_2') \\&\sim \delta(t - t') [\dot{q}_1 q_2, \ddot{q}_1' q_2'] - \Delta(\dot{q}_1 \dot{q}_2, \ddot{q}_1' q_2'),\end{aligned}$$

which leads correctly to Eq. (3.78') up to η^2 . But the difference between (3.78') and (3.80) up to η^2 :

$$\Delta H = \eta^2 (p_2^2 - \nu_2^2 q_2^2) / 2$$

can again be driven away to higher order terms by a canonical transformation on (3.78'):

$$\begin{aligned}p_2'^2 &= p_2^2 / (1 - \eta^2), & q_2'^2 &= (1 - \eta^2) q_2^2, \\H_{\text{tot}}' &= \frac{1}{2} p_2'^2 + \frac{1}{2} p_1'^2 + \frac{1}{2} \nu_1^2 (q_1 - q_2' / \sqrt{1 - \eta^2})^2 + \frac{1}{2} \nu_2^2 q_2'^2 / (1 - \eta^2).\end{aligned}\quad (3.81)$$

Example 3.

$$\begin{aligned}L_{\text{tot}} &= i\hbar \dot{\psi}^* \dot{\psi} - m\psi^* \sigma_z \dot{\psi} + \eta \dot{\psi}^* \sigma_z \dot{\psi} + (\dot{q}^2 - \nu^2 q^2) / 2, \\(\sigma_x, \sigma_z &= \text{Pauli spin matrices}).\end{aligned}\quad (3.82)$$

In this case of coupling between a Bose and a Fermi oscillator, we readily find

$$L = \eta \dot{\psi}^* \sigma_z \dot{\psi}, \quad H = -\eta \dot{\psi}^* \sigma_z \dot{\psi} + \dot{q} \frac{1}{2} \eta^2 (\dot{\psi}^* \sigma_z \dot{\psi})^2.\quad (3.83)$$

An equivalent Lagrangian is given by

$$L' = -\eta \frac{d}{dt} (\dot{\psi}^* \sigma_z \dot{\psi}) \cdot q \equiv -\eta \dot{\sigma}_z q.\quad (3.84)$$

Then

$$\begin{aligned}S_2 &= \Delta(L_1' L_2') = (i/\hbar) \eta^2 \dot{q}^2 [\sigma_z, \dot{\sigma}_z] = (i/\hbar) \eta^2 \dot{q}^2 [\sigma_z, (i/\hbar) [m\sigma_z, \sigma_z]] \\&= 4\eta^2 \dot{q}^2 m\sigma_z / \hbar^2.\end{aligned}$$

From this we anticipate the final result to be

$$H = A\sigma_x + B\sigma_y + C\sigma_z + D, \quad (3.85)$$

so that

$$\begin{aligned} \tilde{L}' &= -\eta q \dot{\tilde{\sigma}}_x = -\eta q (\dot{\sigma}_x - i/\hbar [\sigma_x, H]) \\ &= -\eta q \{ \dot{\sigma}_x - i/\hbar (2i\sigma_x B - 2i\sigma_y C) \} \\ &= 2\eta q / \hbar \{ (m + C)\sigma_y - B\sigma_z \}. \end{aligned}$$

We substitute this in the parametric differential equation for H' and obtain for the coefficients

$$\begin{aligned} dA/d\eta &= dD/d\eta = 0, \\ dB/d\eta &= (-2q/\hbar)(m + C), \quad dC/d\eta = (2q/\hbar)B, \end{aligned}$$

the solution of which, under the initial condition $H=0$ for $\eta=0$, becomes

$$\begin{aligned} B &= -m \sin 2q\eta/\hbar, \quad C = -m(1 - \cos 2q\eta/\hbar), \quad A = D = 0, \\ H' &= -m \{ \sin(2q\eta/\hbar)\sigma_y + (1 - \cos(2q\eta/\hbar))\sigma_z \}. \end{aligned} \quad (3.86)$$

On the other hand, (3.86) is derived from (3.83) by the canonical transformation

$$\begin{aligned} \psi' &= S\psi, \quad S = \exp(iq\eta\sigma_x/\hbar), \\ H'_{\text{tot}} &= S^{-1} \left(\frac{1}{2} \{ (p - \eta\sigma_x)^2 + \nu^2 q^2 \} + m\sigma_z \right) S \\ &= \frac{1}{2} (p^2 + \nu^2 q^2) + m\sigma_z S^2 \\ &= H_0 + m\sigma_z (S^2 - 1) = H_0 + H'. \end{aligned}$$

Appendix

A) On the W.K.B. approximation

Let the transformation function $U(t_0)$ be expressed by

$$U(t_0) = P^* \exp(i/\hbar \int_0^t L dt), \quad L = L_{\text{tot}}, \quad (A1)$$

which is essentially the same relation as Eq. (3.37'). Putting then $1/\hbar \equiv \eta$, we get

$$\begin{aligned} i\partial U / \partial \eta &= -P^* \left(\int L dt, U \right) = -P^*(I, U) \\ &= -\tilde{I}^{(u)} U = -U \tilde{I}^{(u_0)}. \end{aligned} \quad (A2)$$

This suggests that

$$\partial(\eta H) / \partial \eta = -L$$

which, however, is readily verified by the observation

$$(\partial / \partial \eta) \left(\eta H \left(\frac{1}{i\eta} \partial / \partial q, q \right) \right) = H - H_p \frac{1}{i\eta} \partial / \partial q = H - H_p p = -L.$$

Now we assume that the action integral \tilde{I} has been expressed in terms of $\tilde{q}(t)$ and $\tilde{q}(t_0)$ as an ordered function by solving explicitly the equations of motion :

$$\tilde{I}^{(t)} = \sum_n f_n(\tilde{q}(t)) g_n(\tilde{q}(t_0)), \tag{A3}$$

so that

$$\tilde{I}^{(t)} U = \sum f_n(q(t)) U(t t_0) g_n(q(t_0)).$$

The matrix element $\langle q' | U | q'' \rangle \equiv U(q', q'')$ then satisfies the equation

$$i \partial U(q', q'') / \partial \eta = - \sum f_n(q') g_n(q'') U(q', q'') = - I(q' q'') U(q', q''), \tag{A4}$$

which is at once integrated to give

$$U(q', q'' ; \eta) = \exp[i \int_{\eta_0}^{\eta} I(q', q'' ; \eta) d\eta] U_0. \tag{A5}$$

Formally, the equations of motion in quantum mechanics do not differ from the corresponding one in classical theory, and so is also the action integral I , provided that the ordering of operators is taken out of account. A difference arises only when I is rearranged into an ordered function, thereby producing a power series in \hbar which originates from the commutation relations :

$$I_{\text{ord}} = \sum_{n=0}^{\infty} \hbar^n I_n. \tag{A6}$$

Thus

$$\begin{aligned} U &= \exp[-i \int_{\hbar_0}^{\hbar} I_{\text{ord}} d\hbar / \hbar^2] U_0 \\ &= \exp[i I_0 / \hbar - i I_1 \log \hbar + I_1' - \sum_{n=2}^{\infty} \hbar^{n-1} I_n / (n-1)]. \end{aligned} \tag{A7}$$

The first term in the exponent represents the classical approximation :

$$U_0 = \exp(i I_{\text{class}} / \hbar).$$

I_1' corresponds to the initial condition. Though it is independent of \hbar , it does depend on t and q , and has to be determined from other conditions (the normalization factor).

As an example we take*

$$L = m \dot{q}^2 / 2. \tag{A8}$$

The equation of motion $\ddot{q} = 0$ leads to

$$\begin{aligned} \tilde{I} &= \frac{1}{2} (\tilde{q}(t) - \tilde{q}(t_0))^2 / (t - t_0) \\ &= \frac{1}{2} (\tilde{q}(t)^2 + \tilde{q}(t_0)^2 - \tilde{q}(t) \tilde{q}(t_0) - \tilde{q}(t_0) \tilde{q}(t)) / (t - t_0) \\ &= \frac{1}{2} (\tilde{q}(t)^2 + \tilde{q}(t_0)^2 - 2 \tilde{q}(t) \tilde{q}(t_0) - [q(t_0), q(t)]) / (t - t_0) \\ &= \frac{1}{2} (\tilde{q}(t) - \tilde{q}(t_0))^2 / (t - t_0) |_{\text{ord}} - \frac{1}{2} \hbar i, \end{aligned}$$

* A similar observation is made by Schwinger.¹³⁾

so that

$$U = (1/\sqrt{\hbar}) \exp[(im/2\hbar)(q'(t) - q''(t_0))^2/(t - t_0)] U_0. \quad (\text{A9})$$

U_0 is determined from the unitarity condition as

$$U_0 = 1/\sqrt{2\pi i(t - t_0)/m}$$

which reduces U to $\delta(q' - q'')$ in the limit $t - t' \rightarrow 0$.

B) *Hamilton-Jacobi formalism of the parametric differential equation*

The equation (3.20)

$$\partial H / \partial \eta = -L(\dot{q} - i/\hbar[q, H], q) \quad (\text{A10})$$

can be rewritten as

$$\partial H / \partial \eta + L(\dot{q} + \partial H / \partial p, q) = 0, \quad (\text{A11})$$

since $-i/\hbar[q, H] = \partial H / \partial p$. Here \dot{q} is a function of p and q . If we invoke H_{tot} and put

$$\dot{q} + \partial H / \partial p = \partial H_{\text{tot}} / \partial p,$$

we get the result, again writing H instead of H_{tot} :

$$\partial H / \partial \eta + L(\partial H / \partial p, q) = 0. \quad (\text{A12})$$

This can be compared with the usual Hamilton-Jacobi equation

$$\partial S / \partial t + H(\partial S / \partial q, q) = 0. \quad (\text{A13})$$

The correspondence is such that $\eta \rightarrow t$, $p \rightarrow q$, $H \rightarrow S$ and $L \rightarrow H$. In (A12), only the "momentum" $\partial H / \partial p$ enters into the "Hamiltonian", while the variable q merely plays the role of a parameter, as contrasted to the case (A13) where both of the canonical variables p and q in general appear in H . Thus, neglecting q in (A12), and writing

$$q \equiv p, \quad p \equiv \partial H / \partial p = \partial H / \partial q,$$

$$\mathfrak{G} \equiv L, \quad \mathfrak{H} \equiv H,$$

we obtain the Hamilton equations of motion associated with (A12):

$$dp/d\eta = -\partial \mathfrak{G} / \partial q, \quad dq/d\eta = \partial \mathfrak{G} / \partial p. \quad (\text{A14})$$

Because of the mentioned nature of \mathfrak{G} , this can be solved at once:

$$p = \text{const.} = p_0, \quad q = (\partial \mathfrak{G} / \partial p)_0 (\eta - \eta_0) + q_0,$$

so that

$$p_0 = F((q - q_0)/(\eta - \eta_0)).$$

Substituting this in \mathfrak{G} , we get

$$d\mathfrak{S} = -\mathfrak{G}[(q - q_0)/(\eta - \eta_0)] d\eta + p dq, \quad (\text{A15})$$

which fixes \mathfrak{S} up to an additive constant. H_{tot} is obtained simply by transcribing $q - q_0 \rightarrow p$, $\eta - \eta_0 \rightarrow \eta$ in \mathfrak{S} .

The analogy between (A12) and (A13) can further be pushed forward. Thus we observe that the "Lagrangian" \mathfrak{L} corresponding to the "Hamiltonian" \mathfrak{H} is furnished by

$$\mathfrak{L} = p \partial \mathfrak{S} / \partial p - \mathfrak{S}. \quad (\text{A16})$$

This Lagrangian in turn serves, for example, to exhibit the dependence of \mathfrak{S} or H on the "parameter" q , which we shall denote by ξ . In view of the relation (2.9), we have in this case

$$\partial \mathfrak{S} / \partial \xi + \mathfrak{R}(\partial \mathfrak{S} / \partial q) = 0, \quad \mathfrak{R} = - \int \partial \mathfrak{L} / \partial \xi d\eta. \quad (\text{A17})$$

Finally, we may "quantize" the Eq. (A12), and go over to the Schrödinger equation by means of the assumption

$$[p, q] = \hbar/i,$$

which results in

$$i\hbar \partial \psi / \partial \eta = \mathfrak{S} \psi, \quad \psi = \exp(-i\mathfrak{S}\eta/\hbar) \psi_0 \equiv \mathfrak{U} \psi. \quad (\text{A18})$$

If η is taken to be dimensionless, the "Planck constant" \hbar has dimensions of energy. The "transformation function" \mathfrak{U} is expressed as

$$(\langle q' | \mathfrak{U} | q_0'' \rangle) = \exp[i\mathfrak{S}(q', q_0'')/\hbar], \quad (\text{A19})$$

where \mathfrak{S} should agree with the "classical" action \mathfrak{S} in the limit of small \hbar . The implication of this rather academic generalization will be illustrated elsewhere in connection with the expansion of S -matrix in ordered operators.

C) Application to non-canonical systems

We will show that the Lagrangian formalism hitherto discussed, though equivalent to the usual Hamiltonian formalism when and where the latter exists, in fact encompasses a wider range of applicability, and better suited for the purpose of liberating ourselves from the existing quantum mechanics that is based on the usual Hamiltonian formalism.

Example 1. Take

$$L_{\text{tot}} = i\hbar \dot{\psi}^* \dot{\psi} - m \dot{\psi}^* \dot{\psi} + \eta (\dot{\psi}^* \dot{\psi})^2. \quad (\text{A20})$$

The last term represents a non-linear self-action. Though this L does not necessarily seem to contradict the requirements of Fermi statistics, the equation of motion is essentially of the second order nature, and the canonical formulation cannot be given directly. However, starting from the interaction representation, we can obtain the "Hamiltonian" as follows. First we observe that

$$S_1 = L = \eta (\dot{\psi}^* \dot{\psi})^2, \\ S_2 = \mathcal{A}(L_1 L_2) = 8\eta^2 (\dot{\psi}^* \dot{\psi})^3, \text{ etc.}$$

Thus we anticipate the final result to be

$$\mathcal{H} = F(\dot{\psi}^* \dot{\psi}, \eta),$$

which we substitute in the parametric differential equation. First we get

* This is somewhat analogous to the Konopinski-Uhlenbeck type interaction in β -decay, which was considered by Kanesawa and Koba¹⁴⁾ along a similar line as ours.

$$\begin{aligned}\tilde{\phi}^* &= \phi^* - \frac{i}{\hbar} \frac{\partial \mathcal{H}}{\partial \dot{\phi}} \{ \phi, \phi^* \} = \phi^* + \frac{\partial F}{\partial m} \phi^*, \\ \dot{\tilde{\phi}} &= \dot{\phi} - \frac{i}{\hbar} \{ \phi, \phi^* \} \frac{\partial \mathcal{H}}{\partial \phi^*} = \dot{\phi} + \frac{\partial F}{\partial m} \dot{\phi},\end{aligned}$$

where

$$F(\phi^* \dot{\phi}) = F(-im\phi^* \dot{\phi}/\hbar) \equiv F(m)$$

has been understood, since $\dot{\phi}$ can be replaced by $-im\phi/\hbar$ after the operation P^* . Hence

$$\partial F / \partial \eta = -(1 + \partial F / \partial m)^4 (\phi^* \dot{\phi})^2 = (m/\hbar)^2 (1 + \partial F / \partial m)^4, \quad (\text{A21})$$

which can be further converted into

$$\partial G / \partial \zeta = (\partial G / \partial x)^4 / 4, \quad (\text{A22})$$

with

$$\eta/\hbar^2 \equiv \zeta, \quad m + F \equiv G, \quad m \equiv x^2/2.$$

This "Hamilton-Jacobi" equation is equivalent to the one which holds for the Hamiltonian of a hypothetical dynamical system with

$$L_{\text{tot}} = \dot{q}^2/2 - \zeta \dot{q}^4/4, \quad (\text{A23})$$

so that H (or G) can at once be obtained by the usual procedure:

$$\begin{aligned}p &= \partial L / \partial \dot{q} = \dot{q} - \zeta \dot{q}^3 \quad (\sim x), \\ H &= p \dot{q} - L = \dot{q}^2/2 - 3\zeta \dot{q}^4/4 \quad (\sim G).\end{aligned} \quad (\text{A24})$$

H is thus three-valued as a function of p , being given by the roots $\omega_1, \omega_2, \omega_3$ of the equation

$$16\zeta^2 \omega^3 + 8\zeta \omega^2 + \omega(1 - 36\zeta m) + 27\zeta m^2 - m = 0. \quad (\text{A25})$$

As will directly be read off from Eq. (A24), all the roots are real if $0 \leq m\zeta \leq 2/27$, otherwise only one is real. In the limit $\zeta \rightarrow 0$, they become

$$\omega_1 = m, \quad \omega_2, \omega_3 = O(1/\zeta).$$

The eigenvalues of the Hamiltonian corresponding to (A20) are then given by 0, ω_1, ω_2 , and ω_3 , but they are not a simple superposition of two independent frequencies, because evidently no such relations as $\omega_3 = \omega_1 + \omega_2$ hold.

Example 2.

$$L_{\text{tot}} = i\hbar \phi^* \dot{\phi} - m \phi^* \phi + (\dot{q}^2 - \nu^2 q^2)/2 + i\hbar \eta \dot{q} \phi^* \dot{\phi}. \quad (\text{A26})$$

This Lagrangian again does not admit the ordinary canonical formulation for a system composed of a Bose and a Fermi oscillator. By our method, we divide the process of going over to H in two steps. Since ϕ and q are independent and commutable in interaction representation, the operator P^* breaks up into the product $P^*_\psi P^*_q$, so that the "hamiltonianization", implying the reduction of P^* to P , can also be carried out separately and successively for each factor. If we first "hamiltonianize" with respect to ϕ , the result becomes, taking account of (3.57),

$$\mathcal{H}_\psi = -i\hbar\eta\dot{q}\psi^*\dot{\psi}/(1+\eta\dot{q}) = -m\eta\dot{q}\psi^*\psi/(1+\eta\dot{q}). \quad (\text{A27})$$

Next, \mathcal{H}_ψ will further be "hamiltonianized" with respect to q . For this purpose, we take the easier way of ordinary Hamiltonian theory rather than our present method. Thus

$$\begin{aligned} L_q &= (\dot{q}^2 - \nu^2 q^2)/2 + m\eta\dot{q}\psi^*\psi/(1+\eta\dot{q}), \\ p &= \dot{q} + m\eta\psi^*\psi/(1+\eta\dot{q})^2, \\ H &= (\dot{q}^2 + \nu^2 q^2)/2 - (\eta\dot{q}/1+\eta\dot{q})^2 m\psi^*\psi. \end{aligned} \quad (\text{A28})$$

This gives H indirectly as a function of p and q .

Alternatively, we may reverse the steps. Thus we first obtain the Hamiltonian in q :

$$\begin{aligned} H_q &= -\hbar\eta\dot{q}\psi^*\dot{\psi} + (i\hbar\eta)^2(\psi^*\dot{\psi})^2/2 \\ &= (i\hbar\eta\psi^*\dot{\psi} - \dot{q})^2/2 - \dot{q}^2/2, \end{aligned} \quad (\text{A29})$$

which shall next be converted into the complete Hamiltonian by working with the operator \mathcal{H}_ψ^* . Naturally the two alternative methods should lead to the same final result.

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On the Analytic Behaviour of Dyson Transformation Function

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The detailed properties of the Dyson transformation function, which represents the time development of an initial free state at infinite past due to an interaction, adiabatically switched in and off according to a factor $\exp(-\varepsilon|t|/\hbar)$, ($\varepsilon > 0$), were examined in special connection with bound states. The transformation function is unitary and the power expansion in the interaction Hamiltonian is finite and regular in the limit $\varepsilon \rightarrow 0$, except those matrix elements with respect to the lowest energy free state of the system in question. An initial free state except the lowest energy one is transformed at a finite time t into the corresponding unbound eigenstate of the total Hamiltonian with the same energy. Transitions to bound states are due to those initial states whose energy falls in the interval of the order ε from that of the lowest energy state.

§ 1. Introduction and summary

By the elaborate work of Dyson, based upon the covariant formalism of the field theory due to Tomonaga, Schwinger and Feynmann, the mathematical method of dealing with non-stationary problems perturbation theoretically seems to have been completely established. On the other hand, there remains unsolved the problem to establish a formulation of the method of treating bound stationary states, which cannot be treated with perturbation theory, upon the correct field theoretical grounds. In this connection, it may be of considerable value to examine the properties of the transformation function, which played an important role in Dyson formalism and yet has been treated rather superficially so far, in connection with bound states. Some works¹⁾ have been made on this line. However, definite conclusions seem to have not yet been drawn. In this paper we shall be chiefly concerned, with the formal mathematical character of the transformation function.

We begin with the derivation of the equations for unbound and bound stationary state wave functions without use of perturbation theory and define from them the so called Heisenberg-Møller's S matrix,²⁾ say S_H , representing the asymptotic behaviour of the unbound stationary states. The non-stationary solution of the system starting from a given initial free state Φ_α at a time $t=t_0$ can be expressed by means of the usual unitary transformation function $U(t, t_0)$, given in power series in the interaction Hamiltonian. In the limit $t_0 \rightarrow -\infty$, the time integrations appearing in $U(t, t_0)\Phi_\alpha$ can be performed formally unless Φ_α is the lowest energy free state of the system in question,* and $\lim_{t_0 \rightarrow -\infty} U(t, t_0)\Phi_\alpha$ coincides with the corresponding unbound stationary state Ψ_α with the

* We don't refer to the vacuum state by the words "the lowest energy state", but to the state at rest of the particles under consideration, in the case of no interaction.

same energy as Φ_a . In other words, a free state Φ_a under the influence of the suddenly switched-in interaction, coincides with the unbound stationary state after an infinite time elapse. It can also be shown that $\lim_{t \rightarrow \infty, t_0 \rightarrow -\infty} U(t, t_0)$ equals to the Heisenberg-Møller's S matrix S_H . However these conclusions may not be claimed to be definite, since they stand on the tacit assumption of the convergence of the power expansion of the transformation function.

This mathematical difficulties can be overcome by taking an adiabatic case, where the interaction is switched in and off according to a time factor $\exp(-\varepsilon|t|/\hbar)$, ($\varepsilon > 0$), with ε tending to zero. The corresponding transformation function, say $U(t, -\infty; \varepsilon)$, expressed by the power series in the interaction Hamiltonian, can be readily integrated exactly, and is absolutely convergent for any *finite* value of ε . The unitary nature of it is verified independently of the value of ε , and hence it holds in the limit $\varepsilon \rightarrow 0$. In this limit, the *absolute* value series of $U(t, -\infty; \varepsilon)$ are divergent for those states Φ_a whose energy is lower than a certain value. Nevertheless, it can be shown, from the unitary property of $U(t, -\infty; \varepsilon)$, that $\lim_{\varepsilon \rightarrow 0} U(t, -\infty; \varepsilon) \Phi_a$ is convergent, except for the lowest energy state Φ_0 , and it coincides with the corresponding unbound stationary state Ψ_a . For the lowest energy state Φ_0 , $\lim_{\varepsilon \rightarrow 0} U(t, -\infty; \varepsilon) \Phi_0$ diverges, and since the transformation function is unitary, transition to bound states must occur from those states whose energy lies in the interval $E_a \lesssim \varepsilon$. (Here and after we assume the lowest eigenvalue of the free Hamiltonian, i.e., the sum of the rest masses of the system of particles under consideration, to be zero).

These circumstances are confirmed also by calculating the projection of the state $U(t, -\infty; \varepsilon) \Phi_a$ on a bound state Ψ_B , i.e., $\langle \Psi_B | U(t, -\infty; \varepsilon) | \Phi_a \rangle$, which vanishes with vanishing ε for a state with $E_a \neq 0$, and diverges for Φ_0 with $U(t, -\infty; \varepsilon) \Phi_0$ itself.

$\lim_{\varepsilon \rightarrow 0} U(\infty, -\infty; \varepsilon)$, on the contrary, is convergent for any initial state Φ_a , and is essentially equal to Heisenberg-Møller's S matrix S_H .

§ 2. Stationary and non-stationary solutions of Schrödinger equation

Denoting the time independent free and interaction Hamiltonians as H_0 and H respectively, the Schrödinger equation of the total system is, in units $\hbar=c=1$,

$$i\partial\Psi'(t)/\partial t = (H_0 + H)\Psi'(t),$$

which is transformed by the unitary transformation $\Psi'(t) = e^{-iH_0 t} \Psi(t)$ into the wave equation in the interaction representation,

$$i\partial\Psi(t)/\partial t = H(t)\Psi(t), \quad (1)$$

where $H(t)$ is given by

$$H(t) = e^{iH_0 t} H e^{-iH_0 t}. \quad (2)$$

Integrating this equation we obtain

$$\Psi(t) = (1/i) \int_{t_0}^t H(t_1) \Psi(t_1) dt_1 + \Psi(t_0). \quad (3)$$

If the state $\Psi(t)$ represents a stationary state of the total system with the energy E , it must be expressed as

$$\Psi(t) = e^{i(H_0 - E)t} \Psi, \quad (4)$$

where Ψ is the time independent state vector. Introducing this expression into (3) and performing the integration, we obtain

$$\Psi = \frac{1 - e^{i(E - H_0)T}}{E - H_0} H \Psi + e^{i(E - H_0)T} \Psi, \quad T = t - t_0. \quad (5)$$

Since T is arbitrary, we can take the limit $T \rightarrow \infty$. For a *bound* stationary state with $E < 0$, $E - H_0$ never vanishes and this reduces simply to

$$\Psi = \frac{1}{E - H_0} H \Psi, \quad (6)$$

and for an *unbound* stationary states with $E > 0$ to

$$\Psi = -2\pi i \delta_+(H_0 - E) H \Psi + \sum_E \Phi_E, \quad (7)$$

where $\sum_E \Phi_E$ represents an assembly of free states on the energy shell E , and is determined from the boundary conditions. In deriving the latter equation it is essential to use a formula

$$\lim_{T \rightarrow \infty} \frac{1 - e^{i(E - H_0)T}}{E - H_0} = P \frac{1}{E - H_0} - i\pi \delta(E - H_0) = -2\pi i \delta_+(H_0 - E), \quad E \neq 0, \quad (8)$$

which does not hold for $E = 0$, where the integral diverges. (See Appendix I). Taking for $\sum_E \Phi_E$ a normalized free eigenvector Φ_a , which satisfies an equation $H_0 \Phi_a = E_a \Phi_a$, the corresponding $\Psi = \Psi_a$ can be obtained from (7),

$$\Psi_a = \left[1 - 2\pi i \delta_+(H_0 - E_a) H \frac{1}{1 + 2\pi i \delta_+(H_0 - E_a) H} \right] \Phi_a. \quad (9)$$

Heisenberg-Møller's S matrix S_H can be defined from (9) as

$$S_H \Phi_a = \left[1 - 2\pi i \delta(H_0 - E_a) H \frac{1}{1 + 2\pi i \delta_+(H_0 - E_a) H} \right] \Phi_a. \quad (10)$$

Now we examine the non-stationary solution $\Psi(t)$ of the wave equation (1), which satisfies an initial condition $\Psi_a(t_0) = \Phi_a$. The transformation function defined by

$$\Psi_a(t) = U(t, t_0) \Phi_a,$$

satisfies an integral equation

$$U(t, t_0) = 1 + (1/i) \int_{t_0}^t H(t_1) U(t_1, t_0) dt_1,$$

which can be formally expanded into the power series in H ,

$$U(t, t_0) = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{i} \right)^n \int_{t_0}^t H(t_1) dt_1 \int_{t_0}^{t_1} H(t_2) dt_2 \cdots \int_{t_0}^{t_{n-1}} H(t_n) dt_n. \quad (11)$$

We shall exclusively use this expansion, assuming the convergence of it. $\Psi_a(t)$ can be readily calculated using (11). Namely, considering (2),

$$\begin{aligned}
\int_{t_0}^{t_{n-1}} H(t_n) dt_n \Phi_a &= \int_{t_0}^{t_{n-1}} e^{i(H_0 - E_a)t_n} dt_n H \Phi_a \\
&= e^{i(H_0 - E_a)t_{n-1}} \left(\frac{1 - e^{i(E_a - H_0)(t - t_0)}}{E_a - H_0} \right) H \Phi_a \\
&+ e^{i(H_0 - E_a)t_{n-1}} \frac{e^{i(E_a - H_0)(t - t_0)}}{E_a - H_0} (1 - e^{-i(E_a - H_0)(t - t_{n-1})}) H \Phi_a.
\end{aligned}$$

In the limit $t_0 \rightarrow -\infty$, or $T = t - t_0 \rightarrow \infty$, the second term vanishes, as is seen readily applying the formula (8), i.e.,

$$\left(\frac{1}{E_a - H_0} - P \frac{1}{E_a - H_0} \right) e^{i(H_0 - E_a)t_{n-1}} (1 - e^{-i(E_a - H_0)(t - t_{n-1})}) \rightarrow 0.$$

Applying the same formula for the first term, we obtain

$$\int_{t_0}^{t_{n-1}} H(t_n) dt_n \Phi_a \rightarrow e^{i(H_0 - E_a)t_{n-1}} (-2\pi i \delta_+(H_0 - E_a)) H \Phi_a.$$

Integrating successively in the same manner, we obtain

$$\lim_{t_0 \rightarrow -\infty} U(t, t_0) \Phi_a = [1 - 2\pi i \delta_+(H_0 - E_a) H + (-2\pi i \delta_+(H_0 - E_a) H)^2 + \dots] \Phi_a, \quad (12)$$

the time independent term just being equal to (9). Thus we obtain,

$$\lim_{t_0 \rightarrow -\infty} U(t, t_0) \Phi_a = e^{i(H_0 - E_a)t} \Psi_a, \quad E_a \neq 0, \quad (13)$$

which shows that the initial free state Φ_a , after an infinite time elapse following a sudden switching-in of the interaction, transforms into the corresponding unbound stationary state with the energy E_a .

The state at infinite future

$$\Psi_a(\infty) = \lim_{t \rightarrow \infty, t_0 \rightarrow -\infty} U(t, t_0) \Phi_a$$

can be calculated just in the same manner as $\Psi_a(t)$, performing the integrations in (11) successively from right to left, the last integral being,

$$\begin{aligned}
\lim_{t \rightarrow \infty, t_0 \rightarrow -\infty} \frac{1}{i} \int_{t_0}^t e^{i(H_0 - E_a)t_1} dt_1 &= \lim_{t \rightarrow \infty, t_0 \rightarrow -\infty} \frac{(1 - e^{i(H_0 - E_a)t_0}) - (1 - e^{i(H_0 - E_a)t})}{E_a - H_0} \\
&= -2\pi i \delta_+(H_0 - E_a) - 2\pi i \delta_-(H_0 - E_a) = -2\pi i \delta(H_0 - E_a). \quad (14)
\end{aligned}$$

It follows that

$$\Psi_a(\infty) = U(\infty, -\infty) \Phi_a = [1 - 2\pi i \delta(H_0 - E_a) [1 - 2\pi i \delta_+(H_0 - E_a) + \dots]] \Phi_a. \quad (15)$$

Comparing this with (10), we see that

$$U(\infty, -\infty) \Phi_a = S_H \Phi_a, \quad E_a \neq 0. \quad (16)$$

The above conclusions expressed in the equations (13) and (16) are based on the assumption of the convergence of (12) and (15), which is, however, not likely to hold for E_a lower than a certain value. (c.f. § 4.) In spite of these mathematical difficulties arising from the use of the expansion (11), the conclusions (13) and (16) seem to be correct in view of the results of the following paragraphs. (The condition $E_a \neq 0$ may be dropped in (16) but not in (13)).

§ 3. Transformation function for adiabatic case

We replace the time-independent Hamiltonian H by³⁾

$$H_1 = H e^{-\varepsilon |t|}, \quad \varepsilon > 0. \quad (17)$$

The exponential time factor is chosen because of its special simplicity in case of integrations. At infinite past and infinite future, the system lies in a free state of the Hamiltonian H_0 . In this case the transformation function from infinite past to a finite time or from a finite time to infinite future can be easily calculated. We shall define them by the power expansion in the interaction Hamiltonian H , i.e.,

$$U(t, -\infty, \varepsilon) = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{i} \right)^n \int_{-\infty}^t H_1(t_1) dt_1 \int_{-\infty}^{t_1} H_1(t_2) dt_2 \cdots \int_{-\infty}^{t_{n-1}} H_1(t_n) dt_n \quad (18)$$

and

$$U(\infty, t; \varepsilon) = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{i} \right)^n \int_t^{\infty} H_1(t_1) dt_1 \int_t^{t_1} H_1(t_2) dt_2 \cdots \int_t^{t_{n-1}} H_1(t_n) dt_n. \quad (18')$$

Employing a representation with H_0 diagonal $U(t, -\infty; \varepsilon)|a\rangle \equiv U(t, -\infty; \varepsilon)\Phi_a$ can be easily calculated from (2) and (18), performing the time integrations from right to left,

$$U(t, -\infty; \varepsilon)|a\rangle = e^{i(H_0 - E_a)t} \left[1 + \sum_{n=1}^{\infty} e^{n\varepsilon t} \frac{1}{E_a - H_0 + n i \varepsilon} H \frac{1}{E_a - H_0 + (n-1)i\varepsilon} H \cdots \right. \\ \left. \cdots H \frac{1}{E_a - H_0 + i\varepsilon} H \right] |a\rangle, \quad t \leq 0. \quad (19)$$

Also we have, performing the time integrations in (18') from left to right,

$$\langle a|U(\infty, 0; \varepsilon) \\ = e^{i(E_a - H_0)t} \langle a| \left[1 + \sum_{n=1}^{\infty} e^{-n\varepsilon t} H \frac{1}{E_a - H_0 + i\varepsilon} H \cdots H \frac{1}{E_a - H_0 + n i \varepsilon} \right], \quad t \geq 0. \quad (20)$$

Comparing (20) with (19), we find a relation

$$U(\infty, 0; \varepsilon) = U^+(0, -\infty; \varepsilon \rightarrow -\varepsilon), \quad (21)$$

where $\varepsilon \rightarrow -\varepsilon$ means that the substitution $\varepsilon \rightarrow -\varepsilon$ must be made after the time integration are performed. From this relation, we obtain

$$U(\infty, -\infty; \varepsilon) = U(\infty, 0; \varepsilon) U(0, -\infty; \varepsilon) \\ = U^+(0, -\infty, \varepsilon \rightarrow -\varepsilon) U(0, -\infty; \varepsilon). \quad (22)$$

It should be noted that the power expansions (19) and (20) are absolutely convergent for a finite ε , apart from the intrinsic field theoretical divergences in each term. This can be seen from the fact that the n -th term in (19) contains an additional factor $e^{\varepsilon t} (E_a - H_0 + n i \varepsilon)^{-1} H$ compared to the $(n-1)$ -th one, which tends to zero as n increases infinitely, if only ε is kept finite.

The unitarity of the operators $U(t, -\infty; \varepsilon)$ and $U(\infty, t; \varepsilon)$ can be directly proved also from (19) and (20). (See Appendix II).

$$U^+(t, -\infty; \epsilon)U(t, -\infty; \epsilon) = U(t, -\infty; \epsilon)U^+(t, -\infty; \epsilon) = 1, \quad (23)$$

$$U^+(\infty, t; \epsilon)U(\infty, t; \epsilon) = U(\infty, t; \epsilon)U^+(\infty, t; \epsilon) = 1. \quad (23')$$

It must be emphasized that the proof is based solely on the algebraic form of (19) and (20), and independent of the value of ϵ . The unitary relations (23) and (23'), therefore, hold even in the limit of $\epsilon \rightarrow 0$. The unitarity of the operator $U(\infty, -\infty; \epsilon)$ is a direct result from (23) and (23'), if we consider the equation (22),

$$U^+(\infty, -\infty; \epsilon)U(\infty, -\infty; \epsilon) = U(\infty, -\infty; \epsilon)U^+(\infty, -\infty; \epsilon) = 1. \quad (24)$$

Thus the expansions of the transformation functions in powers of H have a well defined mathematical meaning for a finite ϵ , if divergences arising from intermediate integrations are suitably subtracted by renormalization.

We are interested, however, in the limiting form of the transformation function when ϵ tends to zero, for which case the values of the series (19) and (20) might happen to diverge. To find the limiting form of $U(t, -\infty; \epsilon)$, we define an operator $\Lambda(E_a, \epsilon)$,

$$\Lambda(E_a, \epsilon) = 1 + \sum_{n=1}^{\infty} \frac{1}{E_a - H_0 + ni\epsilon} H \cdots \frac{1}{E_a - H_0 + i\epsilon} H, \quad (25)$$

$$\text{for which we have} \quad U(0, -\infty; \epsilon)|a\rangle = \Lambda(E_a, \epsilon)|a\rangle. \quad (26)$$

(We take $t=0$ hereafter for simplicity). It can easily be verified from (25) that $\Lambda(E_a, \epsilon)$, as a function of E_a , satisfies a functional equation

$$\Lambda(E_a, \epsilon) = 1 + \Lambda(E_a + i\epsilon, \epsilon) \frac{1}{E_a - H_0 + i\epsilon} H. \quad (27)$$

If $\Lambda(E_a, \epsilon)$ were finite and analytic even in the limit of $\epsilon \rightarrow 0$ in the neighbourhood of the whole positive real axis in the upper complex E_a plane, we could expand $\Lambda(E_a + i\epsilon, \epsilon)$ in the right hand side of (27) into $\Lambda(E_a, \epsilon) + i\epsilon \partial \Lambda(E_a, \epsilon) / \partial E_a$, and would obtain readily

$$\lim_{\epsilon \rightarrow 0} \Lambda(E_a, \epsilon) = \lim_{\epsilon \rightarrow 0} \left(1 - \frac{1}{E_a - H_0 + i\epsilon} H \right)^{-1}, \quad (28)$$

or by (9), (26) and the relation³⁾ $\lim_{\epsilon \rightarrow 0} (E_a - H_0 + i\epsilon)^{-1} = -2\pi i \delta_+(H_0 - E_a)$

$$\lim_{\epsilon \rightarrow 0} U(0, -\infty; \epsilon)|a\rangle = \frac{1}{1 + 2\pi i \delta_+(H_0 - E_a)H} |a\rangle = \Psi_a. \quad (29)$$

The same equation is obtained if we replace all the integral multiple of $i\epsilon$ appearing in the denominators of the expansion (19) by $i\epsilon$, and assume the convergence of the resulting series. The result (29), expressing that a free state Φ_a is transformed into the corresponding unbound stationary state with the same energy, is apparently false if any bound state exists, since it contradicts with the unitary nature of the transformation function. The equations (28) and (29) were derived from (19) under the assumption that $\Lambda(E_a, \epsilon)$ or $U(0, -\infty; \epsilon)$ were finite and regular in the limit of $\epsilon \rightarrow 0$ for any value of E_a . We must conclude, therefore, that the assumption was false. We know at the same time that the appearance of the integral multiple of $i\epsilon$ in the denominators of

(19) and (20) is essential for the transformation function to be unitary. We shall examine the analytic behaviour of the infinite series (19) more closely in the next paragraph.

§ 4. Analytic behaviour of the transformation function

For a further analysis it may be instructive to examine the simplest case of a non-relativistic particle moving in a one-dimensional δ -function type potential.⁴⁾ In this case, the interaction Hamiltonian is given by

$$H = -(\eta/2)\delta(x), \quad (30)$$

where $\eta/2$ represents the strength of the potential. In momentum representation, it is represented by a constant matrix element

$$\langle p|H|q\rangle = -\eta/\pi. \quad (31)$$

It is known by solving the Schrödinger equation that one bound state Ψ_B exists, with the energy

$$E_B = -\eta^2, \quad (32)$$

and the wave function

$$\langle p|\Psi_B\rangle = N/(\dot{p}^2 + \eta^2), \quad N^2 = 2\eta^3/\pi. \quad (33)$$

(We assign the mass of the particle 1/2 for simplicity). The intermediate integrations in (19) are separated in this case and the following type of integrations appear.

$$\int_{-\infty}^{\infty} \frac{-\eta/\pi}{\dot{p}_a^2 - \dot{p}^2 + mi\epsilon} d\dot{p} = \frac{i\eta}{\sqrt{\dot{p}_a^2 + mi\epsilon}}. \quad (34)$$

The evaluation of this integral was made by employing a contour integral along a large semi-circle in the complex \dot{p} plane and taking the residue. We obtain

$$U(0, -\infty; \epsilon)|a\rangle = |a\rangle + \sum_{n=1}^{\infty} \frac{\eta/\pi}{H_0 - E_a - ni\epsilon} \frac{i\eta}{\sqrt{E_a + (n-1)i\epsilon}} \cdots \frac{i\eta}{\sqrt{E_a + i\epsilon}}. \quad (35)$$

For a finite ϵ , this is absolutely convergent for any value of E_a . For $E_a > \eta^2 = |E_B|$, the series are absolutely convergent even in the limit $\epsilon \rightarrow 0$. However, the absolute value series of (35) apparently diverge in the limit of $\epsilon \rightarrow 0$, for E_a smaller than the absolute value of the ground state energy, i.e., for $E_a \leq \eta^2$. For the point $E_a = 0$ (35) behaves more singularly, each term diverging for a vanishing ϵ . The points $E_a = -i\epsilon, -2i\epsilon, \dots$ are poles as well as branch-points of $U(0, -\infty; \epsilon)|a\rangle$. In the limit $\epsilon \rightarrow 0$, $E_a = 0$ is a limiting point of these infinite number of singular points. Therefore, $E_a = 0$ is a true singular point.

These circumstances may be of quite general nature, not restricted to the special case considered, since in evaluating the intermediate integrations in (18), such factors as $(E_a + mi\epsilon)^{-1/2}$ always appear as residues, if we employ the method of contour integrals. Thus we can infer, generally, that the expansion (19) is absolutely convergent for E_a larger than a certain value (maybe the absolute value of the ground state energy),* but not so

* This can also be inferred from the well-known form of the diagonal matrix element of S_H , i.e., $\Pi_n(\dot{p} + i|a_n|)/(\dot{p} - i|a_n|)$, where \dot{p} means momentum and $-|a_n|^2$ the n -th bound state energy.

for E_a smaller than it, and that the point $E_a=0$ is a true singular point of the transformation function.

However, we shall show in the following, that the transformation function, as a function of the initial state energy E_a , cannot have a diverging value in the limit of $\varepsilon \rightarrow 0$ over a finite region of the variable E_a on account of the unitary property of it. The only singular point that the transformation function diverges must necessarily be the true singular point $E_a=0$. Therefore, the derivation of the equation (29) from (27) is justified except for $E_a=0$. We obtain

$$\lim_{\varepsilon \rightarrow 0} U(0, -\infty; \varepsilon) |a\rangle = \Psi_a \quad \text{for } E_a \neq 0. \quad (36)$$

To show the convergence of the transformation function, we remark the second equation of the unitary relations (23), which can be expressed as

$$\langle a|b\rangle = \int \langle a|U|c\rangle \langle b|U|c\rangle dc, \quad (37)$$

where U means $U(0, -\infty; \varepsilon)$. $\int dc$ runs over all the free states Φ_c . We take an arbitrary function $f(a)$, multiply the above equation by $f(a)f(\bar{b})$, and integrate over an arbitrary small region Δ , with respect to a and b . We obtain

$$\int_{\Delta} |f(a)|^2 da = \int_{\Delta} dc \left| \int_{\Delta} f(a) \langle a|U|c\rangle da \right|^2.$$

The left hand side is finite, and hence $\int_{\Delta} f(a) \langle a|U|c\rangle da$ cannot have an infinite value over a finite range of the initial states Φ_c . Since $f(a)$ is an arbitrary function and Δ is an arbitrary small region and since the unitary relations of U hold also in the limit of $\varepsilon \rightarrow 0$, we obtain the required result.

We can further show that the divergence of $U|a\rangle$ must occur at $E_a=0$ if any bound state exists. We divide the integration in the right hand side of (37) into two parts, one over a small region Δ containing the point $E_c=0$, and the rest \int' .

$$\langle a|b\rangle = \int_{\Delta} \langle a|U|c\rangle \langle b|U|c\rangle dc + \int' \langle a|U|c\rangle \langle c|U|b\rangle dc. \quad (38)$$

We take the limit $\varepsilon \rightarrow 0$, afterwards tending Δ to 0. Since $E_c \neq 0$ in the second integral, $U|c\rangle$ approaches the unbound stationary state Ψ_c by (36). Since Ψ_c 's do not form a complete set,* the second integral does not tend to $\langle a|b\rangle$ with vanishing Δ . Therefore, taking $\Phi_a = \Phi_b$, we obtain

$$\lim_{\Delta \rightarrow 0} \int_{\Delta} \lim_{\varepsilon \rightarrow 0} |\langle a|U|c\rangle|^2 dc \neq 0,$$

which indicates that $\langle a|U|0\rangle$ must be divergent, $|0\rangle$ representing the state Φ_0 with the lowest energy. Therefore, $U|c\rangle$ does not converge uniformly to Ψ_c near $E_c=0$.

It is interesting in this respect to calculate the projection $\rho_B(a)$ of the state $U(0, -\infty; \varepsilon) |a\rangle$ on a bound state Ψ_B ,

$$\rho_B(a) = \langle \Psi_B | U(0, -\infty; \epsilon) | a \rangle. \quad (39)$$

From the unitarity of $U(0, -\infty; \epsilon)$, it is easily proved that

$$\langle c | \Psi_B \rangle = \int \langle c | U(0, -\infty; \epsilon) | a \rangle \rho_B^*(a) da, \quad (40)$$

$$\text{or} \quad \Psi_B = U(0, -\infty; \epsilon) \int \rho_B^*(a) \Phi_a da. \quad (41)$$

$\rho_B^*(a)$, therefore, may be said as a density function of a linear assembly of free states which is transformed into the bound state Ψ_B . Since Ψ_B satisfies an equation (6) with $E = E_B < 0$, we obtain

$$\langle \Psi_B | U | a \rangle = \langle \Psi_B | H \frac{1}{E_B - H_0} U | a \rangle. \quad (42)$$

Substituting for U the expansion (19), we get

$$\begin{aligned} \langle \Psi_B | U | a \rangle &= \langle \Psi_B | H \frac{1}{E_B - H_0} | a \rangle \\ &+ \sum_{n=1}^{\infty} \langle \Psi_B | H \frac{1}{E_B - H_0} \frac{1}{E_a - H_0 + ni\epsilon} H \frac{1}{E_a - H_0 + (n-1)i\epsilon} \dots H \frac{1}{E_a - H_0 + i\epsilon} | a \rangle. \end{aligned}$$

This reduces, on applying a relation

$$\frac{1}{E_B - H_0} \frac{1}{E_a - H_0 + ni\epsilon} = \left(\frac{1}{E_B - H_0} - \frac{1}{E_a - H_0 + ni\epsilon} \right) \frac{1}{E_a - E_B + ni\epsilon},$$

$$\begin{aligned} \text{to} \quad \langle \Psi_B | U | a \rangle &= - \langle \Psi_B | H \left\{ \frac{1}{E_a - E_B} + \sum_{n=1}^{\infty} \frac{1}{E_a - E_B + ni\epsilon} \frac{1}{E_a - H_0 + ni\epsilon} \right. \\ &\quad \left. H \frac{1}{E_a - H_0 + (n-1)i\epsilon} H \dots \frac{1}{E_a - H_0 + i\epsilon} H \right\} | a \rangle + \\ &+ \langle \Psi_B | H \frac{1}{E_B - H_0} \sum_{n=1}^{\infty} \frac{1}{E_a - E_B + ni\epsilon} H \frac{1}{E_a - H_0 + (n-1)i\epsilon} \dots \frac{1}{E_a - H_0 + i\epsilon} H | a \rangle. \end{aligned}$$

Again writing $\langle \Psi_B | H \frac{1}{E_B - H_0} = \langle \Psi_B |$ in the second term, and adding the two terms, we obtain

* In the example (30), we obtain for Ψ , in matrix representation by momentum eigenvalue,

$$\langle p | \Psi | q \rangle = \langle p | q \rangle + (\eta | \pi) 2\pi i \delta_+ (p - q) (1 - i\eta | q |)^{-1},$$

from which we obtain

$$\begin{aligned} \langle p | \Psi^+ \Psi | q \rangle &= \langle p | q \rangle \\ \langle p | \Psi \Psi^+ | q \rangle &= \langle p | q \rangle - (2\eta^2 / \pi^2) (\eta + | \eta |) (p^2 + \eta^2)^{-1} (q^2 + \eta^2)^{-1}. \end{aligned}$$

Therefore,

in attractive potential, $\eta > 0$; $\langle p | \Psi \Psi^+ | q \rangle \neq \langle p | q \rangle$,

in repulsive potential, $\eta < 0$; $\langle p | \Psi \Psi^+ | q \rangle = \langle p | q \rangle$.

$$\rho_B(a) = \langle \Psi_B | U | a \rangle = -i\epsilon \langle \Psi_B | \sum_{n=0}^{\infty} \frac{1}{E_a - E_B + (n+1)i\epsilon} \frac{1}{E_a - E_B + ni\epsilon} H \frac{1}{E_a - H_0 + ni\epsilon} H \cdots | a \rangle.$$

Since $E_B < 0$, $E_a - E_B$ never vanishes, which enables in the limit $\epsilon \rightarrow 0$, to drop $(n+1)i\epsilon$ and $ni\epsilon$ from the factors $(E_a - E_B + (n+1)i\epsilon)^{-1}$ and $(E_a - E_B + ni\epsilon)^{-1}$, respectively. Finally we obtain

$$\lim_{\epsilon \rightarrow 0} \rho_B(a) = \lim_{\epsilon \rightarrow 0} \langle \Psi_B | U | a \rangle = \lim_{\epsilon \rightarrow 0} \frac{-i\epsilon}{(E_a - E_B)^2} \langle \Psi_B | H U | a \rangle = \begin{cases} 0 & \text{for } E_a \neq 0 \\ \infty & \text{for } E_a = 0. \end{cases} \quad (43)$$

The infinity at $E_a = 0$ arises because $E_a = 0$ is the true singular point of U . This indicates that in the integral in (40) or (41) only those states Φ_a with the energies up to the order of ϵ , i.e., $E_a \lesssim \epsilon$, make non-vanishing contributions to the integral. Therefore, for an arbitrary small ϵ ,

$$\Psi_B \sim U(0, -\infty; \epsilon) \int_{E_a \lesssim \epsilon} \rho_B^*(a) \Phi_a da. \quad (44)$$

We reach, in this way, a conclusion that, in the limit $\epsilon \rightarrow 0$, the initial free states except the lowest energy one Φ_a , are transformed into the corresponding eigenstates Ψ_a of the whole Hamiltonian belonging to the continuous spectrum, and that transitions to the bound states occur from those states with the energy $E_a \lesssim \epsilon$.

A remark may be added here. In general, it may not be allowed to put, from (41) and (43),

$$\Psi_B = \lim_{\epsilon \rightarrow 0} U(0, -\infty; \epsilon) \Phi_0 \int \rho_B^*(a) da, \quad (45)$$

since not only $\rho_B^*(a)$ but also $U(0, -\infty; \epsilon) | a \rangle$ are singular at $E_a = 0$ and the infinity of $\rho_B(a)$ is not of the δ -function type. If we tacitly assume that the equation (45) were correct, and put further

$$\int \rho_B^*(a) da = 1 / \rho_B(0) = 1 / \langle \Psi_B | U | 0 \rangle,$$

which would be derived by an analogous assumption from the normalization condition of Ψ_B , i.e.,

$$\int \rho_B^*(a) \rho_B(a) da = 1,$$

then we would get

$$\Psi_B = \lim_{\epsilon \rightarrow 0} \frac{U(0, -\infty; \epsilon) | 0 \rangle}{\langle \Psi_B | U(0, -\infty; \epsilon) | 0 \rangle}.$$

Though this equation might be correct in cases where only one bound state exists, and no zero energy state Ψ_0 is possible, it would in general, be incorrect, since it immediately leads to a contradiction when more than two bound states exist. Since we are dealing

with a representation by the continuous eigenvalue of H_0 we may have to take the limit $\epsilon \rightarrow 0$ always after integration. This is the reason why we keep ϵ finite in the problem of the transformation from the free to bound states.

We turn to the problem of the transformation function from infinite past to infinite future, which is given by (22)

$$\begin{aligned} U(\infty, -\infty; \epsilon) &= U(\infty, 0; \epsilon) U(0, -\infty; \epsilon) \\ &= U^+(0, -\infty; \epsilon \rightarrow -\epsilon) U(0, -\infty; \epsilon). \end{aligned}$$

If the adiabatic theorem were to hold $\lim_{\epsilon \rightarrow 0} U(\infty, -\infty; \epsilon)$ must be equal to unity, which is however, not the case. This stems from the fact that $\lim_{\epsilon \rightarrow 0} U^+(0, -\infty; \epsilon \rightarrow -\epsilon) \neq \lim_{\epsilon \rightarrow 0} U^+(0, -\infty; \epsilon)$. The state $\lim_{\epsilon \rightarrow 0} U(0, -\infty; \epsilon) |a\rangle$ represents an outgoing wave, while $\lim_{\epsilon \rightarrow 0} U(0, -\infty; \epsilon \rightarrow -\epsilon) |a\rangle$ represents an incoming wave. If we put

$$U(0, -\infty; \epsilon) = U(0, -\infty; \epsilon \rightarrow -\epsilon) + \{U(0, -\infty; \epsilon) - U(0, -\infty; \epsilon \rightarrow -\epsilon)\},$$

then, from (22),

$$U(\infty, -\infty; \epsilon) = 1 + U^+(0, -\infty; \epsilon \rightarrow -\epsilon) \{U(0, -\infty; \epsilon) - U(0, -\infty; \epsilon \rightarrow -\epsilon)\}.$$

Therefore, the difference between outgoing and incoming waves remains after the interaction is switched off and the state does not return to the initial one.

For the states Φ_a and Φ_b with energies $E_a, E_b \neq 0$, we can write from (36)

$$\lim_{\epsilon \rightarrow 0} U(0, -\infty; \epsilon) |a\rangle = \lim_{\epsilon \rightarrow 0} \left(1 - \frac{1}{E_a - H_0 + i\epsilon} H \right)^{-1} |a\rangle, \quad E_a \neq 0,$$

$$\lim_{\epsilon \rightarrow 0} \langle b | U(\infty, 0; \epsilon) = \lim_{\epsilon \rightarrow 0} \langle b | \left(1 - H \frac{1}{E_b - H_0 + i\epsilon} \right)^{-1}, \quad E_b \neq 0.$$

Therefore,

$$\lim_{\epsilon \rightarrow 0} \langle b | U(\infty, -\infty, \epsilon) |a\rangle = \lim_{\epsilon \rightarrow 0} \langle b | \left(1 - H \frac{1}{E_b - H_0 + i\epsilon} \right)^{-1} \left(1 - \frac{1}{E_a - H_0 + i\epsilon} H \right)^{-1} |a\rangle,$$

which reduces after some calculation to

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \langle b | U(\infty, -\infty; \epsilon) |a\rangle &= \langle b | a \rangle \\ &+ \left(\frac{1}{E_a - E_b + i\epsilon} - \frac{1}{E_a - E_b} \right) \langle b | H \left(1 - \frac{1}{E_a - H_0 + i\epsilon} H \right)^{-1} |a\rangle \\ &+ \left(\frac{1}{E_b - E_a + i\epsilon} - \frac{1}{E_b - E_a} \right) \langle b | \left(1 - H \frac{1}{E_b - H_0 + i\epsilon} \right)^{-1} H |a\rangle. \end{aligned}$$

Considering $(x + i\epsilon)^{-1} = P(1/x) + i\pi\delta(x)$, a singularity arises from $P \frac{1}{E_a - E_b} - \frac{1}{E_a - E_b}$ at $E_a = E_b$, which is, however, cancelled by the corresponding one in the second term. Hence only the δ -function terms remain, and we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \langle b|U(\infty, -\infty; \varepsilon)|a\rangle &= \langle b|a\rangle + 2\pi i \delta(E_a - E_b) \langle b|H \left(1 - \frac{1}{E_a - H_0 + i\varepsilon} H\right)^{-1} |a\rangle \\ &= \langle b|a\rangle + 2\pi i \delta(E_b - E_a) \langle b|H \frac{1}{1 + 2\pi i \delta_+(H_0 - E_a)H} |a\rangle. \end{aligned} \quad (47)$$

Comparing this expression with the Heisenberg's S_H matrix (10), which was derived from the unbound stationary state wave function, we know both are the same;

$$\lim_{\varepsilon \rightarrow 0} \langle b|U(\infty, -\infty; \varepsilon)|a\rangle = \langle b|S_H|a\rangle, \quad E_a, E_b \neq 0. \quad (48)$$

At points $E_a = 0$ or $E_b = 0$, the transformation function $\langle b|U(\infty, -\infty; \varepsilon)|a\rangle$ cannot behave singularly, contrary to the case of $U(0, -\infty; \varepsilon)|a\rangle$ or $\langle b|U(\infty, 0; \varepsilon)$, because $U(\infty, -\infty; \varepsilon)$ coincides with the unitary matrix S except these isolated points. This can be seen by remarking again the equation (38), replacing U by $U(\infty, -\infty; \varepsilon)$. The second term become by virtue of (48),

$$\lim_{\varepsilon \rightarrow 0} \int' \langle a|S_H|c\rangle \langle c|S_H|b\rangle dc \rightarrow \langle a|b\rangle,$$

since S_H is unitary. Hence taking $\Phi_a = \Phi_b$, we obtain

$$0 = \lim_{\Delta \rightarrow 0} \int_{\Delta} \lim_{\varepsilon \rightarrow 0} |\langle a|U(\infty, -\infty; \varepsilon)|c\rangle|^2 dc,$$

which shows that $U(\infty, -\infty; \varepsilon)|0\rangle$ cannot behave singularly. Also we can show from the relation

$$U^+(\infty, -\infty; \varepsilon)U(\infty, -\infty; \varepsilon) = 1,$$

that $\langle 0|U(\infty, -\infty; \varepsilon)$ cannot be singular. Thus we can say $\lim_{\varepsilon \rightarrow 0} U(\infty, -\infty; \varepsilon)$ is essentially equal to S_H .

In conclusion, the authors wish to express their cordial thanks to Prof. Yoshitaka Mimura and Prof. Kiyoshi Sakuma for their valuable discussions.

Appendix I

To prove the formula (8), we evaluate an integral

$$\begin{aligned} \lim_{T \rightarrow \infty} \int_0^\infty \frac{1 - e^{i(E - E_b)T}}{E - E_b} f(E_b) dE_b &= - \lim_{T \rightarrow \infty} \int_{-E}^\infty \frac{1 - \cos xT}{x} f(x + E) dx - \\ &\quad - i \lim_{T \rightarrow \infty} \int_{-E}^\infty \frac{\sin xT}{x} f(x + E) dx, \end{aligned} \quad (I. 1)$$

where $f(x)$ is an arbitrary function that vanishes sufficiently fast at $x = \infty$. The second term is the so-called Dirichlet's integral and is equal to $i\pi f(E)$ if $E > 0$. To evaluate the first integral, we divide the integral region into three parts

$$\int_{-E}^\infty \frac{1 - \cos xT}{x} f(x + E) dx = \int_{-\Delta}^\Delta + \int_{-\Delta}^{-E} + \int_{\Delta}^\infty, \quad (I. 2)$$

Δ being an arbitrary small number. In the second and last integral, we see that

$$\left. \begin{aligned} \int_{-\Delta}^{-E} \frac{\cos xT}{x} f(x + E) dx &= \frac{1}{T} \left[\frac{\sin xT}{x} f(x + E) \right]_{-\Delta}^{-E} - \int_{-\Delta}^{-E} \sin xT \frac{d}{dx} \left(\frac{f(x + E)}{x} \right) dx \rightarrow 0, \\ \int_{\Delta}^\infty \frac{\cos xT}{x} f(x + E) dx &= \frac{1}{T} \left[\frac{\sin xT}{x} f(x + E) \right]_{\Delta}^\infty - \int_{\Delta}^\infty \sin xT \frac{d}{dx} \left(\frac{f(x + E)}{x} \right) dx \rightarrow 0 \end{aligned} \right\} \quad (I. 3)$$

in the limit $T \rightarrow \infty$. In the first integral of (I. 2), we remark that the integrand $(1 - \cos xT)x^{-1}f(x+E)$ is continuous at $x=0$. Since $(1 - \cos xT)x^{-1}$ is an odd function, the even part of $f(x+E)$ makes not contribution to the integral. Denoting the odd part of $f(x+E)$ as

$$g(x) = \{f(x+E) - f(-x+E)\}/2,$$

it can be expanded into power series of x in the interval $-A \leq x \leq A$, $g(x) = xg'(0) + \dots$. Therefore,

$$\begin{aligned} \int_{-A}^A \frac{1 - \cos xT}{x} f(x+E) dx &= 2 \int_0^A \frac{1 - \cos xT}{x} g(x) dx \\ &= 2 \int_0^A (1 - \cos xT) \{g'(0) + (x/2)g''(0) + \dots\} dx = O(A) \rightarrow 0. \end{aligned}$$

Summing up,

$$\lim_{T \rightarrow \infty} \int_{-E}^{\infty} \frac{1 - \cos xT}{x} f(x+E) dx = \lim_{\Delta \rightarrow 0} \left(\int_{-E}^{-\Delta} + \int_{\Delta}^{\infty} \right) \frac{f(x+E)}{x} dx = \int_{-E}^{\infty} P \frac{1}{x} f(x+E) dx.$$

We obtain, finally,

$$\begin{aligned} \lim_{T \rightarrow \infty} \int_{-E}^{\infty} \frac{1 - e^{i(E-E_b)T}}{E-E_b} f(E_b) dE_b &= \int_{-E}^{\infty} P \frac{1}{x} f(x+E) dx - i\pi f(E) \\ &= \int_0^{\infty} \left\{ P \frac{1}{E-E_b} - i\pi \delta(E-E_b) \right\} f(E_b) dE_b, \end{aligned}$$

which is the content of the formula (8).

For $E=0$, the second integral in (I. 2) drops, and the first integral becomes \int_0^A instead of \int_{-A}^A .

Therefore the even part of $f(\cdot + E)$ contributes to the integral,

$$\int_0^A \frac{1 - \cos xT}{x} f(x+E) dx \sim f(E) \int_0^A \frac{1 - \cos xT}{x} dx = f(E) \int_0^{AT} \frac{1 - \cos y}{y} dy \rightarrow \infty.$$

Therefore, the formula fails for $E=0$.

The second term of the equation (5), namely, $e^{i(E-H_0)T} \Psi = \int e^{i(E-E_b)T} \langle b | \Psi \rangle db$ may seem to vanish in the limit $T \rightarrow \infty$, if we apply the same argument as above. However, this relates with the normalization of Ψ ; we can replace it by the component of Ψ on the energy shell E , since for this component $E-H_0$ vanishes, and the exponential factor does not oscillate.

Appendix II

To prove the unitary relations (23), we write the n -th term of $U(t, -\infty; \epsilon)$, given by (19), as

$$\langle b | U_n(t, -\infty; \epsilon) | a \rangle = e^{-i(E_a - E_b + n\epsilon)t} \frac{H_{n, n-1}}{E_a - E_b + n\epsilon} \frac{H_{n-1, n-2}}{E_a - E_{n-1} + (n-1)\epsilon} \dots \frac{H_{1, 0}}{E_a - E_1 + i\epsilon}, \quad (\text{II. 1})$$

where $H_{n, n-1} = \langle b | H | n-1 \rangle$ etc., and where we have omitted the integral notations over intermediate states, indicated by the twice appearing indices of H . From (II. 1),

$$\langle a | U_n^\dagger(t, -\infty; \epsilon) | b \rangle = e^{-i(E_b - E_a + n\epsilon)t} \frac{H_{n, 1}}{E_a - E_1 - i\epsilon} \frac{H_{1, 2}}{E_a - E_2 - 2i\epsilon} \dots \frac{H_{n-1, n}}{E_a - E_b - n\epsilon}. \quad (\text{II. 2})$$

We expand $U^\dagger(t, -\infty; \epsilon)U(t, -\infty; \epsilon)$ and $U(t, -\infty; \epsilon)U^\dagger(t, -\infty; \epsilon)$ in powers of H , and write the set of terms of the n -th degree in H , as A_n and B_n , respectively, we obtain

$$\langle b | A_0 | a \rangle = \langle b | a \rangle,$$

and for $n \geq 1$,

$$\langle b | A_n | a \rangle = \sum_{k=0}^n \int \langle b | U_k^\dagger(t, -\infty; \epsilon) | c \rangle \langle c | U_{n-k}(t, -\infty; \epsilon) | a \rangle dc.$$

Substituting (II. 1) and (II. 2) into this equation, and changing indices of the integral variables appropriately, we get

$$\begin{aligned} \langle b | A_n | a \rangle &= e^{-i(E_a - E_b + n\epsilon)t} H_{n, n-1} H_{n-1, n-2} \dots H_{1, 0} \\ &\sum_{k=0}^n \{ (E_b - E_{n-1} - i\epsilon) (E_b - E_{n-2} - 2i\epsilon) \dots (E_b - E_k - (n-k)i\epsilon) \}^{-1} \\ &\{ (E_a - E_k + k\epsilon) (E_a - E_{k-1} + (k-1)\epsilon) \dots (E_a - E_1 + i\epsilon) \}^{-1} \\ &(E_n = E_b). \end{aligned}$$

If we put $E_\alpha = F_0$ and $E_k - k\varepsilon = F_k$ ($k=1, 2, \dots, n$), we obtain

$$\sum_{k=0}^n \sum_{l=0}^n \{ (F_n - F_{n-1})(F_n - F_{n-2}) \cdots (F_n - F_k) \}^{-1} \{ (F_0 - F_k)(F_0 - F_{k-1}) \cdots (F_0 - F_1) \}^{-1} = 0,$$

by a simple algebraic relation. Hence the integrand of $A_n(n>1)$ vanishes and also $\langle b|A_n|\alpha\rangle=0$. Therefore

$$\langle b|U^\dagger(t, -\infty; \varepsilon)U(t, -\infty; \varepsilon)|\alpha\rangle = \langle b|\alpha\rangle.$$

We can treat $U(t, -\infty; \varepsilon)U^\dagger(t, -\infty; \varepsilon)$ similarly. In this case, $\langle b|B_0|\alpha\rangle = \langle b|\alpha\rangle$,

and by the same change of indices and the substitution as above, we obtain

$$\langle b|B_n|\alpha\rangle = e^{-i(F_n - F_{n-1} + \dots + F_1)t} H_{n, n-1} H_{n-1, n-2} \cdots H_{1, 0} |\alpha\rangle.$$

$$\sum_{k=0}^n \{ (F_k - F_n)(F_k - F_{n-1}) \cdots (F_k - F_{k+1}) \}^{-1} \{ (F_k - F_{k-1})(F_k - F_{k-2}) \cdots (F_k - F_0) \}^{-1} = 0$$

since $\sum=0$ can be proved by some algebraic manipulation. Hence

$$\langle b|U(t, -\infty; \varepsilon)U^\dagger(t, -\infty; \varepsilon)|\alpha\rangle = \langle b|\alpha\rangle.$$

We can prove (23') similarly.

Note added in proof

The conclusions (13) and (29) concerning $\lim_{t_0 \rightarrow -\infty} U(t, t_0) \Phi_\alpha$ and $\lim_{\varepsilon \rightarrow 0} U(t, -\infty; \varepsilon) \Phi_\alpha$, respectively, which were proved by use of the power expansion of the transformation functions can be easily obtained directly from the integral equation (3). Namely, we employ Fourier integral representation for $\Psi_\alpha(t)$ with respect to t :

$$\Psi_\alpha(t) = \int_{-\infty}^{\infty} d\omega e^{i(H_0 - \omega)t} \Psi_\alpha(\omega). \quad (49)$$

Introducing this expression into (3), and using the formula (8) we obtain in the limit $t_0 \rightarrow -\infty$,

$$\Psi_\alpha(\omega) = -2\pi i \delta_+(H_0 - \omega) H \Psi_\alpha(\omega) + \delta(E_\alpha - \omega) \Psi_\alpha, \quad E_\alpha \neq 0, \quad (50)$$

which expresses the same content as (13) or (12) in view of (49).

For the adiabatic case, we have instead of (3),

$$\Psi_\alpha(t) = \frac{1}{i} \int_{-\infty}^t e^{iH_0 t'} H e^{-iH_0 t' \varepsilon - \varepsilon |t_1|} \Psi(t_1) dt_1 + \Psi_\alpha^*.$$

This yields, from (49),

$$\Psi_\alpha(t) = \begin{cases} \int_{-\infty}^{\infty} e^{i(H_0 - \omega' - i\varepsilon)t} (\omega' - H_0 + i\varepsilon)^{-1} H \Psi(\omega') d\omega' + \Psi_\alpha, & t > 0 \\ \int_{-\infty}^{\infty} [(\omega' - H_0 + i\varepsilon)^{-1} + \{e^{i(H_0 - \omega' + i\varepsilon)t} - 1\}(\omega' - H_0 - i\varepsilon)^{-1}] H \Psi(\omega') d\omega' + \Psi_\alpha, & t < 0. \end{cases} \quad (51)$$

As the inverse relation to (49), we may take

$$\Psi_\alpha(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t' (H_0 - \omega) - \varepsilon' |t|} \Psi_\alpha(t') dt', \quad \varepsilon' > 0, \quad \varepsilon' \rightarrow 0. \quad (52)$$

Introducing (51) into the integrand of (52), carrying out the integrations, and suitably rearranging terms we obtain the equation (40) in the limit $\varepsilon \rightarrow 0$ and $\varepsilon' \rightarrow 0$. Thus the equation (29) is confirmed irrespective of the sign of t .

The above conclusions suggest that the transformation functions $\lim_{t_0 \rightarrow -\infty} U(t, t_0)$ and $\lim_{\varepsilon \rightarrow 0} U(t, -\infty; \varepsilon)$ are essentially equal.

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On the Theory of Cascade Showers, I

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The improved mathematical treatment for the calculation of cascade theory is presented. The solution corresponding to a single incident electron derived by this method is identical with that of Snyder and Scott.

The method is mainly due to the principle of analytic continuation but somewhat different from that of Bhabha-Chakrabarty. It is also applicable to other inhomogeneous integro-differential equations and in particular should be useful for further works on cascade theory. For an example, it is shown the lateral and angular distribution functions are derivable analytically by this method.

§ 1. Introduction

Since the original works on the theory of cascade showers by Bhabha and Heitler¹⁾ and by Carlson and Oppenheimer,²⁾ many contributions have hitherto been published by various authors.³⁾ These cascade theories have become one of the indispensable and the most useful means for the interpretation of cosmic-ray phenomena. It has been required to get more accurate cascade functions for quantitative comparison between theory and the experimental results with increasing accuracy of the experiments. As the historical survey of these developments have been made by many authors,⁴⁾ we will not repeat them here.

Snyder⁵⁾ and Scott⁶⁾ have recently shown that the exact series solution of the diffusion equations of cascade showers can be derived by using the asymptotic forms for the cross section. Since the first terms in their expansions account for the total energy dissipated in the showers, they seem to be useful for the practical applications when the effects of spread of the showers caused by scattering⁷⁾ and the variation of cross sections⁸⁾ with energy of the shower particles can be neglected.

However, their mathematical treatments are somewhat complicated, and we present here the improved mathematical method which can derive the cascade functions identical with those of Snyder⁵⁾ and Scott⁶⁾ more easily. This treatment is mainly due to the principle of the analytic continuation and it is clear that the differences of the solutions between Snyder⁵⁾ and Bhabha-Chakrabarty⁹⁾ come merely from the different analytic continuations used by each authors. The method is also applicable to the problem of the lateral and angular spreads of the cascade showers which has so far scarcely been solved analytically.

The detail features of these distribution functions, the effect of single scattering, comparison with experiments... will be given in the subsequent papers.

§ 2. Series solution including ionization loss

The diffusion equations of the cascade showers are so well known that we will just reproduce them here. Let $\pi(E, t)dE$ and $\gamma(E, t)dE$ be the differential energy spectrum of electrons and photons of cascade showers at the depth t , respectively. Then the equations are, in the notation of Rossi and Greisen,¹⁰⁾

$$\frac{\partial \pi(E, t)}{\partial t} = 2 \int_0^1 \gamma\left(\frac{E}{u}\right) \phi(u) \frac{du}{u} - \int_0^1 \left[\pi(E, t) - \frac{1}{1-v} \pi\left(\frac{E}{1-v}, t\right) \right] \varphi(v) dv + \epsilon \frac{\partial \pi(E, t)}{\partial E} \quad (1)$$

$$\text{and} \quad \frac{\partial \gamma(E, t)}{\partial t} = \int_0^1 \pi\left(\frac{E}{v}, t\right) \varphi(v) \frac{dv}{v} - \sigma_0 \gamma(E, t), \quad (2)$$

where t is measured by radiation unit, ϵ is critical energy, ϕ , φ correspond to the cross section of pair creation, radiation processes, respectively, and σ_0 represents the absorption of photons.

At first, we confine ourselves to the high energy portion of the spectrum, thus the last term of equation (1) can be neglected. Then, the diffusion equations (1) and (2) become homogeneous in E , and can be easily solved exactly by means of Mellin transformation with respect to E .¹¹⁾ The solution corresponding to an initial electron of energy E_0 incident at $t=0$, is given by

$$\begin{aligned} \pi(E, t) &= \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} ds \left(\frac{E_0}{E}\right)^s \frac{1}{E} (H_1(s) e^{\lambda_1(s)t} + H_2(s) e^{\lambda_2(s)t}) \\ &= \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} ds \left(\frac{E_0}{E}\right)^s \frac{1}{E} \phi_0(s, t), \end{aligned} \quad (3)$$

where

$$\begin{aligned} A(s) &= \int_0^1 [1 - (1-v)^s] \varphi(v) dv, \quad B(s) = 2 \int_0^1 u^s \phi(u) du, \quad C(s) = \int_0^1 v^s \varphi(v) dv, \\ \lambda_1(s) &= -\frac{A(s) + \sigma_0}{2} + \frac{1}{2} \{ (A(s) - \sigma_0)^2 + 4B(s)C(s) \}^{\frac{1}{2}}, \\ \lambda_2(s) &= -\frac{A(s) + \sigma_0}{2} - \frac{1}{2} \{ (A(s) - \sigma_0)^2 + 4B(s)C(s) \}^{\frac{1}{2}}, \end{aligned} \quad (4)$$

$$H_1(s) = \frac{\sigma_0 + \lambda_1(s)}{\lambda_1(s) - \lambda_2(s)} \quad \text{and} \quad H_2(s) = \frac{\sigma_0 + \lambda_2(s)}{\lambda_2(s) - \lambda_1(s)}. \quad (5)$$

Next we will consider the case in which ionization loss can not be neglected. Eliminating γ between equations (1) and (2), we get

$$\begin{aligned} \frac{\partial^2 \pi(E, t)}{\partial t^2} + \frac{\partial}{\partial t} \left[\sigma_0 \pi(E, t) + \int_0^1 \left[\pi(E, t) - \frac{1}{1-v} \pi\left(\frac{E}{1-v}, t\right) \right] \varphi(v) dv \right] \\ + \sigma_0 \int_0^1 \left[\pi(E, t) - \frac{1}{1-v} \pi\left(\frac{E}{1-v}, t\right) \right] \varphi(v) dv + 2 \int_0^1 \phi(u) \frac{du}{u} \int_0^1 \pi\left(\frac{E}{uv}\right) \varphi(v) \frac{dv}{v} \\ = \left(\frac{\partial}{\partial t} + \sigma_0 \right) \epsilon \frac{\partial \pi(E, t)}{\partial E}. \end{aligned} \quad (1')$$

Since the formula (3) is the solution of (1) and (2) for $\epsilon=0$, the solution for $\epsilon/E \ll 1$ may be written as follows⁹⁾

$$\pi(E, t) = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} ds \left(\frac{E_0}{E}\right)^s \frac{1}{E} \left(\sum_{n=0}^{\infty} \left(\frac{-\epsilon}{E}\right)^n \phi_n(s, t)\right), \quad (6)$$

where $\phi_n(s, t)$ is defined by

$$\begin{aligned} \left[\frac{\partial^2}{\partial t^2} \{A(s+n) + \sigma_0\} \frac{\partial}{\partial t} + \{A(s+n)\sigma_0 - B(s+n)C(s+n)\} \right] \phi_n(s, t) \\ = \left(\frac{\partial}{\partial t} + \sigma_0 \right) (s+n) \phi_{n-1}(s, t) \end{aligned} \quad (7)$$

with

$$\phi_0(s, t) = H_1(s) e^{\lambda_1(s)t} + H_2 e^{\lambda_2(s)t}. \quad (7')$$

§ 3. Analytic continuation

The series solution (6) does not generally converge for $\epsilon/E > 1$. In order to get the cascade function which converges even for zero energy we first put

$$\sum_{n=0}^{\infty} \left(-\frac{\epsilon}{E}\right)^n \phi_n(s, t) = \lim_{x \rightarrow \infty} \int_0^x e^{-\frac{\epsilon}{E}u} f(u, s, t) du. \quad (8)$$

Since

$$e^{-\frac{\epsilon}{E}} = \sum_{m=0}^{\infty} \frac{1}{\Gamma(m+1)} \left(\frac{-\alpha\epsilon}{E}\right)^m$$

and the integration by terms of the right hand side of the equation (8) is permitted as far as the left hand side is convergent, we get the relation

$$\phi_n(s, t) = \int_0^{\infty} \frac{u^n f(u, s, t)}{\Gamma(n+1)} du. \quad (9)$$

Let $\mathcal{M}(p, s, t)$ be the Mellin integral of $f(u, s, t)$ with respect to u ; i.e.

$$\mathcal{M}(p, s, t) = \int_0^{\infty} u^p f(u, s, t) du \quad (10)$$

and

$$f(u, s, t) = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\mathcal{M}(p, s, t)}{u^{p+1}} dp. \quad (10')$$

Then we get from equation (9)

$$\phi_n(s, t) = \frac{\mathcal{M}(n, s, t)}{\Gamma(n+1)}. \quad (9')$$

Substituting the above relation into equation (8) and changing the orders of integration, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \left(-\frac{\epsilon}{E}\right)^n \phi_n(s, t) &= \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} dp \mathcal{M}(p, s, t) \int_0^{\infty} \frac{e^{-\frac{\epsilon}{E}u}}{u^{p+1}} du \\ &= \frac{1}{2\pi i} \int_{\delta+i\infty}^{\delta-i\infty} dp \left(\frac{\epsilon}{E}\right)^p \mathcal{M}(p, s, t) \Gamma(-p), \end{aligned} \quad (11)$$

where $\mathfrak{M}(p, s, t)$ is defined by,

$$\left[\frac{\partial^2}{\partial t^2} + (A(s+n) + \sigma_0) \frac{\partial}{\partial t} + \{ A(s+n)\sigma_0 - B(s+n)C(s+n) \} \right] \mathfrak{M}(p, s, t) \\ = \left(\frac{\partial}{\partial t} + \sigma_0 \right) (s+p) p \mathfrak{M}(p-1, s, t) \quad (12)$$

$$\text{with} \quad \mathfrak{M}(0, s, t) = H_1(s) e^{\lambda_1(s)t} + H_2(s) e^{\lambda_2(s)t}. \quad (12')$$

If $\epsilon/E \ll 1$, we can evaluate the integral over p in (11) in terms of residues of integrand $p=0, 1, 2, \dots$. This integration gives the same series of the left hand side of (11) as it must be.

Since the relation (11) holds only for the converging domain of the series of (11), the right hand side of (11) may be interpreted as the analytic function obtained by completing the series of (8) following the principle of analytic continuation. In fact it can be proved, by substituting the relation

$$\pi(E, t) = \frac{-1}{4\pi^2} \int_{\delta-i\infty}^{\delta+i\infty} ds \int_{\delta'-i\infty}^{\delta'+i\infty} d\bar{p} \left(\frac{E_0}{E} \right)^s \frac{1}{E} \left(\frac{\epsilon}{E} \right)^p \Gamma(-p) \mathfrak{M}(p, s, t) \quad (13)$$

into (1'), that $\pi(E, t)$ is the solution of (1') for any values of ϵ/E .

For the practical application, one is necessary to evaluate $\mathfrak{M}(p, s, t)$. From eq. (12') and eq. (12), $\mathfrak{M}(p, s, t)$ should be the simple linear function of $e^{\lambda_1(s)t}$, $e^{\lambda_1(s+1)t}$, ..., $e^{\lambda_2(s)t}$, $e^{\lambda_2(s+1)t}$, The value of $\lambda_1(s)$ and $\lambda_2(s)$ are given by formulae (4) and from the character of the function A, B, C , it is easily shown that

$$e^{\lambda_1(s)t} \gg e^{\lambda_2(s)t}, \quad e^{\lambda_1(s)t} \gg e^{\lambda_1(s+1)t}, \quad (14)$$

when t is not so small. Thus we may put $\mathfrak{M}(p, s, t) = \mathfrak{M}(p, s) e^{\lambda_1(s)t}$ and after substitution of this relation into the eq. (12), we get

$$[\lambda_1^2(s) + (A(s+p) + \sigma_0)\lambda_1(s) + \{ A(s+p)\sigma_0 - B(s+p)C(s+p) \}] \mathfrak{M}(p, s) \\ = (\lambda_1(s) + \sigma_0)(s+p) p \mathfrak{M}(p-1, s) \quad (15)$$

with

$$\mathfrak{M}(0, s) = H_1(s).$$

The cascade function with $\mathfrak{M}(p, s, t)$ defined by the above relation is identical with that obtained by Snyder.¹⁰⁾ Furthermore one can easily prove, if we remain smaller terms, i.e. $e^{\lambda_1(s+1)t}$, ..., $e^{\lambda_2(s)t}$, ..., appearing in $\mathfrak{M}(p, s, t)$, that the function (13) agrees with the solutions given by Snyder⁵⁾ and Scott.⁶⁾

Bhabha-Chakrabarty⁹⁾ derived series solutions from (9) following the principle of analytic continuation. Thus it may be interpreted that the differences between the solution of Bhabha-Chakrabarty and that of Snyder and Scott arise merely from the different analytic continuation used by each authors.

§ 4. Distribution function of the cascade showeres

We have so far considered only the longitudinal development of showers. The shower particles, however, spread away from the shower axis due to multiple Coulomb scattering in the matter particles traversed.

This problem of lateral and angular spreads of shower particles is so important for the interpretation of large air showers that many authors¹²⁾ have studied this subject. These results, however, seem not to be very satisfactory for this purpose, and the distribution functions can be derived analytically by the method described in the previous section.

The diffusion equation of the lateral and angular distribution function were given by Landau,¹³⁾ and they are

$$\frac{\partial \pi}{\partial t} = -A'\pi + B'\gamma + \frac{K^2}{4E^2} \left(\frac{\partial^2}{\partial \theta_1^2} + \frac{\partial^2}{\partial \theta_2^2} \right) \pi - \left(\theta_1 \frac{\partial}{\partial y_1} - \theta_2 \frac{\partial}{\partial y_2} \right) \pi + \epsilon \frac{\partial \pi}{\partial E} \quad (16)$$

and

$$\frac{\partial \gamma}{\partial t} = C'\pi - \sigma_0 \gamma - \left(\theta_1 \frac{\partial}{\partial y_1} - \theta_2 \frac{\partial}{\partial y_2} \right) \gamma, \quad (17)$$

where

$y_1, y_2, \theta_1, \theta_2$; Lateral and angular deviations of the shower particles from the shower axis.

A', B', C' ; Integral operator corresponding to the radiation and pair creation process. (c. f. (1) and (2)).

$\pi(E_0, E, y_1, y_2, \theta_1, \theta_2)$ $dE dy_1 dy_2 d\theta_1 d\theta_2$; Number of electrons at the depth t in the energy interval $E, E+dE$ with the lateral and angular deviations y_1, y_2 , and θ_1, θ_2 from the axis of the shower initiated by an incident electron having the energy E_0 .

$\gamma dE dy_1 dy_2 d\theta_1 d\theta_2$; Corresponding numbers of photons.

$K = mc^2(4\pi 137)^{\frac{1}{2}} = 21 \text{ MeV}$.

$\frac{K^2}{4E^2} \left(\frac{\partial^2}{\partial \theta_1^2} + \frac{\partial^2}{\partial \theta_2^2} \right), \left(\theta_1 \frac{\partial}{\partial y_1} + \theta_2 \frac{\partial}{\partial y_2} \right)$; Variation of number of shower particles by the spread of a shower caused by scattering.

Multiplying both side of (16) and (17) by $e^{i(x_1 y_1 + x_2 y_2 + \zeta_1 \theta_1 + \zeta_2 \theta_2)}$ and integrate over y_1, y_2, θ_1 and θ_2 from $-\infty$ to $+\infty$, we get

$$\frac{\partial f}{\partial t} = A'f + B'g - \frac{K^2}{4E^2} (\zeta_1^2 + \zeta_2^2) f + \left(x_1 \frac{\partial}{\partial \zeta_1} + x_2 \frac{\partial}{\partial \zeta_2} \right) f + \epsilon \frac{\partial f}{\partial E}, \quad (16')$$

$$\frac{\partial g}{\partial t} = C'f - \sigma_0 g + \left(x_1 \frac{\partial}{\partial \zeta_1} + x_2 \frac{\partial}{\partial \zeta_2} \right) g, \quad (17')$$

where f and g are the Fourier transformation functions of π and γ with respect to y_1, y_2, θ_1 and θ_2 ; i.e.

$$f(x_1, x_2, \zeta_1, \zeta_2) = \frac{1}{4\pi^2} \iiint_{-\infty}^{+\infty} \pi e^{i(x_1 y_1 + x_2 y_2 + \zeta_1 \theta_1 + \zeta_2 \theta_2)} dy_1 dy_2 d\theta_1 d\theta_2, \quad (18)$$

$$g(x_1, x_2, \zeta_1, \zeta_2) = \frac{1}{4\pi^2} \iiint_{-\infty}^{+\infty} \gamma e^{i(x_1 y_1 + x_2 y_2 + \zeta_1 \theta_1 + \zeta_2 \theta_2)} dy_1 dy_2 d\theta_1 d\theta_2. \quad (19)$$

Elimination of g yields

$$\begin{aligned} & \left\{ \frac{\partial}{\partial t} - \left(x_1 \frac{\partial}{\partial \zeta_1} + x_2 \frac{\partial}{\partial \zeta_2} \right) + \sigma_0 \right\} \left\{ \frac{\partial}{\partial t} - \left(x_1 \frac{\partial}{\partial \zeta_1} + x_2 \frac{\partial}{\partial \zeta_2} \right) \right\} f \\ &= - \left\{ \frac{\partial}{\partial t} - \left(x_1 \frac{\partial}{\partial \zeta_1} + x_2 \frac{\partial}{\partial \zeta_2} \right) + \sigma_0 \right\} \left\{ A' + \frac{K^2}{4E^2} (\zeta_1^2 + \zeta_2^2) - \epsilon \frac{\partial}{\partial E} \right\} f + B' C' f. \end{aligned} \quad (20)$$

For the comparison with experiments, the lateral or angular distribution functions are only important, i.e. the quantities

$$\pi_1(E, \theta_1, \theta_2) = \iint_{-\infty}^{+\infty} \pi dy_1 dy_2, \quad \pi_2(E, y_1, y_2) = \iint_{-\infty}^{+\infty} \pi d\theta_1 d\theta_2. \quad (21)$$

Thus the Fourier transformation functions of these functions are

$$f_1(\zeta_1, \zeta_2) = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \pi_1 e^{i(\zeta_1 \theta_1 + \zeta_2 \theta_2)} d\theta_1 d\theta_2 = 2\pi f(0, 0, \zeta_1, \zeta_2), \quad (22)$$

$$f_2(x_1, x_2) = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \pi_2 e^{i(x_1 y_1 + x_2 y_2)} dy_1 dy_2 = 2\pi f(x_1, x_2, 0, 0). \quad (23)$$

In order to get these functions from eqs. (16') and (17'), we first compute the solution for $K/E \ll 1$ and $\epsilon/E \ll 1$.

$$f_1 = \frac{1}{4\pi^2 i} \int_{\delta-i\infty}^{\delta+i\infty} \left(\frac{E_0}{E} \right)^s \frac{ds}{E} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ -\frac{K^2(\zeta_1^2 + \zeta_2^2)}{4E^2} \right\}^m \left(-\frac{\epsilon}{E} \right)^n \psi_{mn}(s, t), \quad (24)$$

where $\psi_{mn}(s, t)$ is defined by the equation

$$\begin{aligned} & \left[\frac{\partial^2}{\partial t^2} + \{ A(s+2m+n) + \sigma_0 \} \frac{\partial}{\partial t} + \{ A(s+2m+n)\sigma_0 - B(s+2m+n)C(s+2m+n) \} \right] \psi_{mn} \\ &= \left(\frac{\partial}{\partial t} + \sigma_0 \right) (\psi_{m-1, n} + (s+2m+n)\psi_{m, [n-1]}) \end{aligned} \quad (25)$$

with

$$\psi_{00} = H_1(s) e^{\lambda_1(s)t} + H_2(s) e^{\lambda_2(s)t}.$$

As shown in previous section, we put

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ -\frac{K^2(\zeta_1^2 + \zeta_2^2)}{4E^2} \right\}^m \left(-\frac{\epsilon}{E} \right)^n \psi_{mn}(s, t) = \iint_0^{\infty} e^{-\alpha \frac{K^2(\zeta_1^2 + \zeta_2^2)}{4E^2} - \frac{\epsilon}{E} \beta} F(u, \beta, s, t) du d\beta. \quad (26)$$

Let the Mellin integral of $F(u, \beta, s, t)$ be $\mathfrak{M}(p, q, s, t)$; i.e.

$$\mathfrak{M}(p, q, s, t) = \iint_0^{\infty} u^p \beta^q F(u, \beta, s, t) du d\beta$$

and

$$F(\alpha, \beta, s, t) = -\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} d\beta d\eta \frac{\mathfrak{M}(\beta, \eta, s, t)}{\alpha^{\beta+1} \beta^{\eta+1}}.$$

If we integrate by terms of the right hand side of eq. (26), we get

$$\phi_{pq} = \frac{\mathfrak{M}(\beta, \eta, s, t)}{\Gamma(\beta+1) \Gamma(\eta+1)}.$$

Then, we obtain from (25),

$$\begin{aligned} & \left[\frac{\partial^2}{\partial t^2} + \{A(s+2\beta+q) + \sigma_0\} \frac{\partial}{\partial t} + \{A(s+2\beta+q)\sigma_0 - B(s+2\beta+q) \right. \\ & \quad \left. C(s+2\beta+q)\} \right] \mathfrak{M}(\beta, \eta, s, t) \\ & = \left(\frac{\partial}{\partial t} + \sigma_0 \right) \{ \beta \mathfrak{M}(\beta-1, \eta, s, t) + (s+2\beta+q) \eta \mathfrak{M}(\beta, \eta-1, s, t) \} \end{aligned} \quad (27)$$

with

$$\mathfrak{M}(0, 0, s, t) = H_1(s) e^{\lambda_1(s)t} + H_2(s) e^{\lambda_2(s)t}$$

and from (24)

$$\begin{aligned} f_1 &= -\frac{1!}{16\pi^4 i} \iiint_{-\infty}^{+\infty} ds d\beta d\eta \int_0^\infty du d\beta \left(\frac{E_0}{E} \right)^s \frac{1}{E} \frac{\mathfrak{M}(\beta, \eta, s, t)}{\alpha^{\beta+1} \beta^{\eta+1}} e^{-\alpha \frac{K^2(\zeta_1^2 + \zeta_2^2)}{4E^2} - \alpha \frac{\epsilon}{E}} \\ &= -\frac{1}{16\pi^4 i} \iiint_{-\infty}^{+\infty} ds d\beta d\eta \int_0^\infty du \left(\frac{E_0}{E} \right)^s \frac{1}{E} \frac{\mathfrak{M}(\beta, \eta, s, t)}{\alpha^{\beta+1}} \left(\frac{\epsilon}{E} \right)^\eta \Gamma(-\eta) e^{-\alpha \frac{K^2(\zeta_1^2 + \zeta_2^2)}{4E^2}} du. \end{aligned}$$

By the inverse Fourier transformation

$$\begin{aligned} \pi_1 &= \frac{1}{2\pi} \iint_{-\infty}^{+\infty} f_1 e^{-i(\zeta_1 \theta_1 + \zeta_2 \theta_2)} d\zeta_1 d\zeta_2 \\ &= -\frac{1}{8\pi^4 i} \iiint_{-\infty}^{+\infty} ds d\beta d\eta \left(\frac{E_0}{E} \right)^s \frac{1}{E} \left(\frac{E}{K} \right)^2 \left(\frac{\epsilon}{E} \right)^\eta \left(\frac{E^2 \theta^2}{K^2} \right)^{-\eta-1} \\ & \quad \Gamma(\beta+1) \Gamma(-\eta) \mathfrak{M}(\beta, \eta, s, t), \end{aligned} \quad (28)$$

where $\theta^2 = \theta_1^2 + \theta_2^2$.

For the comparison with experiments, the total angular distribution function is important, and is given by

$$\begin{aligned} \Pi_1(0, \theta, t) &= \lim_{\delta \rightarrow 0} \int_0^\infty \pi_1(E, \theta, t) dE \\ &= -\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} ds d\beta \left(\frac{E_0}{E} \right)^s \left(\frac{\epsilon}{K} \right) \left(\frac{E^2 \theta^2}{K^2} \right)^{-\eta-1} \\ & \quad \Gamma(\beta+1) \Gamma(s+2\beta) \mathfrak{M}(\beta, -2\beta-s, s, t). \end{aligned} \quad (29)$$

For the numerical calculation, one should evaluate Π_1 by saddle point method using $\mathfrak{M}(\beta, -2\beta-s, s, t)$ defined by equation (27). The analogous expressions could have

also been derived for $\pi_2(E, r, t)$, $\Pi_2(E, r, t)$, $\gamma_1(F, \theta, t)$ and $\gamma_2(E, r, t)$ in the similar way. Some characters of the distribution functions thus obtained are already described in our previous letter.¹⁴⁾

For the check of our calculations, we compute the differential energy spectrum of electrons from equation (29), and compared it with that obtained by Snyder.⁵⁾

As

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_{\delta}^{\infty} \pi_1 2\pi\theta d\theta &= \lim_{\delta \rightarrow 0} \left\{ -\frac{1}{8\pi^2 i} \iint_{-\infty}^{+\infty} ds d\dot{p} dq \frac{1}{\dot{p}} \left(\frac{E_0}{E} \right)^s \left(\frac{\epsilon}{E} \right)^q \left(\frac{E^2 \dot{\alpha}^2}{K^2} \right)^{-p} \right. \\ &\quad \times \Gamma(\dot{p}+1) \Gamma(-q) \mathfrak{M}(\dot{p}, q, s, t) \\ &= -\frac{1}{4\pi^2} \iint_{-\infty}^{+\infty} ds dq \left(\frac{E_0}{E} \right)^s \frac{1}{E} \left(\frac{\epsilon}{E} \right)^q \Gamma(-q) \mathfrak{M}(0, q, s, t) \end{aligned} \quad (30)$$

and

$$\mathfrak{M}(\dot{p}, s, t) = \mathfrak{M}(0, q, s, t).$$

(30) coincide with (13), as it must be.

§ 5. Conclusion

The solution of Snyder and Scott are derived by the improved mathematical treatment more easily than ever. This method is also applied to the lateral and angular distributions of cascade showers, and they can be computed analytically in approximation B. The detail characters of these functions and some applications of them to cosmic ray phenomena will be reported in the subsequent papers.

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Theory of Super-Quantization of Quantized Field and Its Applications

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A new treatment of quantum field theory is proposed by applying the theory of super-quantization of quantized field. For examples of its application, it is shown that the symmetrical treatment of the four potentials of electromagnetic field is justified without introduction of the indefinite metric for the scalar photon, and the physical interpretation of the regulator theory of Bose field may be given.

§ 1. Introduction

The theory of super-quantization of quantized field has been proposed by the author.¹⁾ In this paper, we shall present the theory in a little more refined form. A main idea of the theory consists in a super-quantization of the state vectors of a quantized field by which a quantized field is treated as something like an elementary particle having infinite degrees of freedom. What relation between the ordinary theory and the present theory may be expected? This relation may be explained in a following simple model. Let us imagine a quantum field as a motion picture projected on a screen in which we see an assembly of elementary particles of the field in motion. We know that an original film of a motion picture consists of an assembly of the separate cuts of film. In contrast with the ordinary quantized field likened to a projected picture on a screen, we may imagine that the super-quantized field corresponds to the separate cuts of the film and a motion of elementary particles of the field will be given by a sequence of cuts of the film in this model.

Both of Fermi and Bose super-quantization might be applied to the state vectors of a quantized field obeying the Fermi statistics or the Bose statistics in an ordinary sense. A faithful correspondence between the ordinary field theory and the super-quantized field theory, however, seems to be obtained only in the case of Fermi quantization of the field state vectors.

It will be seen that the theory of super-quantization is redundant for the Fermi field and the positive energy Bose field, but an application of the theory to a negative energy Bose field will offer a justification for an introduction of a negative energy Bose field into the quantum field theory.

We shall propose a new treatment of quantum electrodynamics in which the symmetrical treatment of the four potentials of electromagnetic field is justified by applying this theory, but without introduction of the indefinite metric for the scalar potential used by several authors.²⁾

An application to the regulator theory of Bose field also is treated. The well-known difficulties of negative energy Bose field accompanied by the theory of regulator will be solved. We see that the anti-field of the negative energy Bose field can be defined and it is expected to become a observable positive energy field just like the positron corresponding to the negative energy electron. In this point, we hope that the theory may play some positive roles in the future theory.

§ 2. Theory of super-quantization of field

1. Introductory remarks

For a sake of simplicity, we take a real scalar Bose field. The field equation and the commutation relation is given in usual manner by

$$\square\varphi - x^2\varphi = 0 \quad (1)$$

and

$$[\varphi(x), \varphi(x')] = i\Delta(x - x') \quad (2)$$

respectively. The notations have usual meanings. We assume that the field is enclosed in a box of unit volume with periodic boundary conditions, and expand the φ as follows:

$$\begin{aligned} \varphi(x) = \sum_k (2k_0)^{-\frac{1}{2}} \{ & C(k) \exp[i(k \cdot x - k_0 t)] \\ & + C^*(k) \exp[-i(k \cdot x - k_0 t)] \}. \end{aligned} \quad (3)$$

For the commutation relations of the $C(k)$ and $C^*(k)$, we obtain

$$[C(k), C^*(k')] = \delta_{k, k'}, \quad (4)$$

while the other commutators $[C(k), C(k')]$, $[C^*(k), C^*(k')]$ vanish. The Lagrangian function and the Hamiltonian density is

$$L = \frac{1}{2} \left\{ \left(\frac{\partial \varphi}{\partial x_\mu} \right)^2 + x^2 \varphi^2 \right\} \quad (5)$$

and

$$H = \frac{1}{2} \left\{ \left(\frac{\partial \varphi}{\partial t} \right)^2 + \left(\frac{\partial \varphi}{\partial x_i} \right)^2 + x^2 \varphi^2 \right\} \quad (6)$$

respectively, where $i=1, 2, 3$ and $\mu=1, 2, 3, 4$.

We obtain for the Hamiltonian omitting the zero point energy

$$\bar{H} = \int H dv = \sum_k k_0 C^*(k) C(k). \quad (7)$$

The quantity:

$$C^*(k) C(k) = n(k) \quad (8)$$

which gives the number of the Bose particles with the momentum k , has the eigen values

0, 1, 2, The constant state vector ϕ of the field in the Heisenberg picture is represented as follows :

$$\phi = \sum_{n(k_1), n(k_2), \dots} \alpha(n(k_1), n(k_2), \dots) \phi_{n(k_1), n(k_2), \dots}(C^*(k_1), C^*(k_2), \dots), \quad (9)$$

where the $\phi_{n(k_1), n(k_2), \dots}$ is the eigenfunction for a state containing $n(k_1), n(k_2), \dots$ particles with the momentum k_1, k_2, \dots respectively. The explicit form of the ϕ and its Hermite conjugate ϕ^* are given by

$$\phi_{n(k)}(C^*(k)) \equiv (n(k_1)! n(k_2)! \dots)^{-\frac{1}{2}} C^*(k)^{n(k_1)} C^*(k_2)^{n(k_2)} \dots, \quad (10)$$

and

$$\phi_{n(k)}^*(C(k)) \equiv (n(k_1)! n(k_2)! \dots)^{-\frac{1}{2}} C(k_1)^{n(k_1)} C(k_2)^{n(k_2)} \dots, \quad (11)$$

where for a brevity, the abbreviated notations

$$n(k) \equiv (n(k_1), n(k_2), \dots)$$

and

$$C^*(k) \equiv (C^*(k_1), C^*(k_2), \dots)$$

are used. The eigenfunction ϕ 's are normalized and orthogonal, for

$$\begin{aligned} (\phi_{m(k)}, \phi_{n(k)}) &= \int \phi_{m(k_1), m(k_2), \dots}^* \phi_{n(k_1), n(k_2), \dots} dC^*(k_1) dC^*(k_2) \dots \\ &= \delta_{m(k_1), n(k_1)} \delta_{m(k_2), n(k_2)} \dots \end{aligned} \quad (12)$$

The operators $C(k)$ and $C^*(k)$ have the usual meaning of an absorption and an emission operator for a particle with a momentum k .

2. Super-quantization

Here, we introduce the super-quantization of the quantized field which is of an analogy to Fock's second quantization method.³⁾

The q -number state vector ϕ is defined by replacing the amplitude $\alpha(n(k))$ with the quantized amplitude $\mathbf{a}(n(k))$ in the equation (9).

The quantized amplitude \mathbf{a} may be defined by the commutation relation

$$[\mathbf{a}(m(k_1), m(k_2), \dots), \mathbf{a}^*(n(k_1), n(k_2), \dots))] = \delta_{m(k_1), n(k_1)} \delta_{m(k_2), n(k_2)} \dots \quad (13)$$

and other commutators vanish, or for the exclusion quantization,

$$\{\mathbf{a}(m(k_1), m(k_2), \dots), \mathbf{a}^*(n(k_1), n(k_2), \dots)\} = \delta_{m(k_1), n(k_1)} \delta_{m(k_2), n(k_2)} \dots \quad (14)^*$$

and other commutators

$$\{\mathbf{a}(m(k)), \mathbf{a}(n(k))\}, \{\mathbf{a}^*(m(k)), \mathbf{a}^*(n(k))\}$$

vanish.

From the reason mentioned in the introduction, we consider only the case of the exclusion commutation relation which is defined by (14).

Then, the quantized state vectors are written as

* The commutator means $\{A, B\} = AB + BA$.

$$\phi(C^*(\mathbf{k})) = \sum_{n(\mathbf{k}_1), n(\mathbf{k}_2), \dots} \mathbf{a}(n(\mathbf{k}_1), n(\mathbf{k}_2), \dots) \phi_{n(\mathbf{k}_1), n(\mathbf{k}_2), \dots}(C^*(\mathbf{k}_1), C^*(\mathbf{k}_2), \dots), \quad (15)$$

and its Hermitian conjugate

$$\phi^*(C^*(\mathbf{k})) = \sum_{n(\mathbf{k}_1), n(\mathbf{k}_2), \dots} \mathbf{a}^*(n(\mathbf{k}_1), n(\mathbf{k}_2), \dots) \phi_{n(\mathbf{k}_1), n(\mathbf{k}_2), \dots}^*(C^*(\mathbf{k}_1), C^*(\mathbf{k}_2), \dots). \quad (15')$$

From the relations (14) and (12), the commutation relations

$$\begin{aligned} \{ \phi(C^*(\mathbf{k}_1), C^*(\mathbf{k}_2), \dots), \phi^*(C^*(\mathbf{k}_1)', C^*(\mathbf{k}_2)', \dots) \} \\ = \delta(C^*(\mathbf{k}_1) - C^*(\mathbf{k}_1)') \delta(C^*(\mathbf{k}_2) - C^*(\mathbf{k}_2)') \dots \end{aligned} \quad (16)$$

are obtained. This relations are time-independent and relativistic invariant, for the above mentioned definitions of ϕ .

In general, the super-quantized field operator \mathbf{O} is defined from the corresponding ordinary field operator O by

$$\mathbf{O} = (\phi, O\phi) = \sum_{m(\mathbf{k})} \sum_{n(\mathbf{k})} \mathbf{a}^*(m(\mathbf{k})) (m(\mathbf{k}) | O | n(\mathbf{k})) \mathbf{a}(n(\mathbf{k})), \quad (17)$$

where the abbreviation as mentioned before

$$m(\mathbf{k}) \equiv (m(\mathbf{k}_1), m(\mathbf{k}_2), \dots) \text{ and } n(\mathbf{k}) \equiv (n(\mathbf{k}_1), n(\mathbf{k}_2), \dots)$$

are used, and the $(m(\mathbf{k}) | O | n(\mathbf{k}))$ is a matrix element of the ordinary field operator given by

$$(m(\mathbf{k}) | O | n(\mathbf{k})) = (\phi_{m(\mathbf{k}_1), m(\mathbf{k}_2), \dots}, O \phi_{n(\mathbf{k}_1), n(\mathbf{k}_2), \dots}). \quad (18)$$

For examples, the super-quantized operators corresponding to the absorption and emission operators $C(\mathbf{k}_i)$ and $C^*(\mathbf{k}_i)$ may be defined as follows:

$$\begin{aligned} C(\mathbf{k}_i) = \sum_{n(\mathbf{k}_1), n(\mathbf{k}_2), \dots} n(\mathbf{k}_i)^{\frac{1}{2}} \mathbf{a}^*(n(\mathbf{k}_1), n(\mathbf{k}_2), \dots, n(\mathbf{k}_i) - 1, \dots) \\ \times \mathbf{a}(n(\mathbf{k}_1), n(\mathbf{k}_2), \dots, n(\mathbf{k}_i), \dots), \end{aligned} \quad (19)$$

and

$$\begin{aligned} C^*(\mathbf{k}_i) = \sum_{n(\mathbf{k}_1), n(\mathbf{k}_2), \dots} (n(\mathbf{k}_i) + 1)^{\frac{1}{2}} \mathbf{a}^*(n(\mathbf{k}_1), n(\mathbf{k}_2), \dots, n(\mathbf{k}_i) + 1, \dots) \\ \times \mathbf{a}(n(\mathbf{k}_1), n(\mathbf{k}_2), \dots, n(\mathbf{k}_i), \dots). \end{aligned} \quad (18')$$

From (19) and the commutation relation (14), we have

$$[C(\mathbf{k}_i), C^*(\mathbf{k}_j)] = \delta_{\mathbf{k}_i, \mathbf{k}_j} \sum_{n(\mathbf{k})} \mathbf{a}^*(n(\mathbf{k})) \mathbf{a}(n(\mathbf{k})), \quad (20)$$

and

$$\text{Other commutators} = 0.$$

The following relation, therefore, is easily obtained,

$$[\varphi(x), \varphi(x')] = i\Delta(x-x')(\phi, \phi), \quad (21)$$

where

$$\varphi(x) = (\phi, \varphi(x)\phi) \quad (22)$$

and

$$(\phi, \phi) = \sum_{n(\mathbf{k})} \mathbf{a}^*(n(\mathbf{k})) \mathbf{a}(n(\mathbf{k})). \quad (23)$$

The operator (23) has a simple meaning such that it gives an operator of total numbers of the fields in the assembly of the identical fields and each $\mathbf{a}^*(n(\mathbf{k}_1), n(\mathbf{k}_2), \dots) \times \mathbf{a}(n(\mathbf{k}_1), n(\mathbf{k}_2), \dots)$ gives the numbers of the field to which the given distribution of particles $n(\mathbf{k}_1), n(\mathbf{k}_2), \dots$ belongs. According to the exclusion quantization (14), the eigenvalue of the operator $\mathbf{a}^*(n(\mathbf{k})) \mathbf{a}(n(\mathbf{k}))$ is zero or one. In other words, there does not exist more than one field which has the same distribution of the particles at the same time in the assembly.

The total Hamiltonian operator of the field assembly is given by

$$\bar{H} = (\phi, \bar{H}\phi) = \sum_{n(\mathbf{k}_1), n(\mathbf{k}_2), \dots} (k_0(\mathbf{k}_1)n(\mathbf{k}_1) + k_0(\mathbf{k}_2)n(\mathbf{k}_2) + \dots) N(n(\mathbf{k}_1), n(\mathbf{k}_2), \dots) \quad (24)$$

and

$$N(n(\mathbf{k}_1), n(\mathbf{k}_2), \dots) = \mathbf{a}^*(n(\mathbf{k}_1), n(\mathbf{k}_2), \dots) \mathbf{a}(n(\mathbf{k}_1), n(\mathbf{k}_2), \dots) \quad (25)$$

whose eigenvalue is zero or one.

The operator $C(\mathbf{k}_i)$ and $C^*(\mathbf{k}_i)$ are the corresponding one of an absorption and an emission of the particle associated with the momentum \mathbf{k}_i , because the $C(\mathbf{k}_i)$ gives a transition such that one field having the particle distribution $n(\mathbf{k}_1), n(\mathbf{k}_2), \dots, n(\mathbf{k}_i), \dots$ is destroyed and another field having the particle distribution $n(\mathbf{k}_1), \dots, n(\mathbf{k}_i) - 1, \dots$ is created, then the numbers of particles with the momentum \mathbf{k}_i belonging to the field decrease by one, and for the $C^*(\mathbf{k}_i)$, vice versa.

We denote an eigen state vector of the super-quantized field by

$$\Psi[N'(n'(\mathbf{k}_1), n'(\mathbf{k}_2), \dots), N''(n''(\mathbf{k}_1), n''(\mathbf{k}_2), \dots) \dots]$$

which belongs to an eigenvalue of $N(n(\mathbf{k}))$ having N' for number of the field of particles distribution $n'(\mathbf{k}_1), n'(\mathbf{k}_2), \dots$ and N'' for numbers of the field of particles distribution $n''(\mathbf{k}_1), n''(\mathbf{k}_2), \dots$ etc..

For this eigenstate, we have

$$\begin{aligned} \bar{H}\Psi[N', N'', \dots] = & \{ (k_0(\mathbf{k}_1)n'(\mathbf{k}_1) + k_0(\mathbf{k}_2)n'(\mathbf{k}_2) + \dots) N' \\ & + (k_0(\mathbf{k}_1)n''(\mathbf{k}_1) + k_0(\mathbf{k}_2)n''(\mathbf{k}_2) + \dots) N'' \\ & + \dots \} \Psi[N', N'', \dots] \end{aligned}$$

where

$$N', N'', \dots = 0 \text{ or } 1.$$

We shall easily see that the ordinary description will be given in terms of the super-quantized field description. If the special assembly of fields contains only one field, i.e., the number N' of the field which has a definite particle distribution $n'(\mathbf{k}_1)$, $n'(\mathbf{k}_2)$, ... is one, all of other number N'' 's are zero, this special assembly of fields is originally equivalent to the ordinary quantized field.

3. Interaction with other field

Even if we introduce an interaction of the field φ with any other field denoted by $A(x)$, whose special properties are not needed here, we can see that the above equivalence between the super-quantized theory and the ordinary theory is conserved.

We denote the interaction energy density

$$H'(x) = g\varphi(x)A(x).$$

If the corresponding super-quantized operator is defined as follows:

$$\mathbf{H}'(x) = g\varphi(x)A(x) \quad (26)$$

and

$$\varphi(x) = (\psi, \varphi(x)\psi),$$

the numbers of the field φ will not be changed by this interaction, because the operator φ has a bilinear form of the field absorption and emission operator α , α^* , then in any change of the field assembly state, the total numbers of the field φ are conserved, i.e.,

$$\sum N'(n'(\mathbf{k}_1), n'(\mathbf{k}_2), \dots) = \text{const.}$$

We start from the initial condition at which we have only one field, then the above mentioned equivalence will be easily proved.

In the interaction representation of the super-quantized theory of the field φ , the equation of motion may be given by

$$i \frac{\delta \Psi[\sigma]}{\delta \sigma(x)} = \mathbf{H}'(x) \Psi[\sigma], \quad (27)$$

where \mathbf{H}' is defined by (26). The state vector $\Psi[\sigma]$ in the equation (27) does not refer to the particle assembly state of the field φ , but the φ field assembly state.

In general, the total numbers of the fields in the state $\Psi[\sigma]$ are defined by

$$\sum_{n'(\mathbf{k})} N(n'(\mathbf{k})) \Psi[\sigma] = \sum_{n'(\mathbf{k})} N'(n'(\mathbf{k})) \Psi[\sigma], \quad (28)$$

then if the initial condition

$$\sum_{n'(\mathbf{k})} N'(n'(\mathbf{k})) = 1 \quad (19)$$

is given, we can prove that

$$[C(\mathbf{k}_i), C^*(\mathbf{k}_j)] \Psi[\sigma] = \delta_{\mathbf{k}_i, \mathbf{k}_j} \Psi[\sigma],$$

and

$$[\varphi(x), \varphi(x')] \Psi[\sigma] = i\Delta(x-x') \Psi[\sigma].$$

We see, therefore, that in this case, the super-quantized operator $\varphi(x)$ has a meaning as equivalent as that of the ordinary operator $\varphi(x)$ and the state vector $\Psi[\sigma]$ describing a change of the special assembly state which includes only one field just corresponds to the ordinary state vector $\psi[\sigma]$ describing a change of the state of the field in terms of the particle state.

The expectation value of any operator O on a space-like surface will be defined as

$$\langle O \rangle = (\Psi[\sigma], (\phi, 0\phi)\Psi[\sigma]). \quad (30)$$

This expectation value will have a same value as the corresponding one in Tomonaga-Schwinger's theory, if the field assembly state $\Psi[\sigma]$'s are restricted to one defined by (28) and (29). For an example, we can prove

$$\begin{aligned} \langle [\varphi(x), \varphi(x')] \rangle &= i\Delta(x-x') \langle (\phi, \phi) \rangle \\ &= i\Delta(x-x'). \end{aligned}$$

Following to the equation of motion (37), the change of a state is described in manners of a destruction of one field and a subsequent creation of another field. Therefore, we see that an ordinary motion of particle assembly will be described in terms of the transitions between the states of such a special field assembly to which only one field always belongs.

It must be noted that the vacuum in the super-quantized theory is not a zero field state, but a state such that there exists only one vacuum field defined in a sense of the ordinary theory, namely for the vacuum state, we have

$$\begin{aligned} N' &= 1 \quad \text{only for } n'(k_1) = n'(k_2) = \dots = 0, \\ \text{all other } N'' &= 0. \end{aligned}$$

§ 3. Application to quantum electrodynamics

1. Introductory remarks

We follow to Gupta's representation of quantum electrodynamics.⁽⁴⁾ The electromagnetic potentials are represented by

$$A_\mu = e_\mu^{(1)} A^{(1)} + e_\mu^{(2)} A^{(2)} + e_\mu^{(3)} A^{(3)} + e_\mu^{(0)} A^{(0)}. \quad (3)$$

where all $A^{(\alpha)}$'s are scalar and Hermitian, but it should be noted that we take also $A^{(0)}$ to be Hermitian in contrast to Gupta's one of anti-Hermitian $A^{(0)}$, and $e_\mu^{(1)}, e_\mu^{(2)}, e_\mu^{(3)}, e_\mu^{(0)}$ are orthogonal unit four vectors, $e_\mu^{(i)}, i=1, 2, 3$, being space-like and $e_\mu^{(0)}$ time-like. Following to Gupta, the relations

$$e_\mu^{(1)} e_\nu^{(1)} + e_\mu^{(2)} e_\nu^{(2)} + e_\mu^{(3)} e_\nu^{(3)} - e_\mu^{(0)} e_\nu^{(0)} = \delta_{\mu\nu}$$

are satisfied. The commutation relations are given by

$$\begin{aligned} [A^{(i)}(x), A^{(i)}(x')] &= iD(x-x'), \quad i=1, 2, 3, \\ [A^{(0)}(x), A^{(0)}(x')] &= -iD(x-x'). \end{aligned} \quad (32)$$

Each $A^{(\alpha)}$ is expanded as same as (3) respectively,

$$A^{(\alpha)} = \sum_{\mathbf{k}} \sqrt{\frac{2\pi}{k}} \{ C_{\alpha}(\mathbf{k}) \exp [i(\mathbf{k} \cdot \mathbf{r} - kt)] + C_{\alpha}^*(\mathbf{k}) \exp [-i(\mathbf{k} \cdot \mathbf{r} - kt)] \}, \quad (33)$$

where $\alpha = 1, 2, 3, 0$ and C_{α}^* 's are hermite-conjugate to C_{α} 's.

From (32) and (33), the following commutation relations are obtained

$$[C_i(\mathbf{k}), C_i^*(\mathbf{k}')] = \delta_{\mathbf{k}, \mathbf{k}'},$$

and

$$[C_0(\mathbf{k}), C_0^*(\mathbf{k}')] = -\delta_{\mathbf{k}, \mathbf{k}'}. \quad (34)$$

As the usual manner, the Lagrangian density is taken to be

$$L = -\frac{1}{8\pi} \left\{ \left(\frac{\partial A_{\mu}}{\partial x_{\mu}} \right)^2 + \frac{1}{2} \left(\frac{\partial A_{\mu}}{\partial x_{\nu}} - \frac{\partial A_{\nu}}{\partial x_{\mu}} \right)^2 \right\},$$

then, we get the Hamiltonian, omitting the zero-point energy,

$$\bar{H} = \sum_{\mathbf{k}} k \{ C_1^*(\mathbf{k}) C_1(\mathbf{k}) + C_2(\mathbf{k}) C_2(\mathbf{k}) + C_3(\mathbf{k}) C_3(\mathbf{k}) - C_0(\mathbf{k}) C_0^*(\mathbf{k}) \}. \quad (35)$$

We see that the radiation field consists of four scalar fields.

2. Super-quantization of electromagnetic field

Following to § 2, we introduce the quantized state vectors for each $A^{(\alpha)}$:

$$\begin{aligned} \phi_i(C_i^*(\mathbf{k})) = \sum_{n_i(\mathbf{k}_1), n_i(\mathbf{k}_2), \dots} \mathbf{a}_i(n_i(\mathbf{k}_1), n_i(\mathbf{k}_2), \dots) \phi_{n_i(\mathbf{k}_1), n_i(\mathbf{k}_2), \dots} \\ (C_i^*(\mathbf{k}_1), C_i^*(\mathbf{k}_2), \dots), \quad i=1, 2, 3, \end{aligned} \quad (36)$$

and

$$\begin{aligned} \psi_0(C_0(\mathbf{k})) = \sum_{n_0(\mathbf{k}_1), n_0(\mathbf{k}_2), \dots} \mathbf{a}_0(n_0(\mathbf{k}_1), n_0(\mathbf{k}_2), \dots) \phi_{n_0(\mathbf{k}_1), n_0(\mathbf{k}_2), \dots} \\ (C_0(\mathbf{k}_1), C_0(\mathbf{k}_2), \dots), \end{aligned} \quad (36')$$

where $\phi_{n_i(\mathbf{k})}(C_i^*(\mathbf{k}))$ and $\phi_{n_0(\mathbf{k})}(C_0(\mathbf{k}))$ is the eigenfunction for the state containing $n_{\alpha}(\mathbf{k}_1), n_{\alpha}(\mathbf{k}_2), \dots$ α -scalar photons respectively.

The quantized amplitudes are defined by

$$\{ \mathbf{a}_{\alpha}(m_{\alpha}(\mathbf{k})), \mathbf{a}_{\beta}^*(n_{\beta}(\mathbf{k})) \} = \delta_{\alpha\beta} \delta_{m_{\alpha}(\mathbf{k}), n_{\alpha}(\mathbf{k})}. \quad (37)$$

The super-quantized quantities of each α -field are taken to be

$$\mathbf{O}_{\alpha} = (\phi_{\alpha}, \mathbf{O}_{\alpha} \phi_{\alpha}).$$

For examples, we have

$$\begin{aligned} C_i(\mathbf{k}_s) = \sum_{n_i(\mathbf{k}_1), n_i(\mathbf{k}_2), \dots} n_i(\mathbf{k}_s)^{\frac{1}{2}} \mathbf{a}_i^*(n_i(\mathbf{k}_1), n_i(\mathbf{k}_2), \dots, n_i(\mathbf{k}_s) - 1, \dots) \\ \times \mathbf{a}_i(n_i(\mathbf{k}_1), n_i(\mathbf{k}_2), \dots, n_i(\mathbf{k}_s), \dots), \end{aligned}$$

$$C_i^*(\mathbf{k}_s) = \sum_{n_i(\mathbf{k}_1), n_i(\mathbf{k}_2), \dots} (n_i(\mathbf{k}_s) + 1)^{\frac{1}{2}} a_i^*(n_i(\mathbf{k}_1), n_i(\mathbf{k}_2), \dots, n_i(\mathbf{k}_s) + 1, \dots) \\ \times a_i(n_i(\mathbf{k}_1), n_i(\mathbf{k}_2), \dots, n_i(\mathbf{k}_s), \dots), \quad (38)$$

for $i = 1, 2, 3$, and

$$C_0(\mathbf{k}_s) = \sum_{n_0(\mathbf{k}_1), n_0(\mathbf{k}_2), \dots} (n_0(\mathbf{k}_s) + 1)^{\frac{1}{2}} a_0^*(n_0(\mathbf{k}_1), n_0(\mathbf{k}_2), \dots, n_0(\mathbf{k}_s) + 1, \dots) \\ \times a_0(n_0(\mathbf{k}_1), n_0(\mathbf{k}_2), \dots, n_0(\mathbf{k}_s), \dots), \\ C_0^*(\mathbf{k}_s) = \sum_{n_0(\mathbf{k}_1), n_0(\mathbf{k}_2), \dots} n_0(\mathbf{k}_s)^{\frac{1}{2}} a_0^*(n_0(\mathbf{k}_1), n_0(\mathbf{k}_2), \dots, n_0(\mathbf{k}_s) - 1, \dots) \\ \times a_0(n_0(\mathbf{k}_1), n_0(\mathbf{k}_2), \dots, n_0(\mathbf{k}_s), \dots). \quad (39)$$

The modified supplementary conditions given by Gupta⁴⁾ are

$$\frac{\partial A_\mu^+}{\partial x_\mu} \psi = 0, \quad \text{or} \quad \psi^* \frac{\partial A_\mu^-}{\partial x_\mu} = 0$$

where A_μ^+ is a positive frequency part of A_μ . In the super-quantized theory, we assume that the supplementary conditions are given by

$$\frac{\partial A_\mu^+}{\partial x_\mu} \Psi = 0 \quad (40)$$

or

$$\Psi^* \frac{\partial A_\mu^-}{\partial x_\mu} = 0 \quad (40')$$

where

$$A_\mu^\pm = e_\mu^{(1)} A^{\pm(1)} + e_\mu^{(2)} A^{\pm(2)} + e_\mu^{(3)} A^{\pm(3)} + e_\mu^{(0)} A^{\pm(0)}, \quad A^{(\alpha)} = (\phi_\alpha, A^{(\alpha)} \phi_\alpha). \quad (41)$$

If we choose $e_\mu^{(\alpha)}$ to satisfy the following condition for a plane wave component

$$k_\mu e_\mu^{(1)} = k_\mu e_\mu^{(2)} = 0, \quad k_\mu e_\mu^{(3)} = -k_\mu e_\mu^{(0)}, \quad (42)$$

the supplementary conditions (40), (40') will become

$$\{C_3(\mathbf{k}) - C_0(\mathbf{k})\} \Psi = 0 \quad (43)$$

and

$$\Psi^* \{C_3^*(\mathbf{k}) - C_0^*(\mathbf{k})\} = 0. \quad (43')$$

The compatibility of these conditions are easily proved, because we have

$$[C_3(\mathbf{k}') - C_0(\mathbf{k}'), C_3(\mathbf{k}) - C_0(\mathbf{k})] = 0.$$

3. Vacuum definition and modified supplementary conditions

The usual symmetrical treatment of four potentials of electromagnetic field leads the vacuum definition

$$A_{\mu}^{\dagger} \Psi_{vac.} = 0, \quad \mu = 1, 2, 3, 4,$$

or

$$A^{+(\alpha)} \Psi_{vac.} = 0. \quad (44)$$

For each plane wave component, it becomes

$$C_{\alpha}(\mathbf{k}) \Psi_{vac.} = 0. \quad (45)$$

A physical meaning of this condition is as follows: A vacuum is such a state that there exists only one i -th field ($i=1, 2, 3$) which has the zero-particle distribution, i.e.,

$$N_i(n_i(\mathbf{k}_1)=0, n_i(\mathbf{k}_2)=0, \dots)=1, \quad \text{all other } N_i(n_i(\mathbf{k}_1), n_i(\mathbf{k}_2), \dots)=0,$$

where $N_i(n_i(\mathbf{k}_1), n_i(\mathbf{k}_2), \dots)$ is number of the i -th field having a particle distribution $n_i(\mathbf{k}_1), n_i(\mathbf{k}_2), \dots$ in the super-quantized system. But we must take a special care for the 0-th field. From the commutation relation (32) and the Hamiltonian (35), we see that the 0-th field is a negative energy field and the emission and absorption character of $C_0(\mathbf{k})$ and $C_0^*(\mathbf{k})$ is interchanged comparing with the corresponding one of other fields. This was one of the well-known difficulties of the negative energy Bose field in the ordinary theory. But, in the present theory we consider a quantized field assembly which obeys Fermi statistics, we will see that the vacuum condition

$$C_0(\mathbf{k}) \Psi_{vac.} = 0 \quad (46)$$

can be satisfied, if the vacuum for the 0-th field is defined as follows; the vacuum consists of infinite numbers of the 0-th field which occupy all of the states having any particle distribution except the zero particle distribution one, i.e.,

$$N_0(n_0(\mathbf{k}_1)=0, n_0(\mathbf{k}_2)=0, \dots)=0, \quad \text{all other } N_0(n_0(\mathbf{k}_1), n_0(\mathbf{k}_2), \dots)=1.$$

This situation is very similar to the negative energy electron sea in the electron vacuum. The above mentioned vacuum definition for the 0-th field satisfies the condition (46).

We see that the supplementary conditions are satisfied for the simultaneous vacuum state of the 3rd and the 0-th field. We, therefore, replace the supplementary conditions by the stronger one:

$$C_3(\mathbf{k}) \Psi = 0, \quad C_0(\mathbf{k}) \Psi = 0. \quad (47)$$

This means that the pure radiation field is defined as the simultaneous vacuum state of of the 3rd and the 0-th field.

4. Commutation relations and expectation values

From the equations (37) and (31), we obtain for the commutation relations of the super-quantized four potentials

$$[A_{\mu}(x), A_{\nu}(x')] = i \{ e_{\mu}^{(1)} e_{\nu}^{(1)}(\phi_1, \phi_1) + e_{\mu}^{(2)} e_{\nu}^{(2)}(\phi_2, \phi_2) + e_{\mu}^{(3)} e_{\nu}^{(3)}(\phi_3, \phi_3) \\ - e_{\mu}^{(0)} e_{\nu}^{(0)}(\phi_0, \phi_0) \} D(x-x'). \quad (48)$$

For the real radiation field, we must put the conditions:

$$(\phi_1, \phi_1)\Psi = (\phi_2, \phi_2)\Psi = (\phi_3, \phi_3)\Psi = \Psi. \quad (49)$$

This means that number of the i -th field is one. Although the eigen-value of (ϕ_0, ϕ_0) is really infinite as we see from the vacuum definition of the 0-th field, we may normalize this number of the 0-th field as one, namely we define

$$(\phi_0, \phi_0)\Psi = \Psi. \quad (50)$$

We then obtain

$$\langle [A_\mu(x), A_\nu(x')] \rangle = i\delta_{\mu\nu}D(x-x'). \quad (51)$$

This expectation value of the commutation relation has the same value as that of the ordinary theory. The expectation values of other quadratic form can be computed and they also become to be as same as the corresponding one in the ordinary theory.

For an example, we obtain

$$\langle \{A_\mu(x), A_\nu(x')\} \rangle_{\text{vacuum}} = D^{(1)}(x-x'). \quad (52)$$

These results also certify an equivalence of the super-quantized theory to the ordinary quantum electrodynamics. It should be noted that the supplementary conditions are given in the physically simple form of (47) without any inconsistency.

5. Interaction with electrons

The theory of interaction with electrons is given in a straightforward generalization of the ordinary one. In the present theory, the interaction operator of electromagnetic field with electrons is given by

$$H'(x) = -j_\mu(x)A_\mu(x) \quad (53)$$

where j_μ represents the usual operator of electron charge and current, and the super-quantized potentials A_μ defined in (2). The interaction representation gives the equation of motion:

$$i\frac{\partial[\Psi\sigma]}{\partial\sigma(x)} = H'(x)\Psi[\sigma] \quad (54)$$

with the supplementary condition

$$\left[\frac{\partial A_\mu^+(x)}{\partial x_\mu} - \int_\sigma D^+(x-x')j_\mu(x')d\sigma'_\mu \right] \Psi[\sigma] = 0, \quad (55)$$

in which the used notations are the usual one.

As a physical illustration of the theory, we shall consider the calculation of the self-energy of a free electron at rest by means of usual second order perturbational method. In the zero-order approximation, the superquantized electromagnetic field is in the vacuum state as mentioned above. The second order interactions will take place through the following processes:

A. First, the $A^{(i)}$ field being in its own zero-particle state is absorbed, subsequently the $A^{(i)}$ -field having the particle distribution: $(n_i(\mathbf{k}_1)=0, n_i(\mathbf{k}_2)=0, \dots, n_i(\mathbf{k}_s)=1,$

$n_s(\mathbf{k}_{s+1})=0, \dots$), is created and the electron takes the recoil $(-\mathbf{k}_s)$; then this created field is absorbed, subsequently the original vacuum field is again created and the electron comes to rest.

These processes correspond to the ordinary emission and absorption processes of a transverse and a longitudinal photon with a momentum \mathbf{k}_s respectively.

B. First, one of the $A^{(0)}$ -field occupying the state of the particle distribution: $(n_0(\mathbf{k}_1)=0, n_0(\mathbf{k}_2)=0, \dots, n^0(\mathbf{k}_s)=1, n^0(\mathbf{k}_{s+1})=0, \dots)$, originally in the vacuum is absorbed, subsequently the zero-particle distribution $A^{(0)}$ field which was absent in the vacuum state is created. and the electron takes the recoil \mathbf{k}_s ; then the latter field is absorbed, the original field is again created and the electron comes to rest. This corresponds to the absorption and emission process of negative energy scalar photon. From our definition of the vacuum state of $A^{(0)}$ -field, however, the process can be interpreted as the emission and absorption process of an anti-scalar photon with positive energy. It will be seen that all other processes can not contribute to the second-order perturbational calculation, especially for $A^{(0)}$ -field, all processes except B are forbidden according to Pauli's exclusion principle.

In this calculation, we can obtain the well-known self-energy expression of an electron without the ambiguities remarked by several authors.⁵⁾

§ 4. Regulator theory of Bose field

According to Pais and Uhlenbeck,⁶⁾ the generalized Bose field theory—for an example, the generalized quantum electrodynamics⁷⁾—, may be said to be a realization of the regulator type of theory proposed by Pauli and Villars,⁸⁾ but it was shown that the theory is physically inconsistent. This inconsistency of the theory, however, are considered to be only connected with the negative energy Bose field. We could solve this type of difficulty by means of the super-quantization theory and the appropriate vacuum definition for the negative energy Bose field as shown in the preceding section. We shall show that the same procedure will be applicable also to the generalized field theory and then, the regulator theory becomes to be physically an acceptable scheme. We consider the generalized field theory (scalar and neutral). The generalized scalar potential is defined as follows:

$$\varphi = \sum_{k=1}^N \zeta_k \varphi^{(k)} \quad (56)$$

which obeys the field equation

$$\prod_{k=1}^N \left(1 - \frac{\square}{\mu_k^2} \right) \varphi = 0, \quad (57)$$

where \square is the D'Alembertian and μ_k is the mass of the k -th associated field $\varphi^{(k)}$. The associated field $\varphi^{(k)}$ satisfies the equation

$$(\square - \mu_k^2) \varphi^{(k)} = 0, \quad (58)$$

and the commutation relation

$$[\varphi^{(k)}(x), \varphi^{(l)}(x')] = i\delta_{kl} \frac{1}{\zeta_k} \Delta_k(x-x'), \quad (59)$$

where Δ_k is the four dimensional δ -function involving the mass μ_k . The generalized field should be regularized, i.e., it satisfies the regularized commutation relation

$$[\varphi(x), \varphi(x')] = i\Delta_R(x-x') \quad (60)$$

where Δ_R is the regularized Δ -function.

This requirement will be fulfilled by some regularization condition for ζ_k . For examples, we may put

$$\sum_{k=1}^N \zeta_k \mu_k^{2n} = 0, \quad n=0, 1, \dots, m \leq N, \quad (61)$$

Other types of condition also are possible.

It is clear that this condition can not be satisfied by positive values of ζ_k 's for real masses of μ_k 's. We have, therefore, some positive ζ_k 's and other negative ζ_i 's (we assume all of ζ_k to be real).

For a negative ζ_i , the commutation relation changes its sign comparing with the ordinary commutation relation. This means that the field $\varphi^{(i)}$ having a negative ζ_i is a negative energy Bose field. Thus, the generalized field φ is considered to be a mixed field of positive and negative energy fields. The case is very similar to the electromagnetic field which was treated in the preceding section. The super-quantization of the generalized field can be performed in the parallel manner to that. We introduce the quantized state vectors φ_k for each $\varphi^{(k)}$ which is defined as (15) and (15'). We denote the super-quantized quantities by gothic letters as before. We have

$$\varphi^{(k)} = (\phi_k, \varphi^{(k)} \phi_k) \quad (62)$$

and

$$\varphi = \sum_k \zeta_k \varphi^{(k)}. \quad (63)$$

For vacuum definition, we put the conditions:

$$\varphi^{+(k)} \Psi_{vac} = 0, \quad (64)$$

where $\varphi^{+(k)}$ is a positive frequency part of $\varphi^{(k)}$. We have seen that these conditions are consistently fulfilled. As before, number of the positive energy field is one and infinite numbers of the negative energy field is normalized to one, we obtain

$$(\Psi, [\varphi(x), \varphi(x')] \Psi) = i\Delta_R(x-x') \quad (65)$$

which corresponds to the ordinary regularized commutation relation. If there is an interaction with other field, the sufficiently regularized Δ_R will give all finite results which arise from the interaction fluctuation of the field φ .

If we have not the supplementary condition:

$$\varphi^{+(k)} \Psi = 0, \quad (66)$$

in this case, the field $\varphi^{(k)}$ will become to be observable as an assembly of Bose particles with mass μ_k . However, this situation gives no physical inconsistency even for the negative energy field.

The vacuum for a negative energy field is defined as such state that only the zero-particle state is unoccupied and all other state are occupied. Then, if it happens that the zero-particle state is occupied and one of the states occupied in the vacuum comes to be vacant, we observe a field in which some particle distribution with positive energy corresponding to an anti-particle of the original negative energy one is realized. We may, therefore, expect a mass spectrum of Bose particles associated to a regularized field. It may be imagined to have any relation between this mass spectrum and that of the observed mesons in cosmic rays.

§ 5. Conclusions

The super-quantization theory affords a physically consistent reinterpretation of negative energy Bose field which serves to give a physical scheme for the symmetrical treatment of electromagnetic potentials and the theory of regulator of Bose field.

It is regretful that the present theory can offer no clue to solve the difficulties associated to the regulator theory of Fermi field.⁶⁾ This problem remains to study in future.

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Non-Additivity of Nucleon Magnetic Moment in the Deuteron

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The modification of nucleon's anomalous magnetic moment by the proximity of another nucleon has been estimated employing the pseudoscalar meson theory. Contribution of this effect to the deuteron magnetic moment turns out to be of the order of $\sim 0.12|U|/\kappa$, where κ means the rest energy of pion and U the potential energy of the deuteron (~ 14 Mev.). The precise prediction of this value is impossible because of the lack of any satisfactory form of meson theory. The main conclusion reached is that the magnetic moment of the deuteron does not determine the fraction of D state mixture. Including the effect of relativistic correction (which induces the uncertainty of $\sim 1.5\%$), the D state percentage of the deuteron would be from 2 to 8% or more.

§ 1. Introduction

The observed value of the deuteron magnetic moment

$$\mu_d = 0.857$$

is found to differ markedly from the sum of that of proton and neutron, *viz.*,

$$\mu_d^0 = \mu_n + \mu_p = 0.879.$$

This difference has been employed to determine the rate of D state admixture in the deuteron. In fact, if the D state percentage were denoted by P_D , one finds a magnetic moment

$$\mu_d = \mu_d^0 - \frac{3}{2} \left(\mu_d^0 - \frac{1}{2} \right) P_D, \quad (1)$$

from which one gets

$$P_D = 4\%. \quad (2)$$

This value has been a valuable information in determining the interaction between proton and neutron in the early stage of the nuclear theories, but gradually it proved to be a rather severe restriction. Indeed, the fact that the D admixture cannot be larger than (2), together with the value of quadrupole moment of the deuteron imposes a lower bound of the range of $n-p$ interaction to be 2×10^{-13} cm¹). If the nuclear forces were assumed to arise from the exchange of pions, the rather small range makes it difficult to account for both the observed quadrupole moment and the small D state percentage. Investigations of deuteron problem²⁾ using the nuclear potentials derived from the pseudoscalar meson theory showed that if constants were adjusted so as to give the correct quadrupole moment, the D state percentage was found to be considerably larger than 4%.

One must assume either that meson theory of nuclear forces is incorrect or that there are other contributions to the deuteron magnetic moment and the amount of D state is

not necessarily 4%. Recently, Low and Salpeter³⁾, in their calculation on the hyperfine structure of deuteron, suggested that the value of 4% for P_D would be incorrect. It is, therefore, of some interest to investigate the other effect on the deuteron magnetic moment, and to see how much it is.

One conceivable effect is the relativistic correction for the motion of nucleons. This is rather difficult to estimate since the relativistic transformation properties of the interactions are involved. An investigation of the various model indicates⁴⁾ that the correction to eq. (1) is uncertain by $\sim \pm 0.008$, which implies the uncertainty in P_D of $\sim \pm 1.5\%$.

Another effect is the non-additivity of the nucleon magnetic moment. The charge cloud of mesons surrounding a nucleon, which gives rise to the anomalous magnetic moment, will be disturbed by the presence of another nucleon in its neighbourhood, thereby results in the modification of the intrinsic magnetic moments. This effect has long been realized, but does not seem to have been treated so far*. The reason is that the exchange magnetic moment derived from meson theory in the lowest order in the coupling constant has a vanishing contribution to the deuteron, therefore no additional term to (1) arises in this approximation. In this paper this effect will be investigated. The well-known deficiencies of all forms of meson theory at the present time makes it impossible to evaluate the precise value, but we can estimate the order of this effect and can see to what extent the value (2) for P_D is reliable. It will be shown that the contribution to the deuteron magnetic moment does not come from the current of mesons interchanged between nucleons, but from nucleon's magnetic moment, and it is the order of magnitude of $0.005 \sim 0.02$. Consequently P_D would be $4 + (1 \sim 4)\%$.

§ 2. Modification of the intrinsic magnetic moment by the proximity of another nucleon

As is well known the existence of charged exchange forces between nucleons requires the flow of charged mesons between the interacting neutrons and protons. These mesons will then interact with electromagnetic field and give rise, in general, to a contribution to the magnetic moment of the nucleus. However, this magnetic moment vanishes for self-conjugate nuclei and consequently for deuteron. This is due to a symmetry between positive and negative mesons which can explicitly be demonstrated as follows.

We shall for the time being assume charge symmetrical theory and adopt the isotopic space formalism as introduced by Møller and Rosenfeld⁵⁾. Then

$$\bar{\psi} \tau_1 \psi, \bar{\psi} \tau_2 \psi, \bar{\psi} \tau_3 \psi \quad (3)$$

behaves as a vector for the rotation in the isotopic space (this fact will be abbreviated as 'to form a τ -vector'). The three component of meson field, say, scalar field,

$$\phi_1, \phi_2, \phi_3 \quad (4)$$

also forms a τ -vector. The electromagnetic interaction of mesons,

* R. K. Osborne and I. L. Foldy (Phys. Rev. **79** (1950), 795) have determined possible types of this effect phenomenologically, but involves arbitrary functions. The present paper contains the case in which nuclear potential is not of charge exchange type, a case not treated (but alluded) by them.

$$(\phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1) A_\mu \quad (5)$$

is therefore the ε -component of a τ -vector. The meson-nucleon interaction

$$f \bar{\psi} \mathbf{0} \tau_i \psi \phi_i \quad (6)$$

is an inner product of two τ -vectors, then is a τ -scalar. The exchange magnetic moment consists of a number of products of (6) and a (5), so is the ε -component of a τ -vector, irrespective of the order of approximation. The expectation value of this operator therefore vanishes for deuteron which is a τ -scalar (charge singlet state). Also exchange magnetic moment arising from meson-nucleon-photon interaction,

$$\bar{\psi} (\phi_1 \tau_2 - \phi_2 \tau_1) \boldsymbol{\sigma} \mathbf{A} \psi \quad (7)$$

is the ε -component of a τ -vector, thus has a vanishing expectation value for deuteron. The interaction of nucleon's magnetic moment with electromagnetic fields is proportional to

$$(1 - \tau_3)/2, \quad (8)$$

thus composed of a τ -scalar and a τ -vector. The latter, as before, cannot contribute to the deuteron magnetic moment. Only the former term, $1/2$, has a contribution to the deuteron moment.

This term is intimately connected with the non-vanishing value of

$$\Delta = \mu_n + (\mu_p - 1) = -0.12. \quad (9)$$

Indeed, the anomalous magnetic moment of a nucleon arising from (6), (7) and the second term of (8), being the ε -component of a τ -vector, must necessarily be proportional to τ_3 , for one cannot construct any τ -vector from $\bar{\psi}$ and ψ other than (3). Thus these are equal and opposite for proton and neutron⁽⁶⁾, and does not contribute to Δ . Only the τ -scalar term, $1/2$ of (8), gives equal sign and equal amount for proton and neutron moment and gives rise to Δ . If this magnetic moment were modified, say, by 10%, by the proximity of another nucleon, the additional magnetic moment of the deuteron will be $\Delta \times 10\% = 0.012$.

We have up to now assumed charge symmetry merely for the sake of elegance and easier understanding. However, what is essentially needed in the proof is the invariance of the theory under the rotation not of whole τ -space but only around the ε -axis, that is the gauge invariance. Therefore above statements hold for all types of meson field, even in the case of charge asymmetry.

The summary of results of the above argument is that the non-additivity of nucleon's magnetic moment, if it exists at all, comes from the $1/2$ term of (8), and has nothing to do with other interactions (6), (7) and $-\tau_3/2$ term of (8). The algebraic sum Δ of anomalous magnetic moments of proton and neutron also comes just from $1/2$ term of (8), so can be used as a measure for the deuteron magnetic moment in question. So in our discussion we may assume as if both proton and neutron had charge $+e/2$, and as if mesons carried no charge. This is a considerable simplification of our calculation.

Having traced the origin of the non-additivity effect, we shall here roughly estimate the order of its magnitude. The anomalous magnetic moment Δ arises in the following

way. A nucleon emits a meson with angular momentum, and reverses its spin direction. This state has negative magnetic moment, and the fraction of time which the nucleon spends in this state is proportional to

$$G^* G / \omega^2,$$

where G means the matrix element of meson-nucleon interaction, and ω is the energy of the emitted meson. Nucleon's kinetic energy is neglected. Therefore anomalous magnetic moment is proportional to

$$-\sum G^* G / \omega^2 \quad (10)$$

summed over all virtual mesons. The twice of this expression is to be equated with the \mathcal{A} of (9).

How is this value changed if nucleons were bound to the deuteron? Consider a deuteron and assume that nuclear force of the type of $\sigma^{(1)} \sigma^{(2)} V'_{12}$ is acting between nucleons. One of the constituent nucleons emits a meson and become a nucleon with spin direction reversed. But in this state the potential changes its sign so that additional term $2U$ appears in the energy denominator of (10), where U denotes (minus of) the potential energy of the deuteron. The anomalous magnetic moment is thus proportional to

$$-\sum G^* G / (\omega + 2U)^2 \quad (11)$$

instead of (10). The occurrence of $2U$ in this expression manifests itself as the non-additivity of nucleon magnetic moment, which is, if expanded in powers of U/ω ,

$$\delta\mu_{\infty} + \sum G^* G / \omega^3 \cdot U = (\sum G^* G / \omega^2) \cdot U / \langle \omega \rangle_{av},$$

in which the second version involves the mean value theorem. The first parenthesis is evidently $|\mathcal{A}|$ of (9). Therefore the additional contribution to the deuteron magnetic moment is positive and is given by

$$\delta\mu \sim 0.12 \cdot U / \langle \omega \rangle_{av}.$$

If the acting potential is of the form $\tau\tau V'$, the potential energy does not necessarily change its sign when one of the nucleon changes its spin direction. It changes its sign when a τ -spin is reversed. This occurs twice in three transitions if we assume charge symmetry. $\delta\mu$ in this case, therefore, would be $2/3$ times of the previous case. If the potential were $\sigma\sigma\tau\tau V'$, by quite similar arguments, we can infer that $\delta\mu$ would be $1/3$ of the case when the potential were $\sigma\sigma V'$.

$\langle \omega \rangle_{av}$ would not be very much larger than the meson rest mass κ , and U is, as we shall see below, about 14 Mev. Thus we see that $\delta\mu \sim 0.01$ and the consequent change in P_D is $\sim 2\%$. The above arguments may seem quite rough, but essential points will turn out to be entirely correct. The following section is devoted to the verification and confirmation of this statements.

§ 3. Calculation of the deuteron magnetic moment

In this section we shall carry through the explicit calculation of $\delta\mu$ explained in the previous section with the use of symmetrical pseudoscalar meson theory.

Before starting the calculation, we must first normalize the constants so as to fit the free nucleon moment Δ . The calculation based on the second order perturbation theory gives a convergent but too large value for Δ . This difficulty is due to the enormous contribution of high momentum virtual mesons, and here we take a view that these high momentum should be cut off at about $2 \sim 3\pi$, as suggested from the cut off radius in the meson theory of nuclear forces. In the following section we shall carry out an alternative calculation without cut off.

As rather low momenta are involved, kinetic energies of nucleons can be neglected as compared with meson's energy. The anomalous magnetic moment of a nucleon, arising from the $1/2$ term of (8), can be easily calculated, yielding*

$$\frac{\Delta}{2} = -\frac{g^2}{4\pi} \frac{2}{\pi} \int_0^{k_0} \frac{k^2 d^3 k}{x^2 (k^2 + x^2)^{3/2}}.$$

Here, and in the following, we take the cut off momentum $k_0 = 2\pi$. Then

$$\frac{\Delta}{2} = -0.6 \frac{g^2}{4\pi}.$$

Equating this with the half of (9), we have the coupling constant

$$\frac{g^2}{4\pi} = 0.1, \quad \text{for } k_0 = 2\pi. \quad (12)$$

The hamiltonian with which we start is

$$H_0 = \sum_{i=1}^3 \sum_k \omega_k a_k^{(i)*} a_k^{(i)} + \sum_k G_k^{(i)} a_k^{(i)} + \sum_k G_k^{(i)*} a_k^{(i)*} + H',$$

where

$$\omega_k^2 = k^2 + x^2.$$

G represents the meson-nucleon interaction,

$$G_k^{(i)} = \frac{g}{x} \frac{i}{\sqrt{2\omega}} \sum_{\nu=1}^2 \tau_i^{(\nu)} (\sigma^{(\nu)} \mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}^{(\nu)})$$

in which ν labels the two nucleons, and H' represents the nuclear potential V and the interaction of nucleon's magnetic moment with magnetic field,

$$H' = -\frac{\mu \mathcal{H}}{2} (\sigma_3^{(1)} + \sigma_3^{(2)}) + V \quad (\mu: \text{nuclear magneton})$$

where both nucleons are assumed to have charge $e/2$ and magnetic field \mathcal{H} is assumed to point in the z -direction. We are interested in the modification of anomalous magnetic moment by the potential, that is a term bilinear in μ and V . Two successive canonical transformations with unitary operators

$$U = \exp \sum \frac{1}{\omega} (G^* a^* - G a)$$

* Hereafter we use natural units, i.e., \hbar and $c=1$.

and

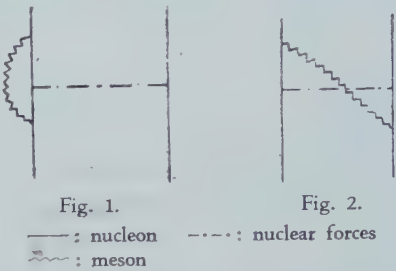
$$U'=\exp \sum \frac{1}{\omega^2} (a^* [H', G^*]+a [H', G])$$

suffices k and i have been omitted) eliminate terms linear in a and gives the desired modified hamiltonian

$$H=\sum \frac{1}{\omega^3} [H', G^{(i)}] [H', G^{(i)*}],$$

from which we have to extract terms proportional to μV .

There are two types of this form, schematically indicated in Figs. 1 and 2. The first type represents the modification of anomalous magnetic moment by the nuclear potential, and the second, so to say, the exchange magnetic moment. We shall not follow the detailed calculations here. For the first type, we obtain the magnetic moment



$$\delta\mu=\frac{g^2}{4\pi}\frac{1}{2\pi}\int\frac{k^2d^3k}{x\omega^4}\frac{1}{x}\left(-4\sigma_3^{(1)}V+\frac{1}{3}(\sigma_3^{(1)}\sigma_\rho^{(1)}\tau_i^{(1)}V\sigma_\rho^{(1)}\tau_i^{(1)}-\sigma_\rho^{(1)}\tau_i^{(1)}\sigma_3^{(1)}V\sigma_\rho^{(1)}\tau_i^{(1)})+c.c.\right)$$

+corresponding quantity for particle ‘2’.

(13)

Inserting numerical values for g and k_0 given in (12), we have

$$\delta u=0.009\frac{1}{x}\left(-4\sigma_3V+\frac{1}{3}[\sigma_3,\sigma_\rho\tau_i]V\sigma_\rho\tau_i+c.c.\right)+\text{that for ‘2’}.$$

(14)

We shall evaluate expectation values of this expression for deuteron, in the case of central forces. Expectation values of the parenthesis of (14) for various types of potential are tabulated in the following:

V	$\langle (-4\sigma_3V+\cdots) \rangle_{av}$
1	0
$\sigma\sigma$	$16U$
$\tau\tau$	$32/3U$
$\sigma\sigma\tau\tau$	$16/3U$

where U denotes (minus of) the potential energy of the deuteron,

$$U=-\langle V \rangle_{\text{deuteron}}.$$

The $\delta\mu$ for potentials of the type $\sigma\sigma$, $\tau\tau$ and $\sigma\sigma\tau\tau$ respectively, are in the proportion of 3 : 2 : 1, thus confirming the expectation of the previous section. For the evaluation of U , we observe that

$$U=\text{binding energy}+\text{kinetic energy}.$$

The ground state of the deuteron may approximately be written

$$\varphi(r)=A/r\cdot(e^{-\alpha r}-e^{-\beta r}),$$

(15)

in which a can be determined from the binding energy of the deuteron. With this wave function

$$K = \text{kinetic energy} = \frac{a\beta}{M}, \quad (M: \text{nucleon mass})$$

β adjusted to fit the triplet effective range is $7a$ which corresponds $K=17$ Mev. Exact solution of Hulthen's potential with range $1/x$, gives $\beta=a+x$ and $K=9$ Mev. We shall in the following take the value

$$U=14 \text{ Mev.}$$

The magnetic moment of the deuteron is then

$$\begin{aligned} \delta\mu &= 0.027 & \sigma\sigma, \\ &= 0.018 & \tau\tau, \\ &= 0.009 & \sigma\sigma\tau\tau. \end{aligned} \quad (16)$$

The magnetic moment coming from the process of Fig. 2 is, for pure 3S state,

$$\delta\mu = -a \frac{g^2}{4\pi} \frac{4}{\pi^2} \frac{1}{x^2} \int d^3k \frac{k^2}{\omega^4} e^{ikx} V'(|x|), \quad (17)$$

if $V=a\sigma\sigma V'$ or $V=a\sigma\sigma\tau\tau V'$, and vanishes for Wigner force or if $V=a\tau\tau V'$. With Yukawa potential and the wave function (15), together with (12), we have

$$\delta\mu = -0.026 \cdot a.$$

The constant a , adjusted to the binding energy of the deuteron, is 0.1 for $\sigma\sigma\tau\tau$ -type, and -0.3 for $\sigma\sigma$ -type. Added with (16), the magnetic moment of the deuteron is

	type of the potential
$\delta\mu=0$	1,
$=0.018$	$\tau\tau$,
$=0.035$	$\sigma\sigma$,
$=0.0064$	$\sigma\sigma\tau\tau$.

(18)

If the potential were written in the form

$$V = (a_0 + a_\sigma\sigma\sigma + a_\tau\tau\tau + a_{\sigma\tau}\sigma\sigma\tau\tau) V'$$

normalized such that $3(a_\tau + a_{\sigma\tau}) - a_0 - a_\sigma = 1$, the magnetic moment is

$$\delta\mu = -0.035a_\sigma + 0.058a_\tau + 0.019a_{\sigma\tau}.$$

At present we have no conclusive form of nuclear potential, and cannot obtain a unique value for $\delta\mu$. For the Serber type potential with half exchange character,

$$a_0 = -0.32, \quad a_\sigma = 0.067, \quad a_\tau = 0.145, \quad a_{\sigma\tau} = 0.106$$

and

$$\delta\mu = 0.0083.$$

Numerical values would change by several percent according to the different choice of the shape of potential.

In the above calculation we have neglected the retardation effect of nuclear forces and replaced it by an instantaneous phenomenological potential. This procedure can be justified in our approximation (which neglect the kinetic energy of nucleon). In fact, if we regard nuclear forces as arising from exchange of mesons, and calculate the $\delta\mu$ in the fifth order perturbation (i.e., in order μ_S^4), the results are exactly (13) and (17) with V replaced by the second order meson potential.

§ 4. The relativistic calculation

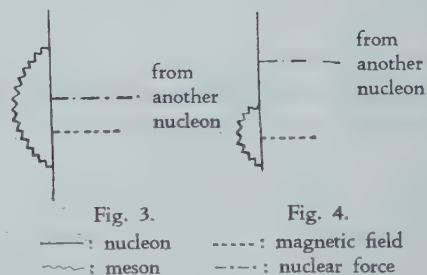
Above calculations were made with cut off, based on the choice of constants (12). These values are, however, not correct ones since with these constants the anomalous magnetic moment of nucleon, due to meson's charge, turns out to be too small—about 1/5 of the observed value. Therefore results obtained in the previous section may be an underestimation of the true one. In this section we shall try a complete second order perturbation calculation without cut off. This leads to a rather large value of A , however, we can hope that higher order effect would cancel greater part of the second order result, or we can assume Pauli type magnetic moments for nucleons compensating so as to give the correct values. In fact, Nakabayashi and Sato⁷⁾, in their calculation with pseudoscalar theory with pseudoscalar coupling, found the fourth order contribution of 0.7 to A , as compared with the second order one, -1.5 .

Processes we are to calculate are shown in Fig. 3 and 4. ((17) is already convergent without cut off, and does not appreciably change when k_0 is taken as infinity so we omit the process Fig. 2). These are similar to that of the fourth order correction to γ - π process, but simplified by the fact that mesons do not interact with electromagnetic field. We shall introduce further simplification by assuming that $M \gg \kappa$.

In this case, deflection of nucleons by nuclear potential can be neglected, which means that the nuclear potential does not change within distances of the order of $1/M$. Nuclear potentials adopted are central ones of the form 1, $\tau\tau$, $\gamma_5\gamma_5$, and $\tau\tau\gamma_5\gamma_5$. The choice of $i\gamma_5$ instead of $i\gamma_5\gamma_\mu = \sigma$ is for the simplification of calculation: for the latter interaction the evaluation turned out to be rather complicated.

An elementary application of Feynman and Dyson's evaluation method leads to the effective hamiltonian of the form $\bar{\psi} F_{\mu\nu} \sigma_{\mu\nu} \phi \phi$ or $i\bar{\psi} F_{\mu\nu} \sigma_{\mu\nu} \gamma_5 \phi \phi$ where ϕ denotes the external meson field, which is to be replaced by nuclear potential. Expectation values of this expression lead to the deuteron magnetic moment tabulated below:

meson-nucleon interaction	type of potential			
	1	$\tau\tau$	$\gamma_5\gamma_5$	$\tau\tau\gamma_5\gamma_5$
pseudoscalar	$\rightarrow 0.6$	1.0	-1.8	-0.2
pseudovector	0.6	1.4	0	0.8
	$\times \frac{f^2}{4\pi} \left(\frac{\kappa}{2M} \right)^2$			
	$\times \frac{g^2}{4\pi}$			



The intuitive expectation of § 2, that $\delta\mu$ would be positive is not justified here. This would certainly be due to the unusual behaviour of the odd operator γ_5 of the potential.

For usual values of $g^2/4\pi$, i.e., $\sim 0.05-0.1$, the $\delta\mu$ is about $0.03 \sim 0.1$. This would surely be an over estimation, since, as stated before, this form of treatment gives too large value of A , and also we can expect that higher order correction to $\delta\mu$ would give negative contribution.

§ 5. Effect of tensor forces

So far we have treated only central forces and pure 3S state. This can be valid if the contribution of tensor forces on the binding energy of the deuteron can be neglected as compared with that of central one. In this section we shall briefly evaluate the expectation values of (14) when tensor forces are included.

First, for tensor operator S_{12} ,

$$\begin{aligned} -4\sigma_3 V + \frac{1}{3} [\sigma_3, \sigma_\tau] V \sigma_\tau + c.c. &= -8(\sigma_3 V + V \sigma_3) \quad \text{if } V = S_{12}, \\ &= -\frac{8}{3} (\sigma_3 V + V \sigma_3) \quad \text{if } V = \tau \tau S_{12}. \end{aligned}$$

Next task is the evaluation of expectation values of $\sigma_3 S_{12} + S_{12} \sigma_3$ for $S+D$ mixture $\phi_S + \phi_D$. With the explicit use of angular wave functions one can show that

$$(\phi_S^* + \phi_D^*) (\sigma_3 S_{12} + S_{12} \sigma_3) (\phi_S + \phi_D) = (\phi_S^* S_{12} \phi_D) - (\phi_D^* S_{12} \phi_D).$$

Thus, if nuclear potential were expressed as

$$\begin{aligned} V &= (a_0 + a_\sigma \sigma\sigma + a_\tau \tau\tau + a_{\sigma\tau} \sigma\sigma\tau\tau) V' + (b_0 + b_\tau \tau\tau) S_{12} V'', \\ \delta\mu &= 0.017 \frac{1}{x} (-16(a_\sigma - 2a_\tau - a_{\sigma\tau}) \langle V' \rangle - 8(b_0 - b_\tau) \langle S V'' \rangle). \end{aligned} \quad (19)$$

where

$$\langle V' \rangle = (\phi_S^* V' \phi_S), \quad \langle S V'' \rangle = (\phi_S^* S_{12} V'' \phi_D).$$

In (19), $D-D$ term were neglected and contributions from the process of Fig. 2 are not included.

$$(a_0 + a_\sigma - 3(a_\tau + a_{\sigma\tau})) \langle V \rangle + 2(b_0 - 3b_\tau) \langle S V'' \rangle$$

represents the potential energy of the deuteron which is about 14 Mev. If main part of the deuteron binding is due to the b_τ term, $\delta\mu$ is considerably small—about 0.002. For usual potentials $\delta\mu$ would be ~ 0.01 .

§ 6. Concluding remarks

The additional magnetic moment of the deuteron, due to the non-additivity of nucleon magnetic moments has been estimated and turned out to be $\sim 0.005-0.02$ or more. The precise value cannot be expected because of the lack of our knowledge about the nature of nuclear forces as well as of the deficiencies of current meson theory. However it is very likely that there exists an additional magnetic moment of the deuteron of about 0.01. This must be added to the right-hand side of eq. (1). This implies that the D state

percentage is not 4% but 6%. Considering the possible contribution coming from relativistic effect, one may conclude that eq. (1) for deuteron magnetic moment is quite uncertain and cannot determine the D state percentage P_D .

The magnetic moment $\delta\mu$ calculated here critically depends on the exchange character of nuclear forces and the mean energy of virtual mesons contributing the anomalous magnetic moment of a nucleon. If the non-additivity were experimentally exactly known, it would be a valuable information to test meson theory and to determine the type of nuclear forces.

One can also expect a change in the quadrupole moment of the deuteron due to the virtual transition of nucleons emitting and reabsorbing mesons. However it is not likely that it is large enough to be of interest.

It is an embarrassing situation that meson theory in its present form is not reliable. Yet one cannot be allowed to dispense with mesons in nuclear phenomena. Deuteron problems seem to be unsuccessful without the help of virtual mesons (e.g., photomagnetic disintegration).

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Relativistic Two-Body Problem*

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Starting from Feynman theory an inhomogeneous and a homogeneous integral equations are obtained relativistically for scatterings and bound states of two particles respectively in connection with the S -matrix theory.

§ 1. Introduction and summary

Since the recent improvements of the quantum field theories have made possible to perform extensive calculations relativistically, these elegant techniques have been applied to the open systems by many authors. However, no relativistic treatment of bound systems has been made as yet except for Heisenberg-Møller's S -matrix theory⁽¹⁾⁽²⁾, and an obstacle to the research in this field was the concept of "potential", which was very useful in the non-relativistic domain.

The usual method is as follows. First by eliminating freedoms of a quantized field carrying interactions between particles one calculates an effective interaction Hamiltonian between two particles, which, however, is not diagonal in configuration space. Then a Schrödinger's eigen-value problem is obtained by inserting as a potential a diagonal part of that effective interaction Hamiltonian into a Schrödinger or Dirac equation, neglecting the retardation and recoil effects. But the relativistic characters are destroyed by this approximations, and potentials have their meanings only in the non-relativistic domain. Indeed, Coulomb potential has given the energy-levels of hydrogen atom with an excellent accuracy. But in the case of meson theory of nuclear forces the situation would not be the same as in the electromagnetic case, on account of the fact that meson has a charge and a large mass and that the coupling constant is larger than the latter. The effects neglected in the adiabatic approximation would play a considerable role.

On the other hand, Heisenberg-Møller's S -matrix theory can give discrete eigen-values of bound systems relativistically in principle. But on its application to Dyson's expression⁽³⁾

* The preliminary treatments of this work are seen in Japanese journals; H. Kita, *Soryushiron-kenkyu*, vol. 2, no. 3 (1950), 57, and H. Kita and Y. Munakata, *ibid.*, vol. 2, no. 5 (1950), 48. See also Y. Nambu, *ibid.*, vol. 2, no. 5 (1950), 57, and Y. Nambu, *Prog. Theor. Phys.* 5 (1950), 614. Independently of us, Bethe and Salpeter have found the same equations; H. A. Bethe and E. E. Salpeter, *Phys. Rev.* 82 (1951), 309 (A). Recently Gell-Mann and Low have derived them in an interesting way from field theory; M. Gell-Mann and F. Low, *Phys. Rev.*, (in publishing). We wish to express our thanks to Drs. Gell-Mann and Low, who have made us available their paper before publication, which was sent to Prof. Nambu.

of S -matrix we find a difficulty which arises from the fact that it is given as a power series expansion of the coupling constant and that one may at most calculate it up to the first or the second Born approximation.

Thus it seems necessary and meaningful to look for an alternative formulation of the relativistic two-body problem. And also such investigation of this problem would give some suggestions to the point of view as to the mutual and internal structures of elementary particles.

Then how should one formulate this problem?

In this paper we start from Feynman⁽⁴⁾ theory, which handles the "kernels" eliminating a quantized field, and obtain an inhomogeneous integral equation for scattering of two particles and investigate its general properties (§ 2), where the homogeneous part is an incident plane wave and the kernel of this equation can be written down directly according to Feynman's diagram. Then a homogeneous integral equation with the same kernel as the former is derived for bound states of two particles in connection with S -matrix theory (§ 3).

§ 2. Total scattering amplitude and inhomogeneous integral equation.

We consider, for example, the system of an electron and a proton interacting through an electromagnetic field. In other cases also our formulation can be extended directly. In the following we use the same notations as Feynman if not otherwise stated. Suffices 1 and 2 refer to the electron and the proton respectively. After Feynman's procedure we can ask for the amplitude, $\Psi_k(x_1, x_2)$, that an electron and a proton with original free wave functions were scattered k times by irreducible interactions either forward or backward in time to arrive at x_1 and x_2 . Here by irreducible interactions we mean those interactions which cannot be obtained by their iterations, as represented by Fig. 1. Then the amplitude after one more irreducible scattering is given as follows.

$$\Psi_{k+1}(x_1, x_2) = \iint G(x_1, x_2; x'_1, x'_2) \Psi_k(x'_1, x'_2) d\tau'_1 d\tau'_2 \quad (1)$$



Fig. 1

where

$$\begin{aligned} G(x_1, x_2; x'_1, x'_2) = & ie^2 K_+(x_1, x'_1) K_+(x_2, x'_2) \gamma_{\mu 1} \gamma_{\mu 2} \delta_+ \{ (x'_1 - x'_2)^2 \} \\ & - ie^2 \int K_+(x_1, x''_1) \gamma_{\mu 1} K_+(x''_1, x'_1) \gamma_{\mu 1} \delta_+ \{ (x''_1 - x'_1)^2 \} \delta(x_2 - x'_2) d\tau''_1 \\ & - ie^2 \int K_+(x_2, x''_2) \gamma_{\mu 2} K_+(x''_2, x'_2) \gamma_{\mu 2} \delta_+ \{ (x''_2 - x'_2)^2 \} \delta(x_1 - x'_1) d\tau''_2 \end{aligned}$$

$$\begin{aligned}
 & + (ie^2)^2 \iint \dot{K}_+(x_1, x_1'') K_+(x_2, x_2'') \gamma_{\mu 1} \gamma_{\nu 2} K_+(x_1'', x_1') K_+(x_2'', x_2') \gamma_{\nu 1} \gamma_{\mu 2} \times \\
 & \quad \times \delta_+ \{ (x_1' - x_2'')^2 \} \delta_+ \{ (x_2' - x_1'')^2 \} d\tau_1'' d\tau_2'' \\
 & + \dots
 \end{aligned} \quad (2)$$

(Kernels with arguments x_1, x_1', \dots refer to the first particle and those with argument x_2, x_2', \dots refer to the second particle.)

An equation for the total scattering amplitude for the two particle arriving at x_1 and x_2 either directly or after any number of scattering is obtained by summing (1) over all k from 0 to ∞ ;

$$\Psi(x_1, x_2) = \sum_{k=0}^{\infty} \Psi_k(x_1, x_2) \quad (3)$$

$$\Psi(x_1, x_2) = \Psi_0(x_1, x_2) + \iint G(x_1, x_2; x_1', x_2') \Psi(x_1', x_2') d\tau_1' d\tau_2' \quad (4)$$

This equation (4) is also obtained from an equation:

$$\begin{aligned}
 K(x_1, x_2; x_1'', x_2'') &= K_+(x_1, x_1'') K_+(x_2, x_2'') \\
 &+ \iint G(x_1, x_2; x_1', x_2') K(x_1', x_2'; x_1'', x_2'') d\tau_1' d\tau_2',
 \end{aligned} \quad (5)$$

by defining $\Psi(x_1, x_2)$:

$$\Psi(x_1, x_2) \equiv \lim_{\substack{t_1'' \rightarrow -\infty \\ t_2'' \rightarrow -\infty}} \iint K(x_1, x_2; x_1'', x_2'') \beta_1 \beta_2 \Psi_0(x_1'', x_2'') d\mathbf{x}_1'' d\mathbf{x}_2'' \quad (6)$$

and using

$$\Psi_0(x_1, x_2) = \lim_{\substack{t_1'' \rightarrow -\infty \\ t_2'' \rightarrow -\infty}} \iint K_+(x_1, x_1'') K_+(x_2, x_2'') \beta_1 \beta_2 \Psi_0(x_1'', x_2'') d\mathbf{x}_1'' d\mathbf{x}_2'' \quad (6)$$

It is easily seen by solving (5) successively that $K(x_1, x_2; x_1', x_2')$ is Feynman's two-body kernel. Evidently (4) is relativistically covariant in form.

Now it is well known that $K(x_1, x_2; x_1', x_2')$ is connected with an element of S -matrix between an initial state $\Psi_0(x_1, x_2)$ and a final state $\chi_0(x_1, x_2)$ by the relation;

$$\begin{aligned}
 (k'_1, k'_2 | R | k_1, k_2) &= \lim_{\substack{t_1' \rightarrow +\infty \\ t_2' \rightarrow +\infty}} \int \dots \int \bar{\chi}_0(x_1', x_2') \beta_1 \beta_2 \{ K(x_1', x_2'; x_1, x_2) \\
 &\quad - K_+(x_1', x_1) K_+(x_2', x_2) \} \beta_1 \beta_2 \Psi_0(x_1, x_2) \\
 &\quad \cdot d\mathbf{x}_1' d\mathbf{x}_2' d\mathbf{x}_1 d\mathbf{x}_2,
 \end{aligned} \quad (8)$$

where initial and final states are given by the following Ψ_0 and χ_0 respectively;

$$\begin{aligned}
 \Psi_0(x_1, x_2) &\equiv \Psi_0(k_1, k_2) \exp(-ik_1 \cdot x_1 - ik_2 \cdot x_2) \\
 \chi_0(x_1, x_2) &\equiv \chi_0(k'_1, k'_2) \exp(-ik'_1 \cdot x_1 - ik'_2 \cdot x_2) \\
 \bar{\chi}_0 &\equiv \chi_0^* \beta_1 \beta_2
 \end{aligned} \quad (9)$$

and for the convenience of later arguments, R is used;

$$R \equiv S - 1 \quad (10)$$

From the general theory of S -matrix⁽²⁾, it is known that S commutes with a certain number of constants of collision, among which it is convenient in the case of two particles to take the following six quantities as a complete set of collision constants;

$$P_\mu, L, m, \dots \quad (11)$$

where P_μ is a total kinetic energy-momentum four vector and L and m are Møller's operators which are reduced to the square of the total angular momentum and its z -component respectively in the center of mass system. Therefore it is obvious from (8) that when $\Psi(x_1, x_2)$ is expanded by eigen-functions of (11) its coefficients are separated and each satisfies its integral equation independently.

§ 3. Analytic continuation and homogeneous integral equation

We can write

$$\begin{aligned} \Psi(x_1, x_2) &= \varphi(x) \exp(-iP \cdot X), \\ \Psi_0(x_1, x_2) &= \varphi_0(x) \exp(-iP \cdot X) = \Psi_0(k_1, k_2) \exp\left\{-iP \cdot X - \frac{i}{2}(k_1 - k_2) \cdot x\right\}, \\ \chi_0(x_1, x_2) &= \chi_0(k'_1, k'_2) \exp\left\{-iP' \cdot X - \frac{i}{2}(k'_1 - k'_2) \cdot x\right\}, \end{aligned} \quad (12)$$

where

$$\begin{aligned} X_\mu &= \frac{1}{2}(x_{\mu 1} + x_{\mu 2}), \\ x_\mu &= x_{\mu 1} - x_{\mu 2}, \end{aligned} \quad (13)$$

and

$$\begin{aligned} P_\mu &= (P_0, \mathbf{P}), & P'_\mu &= (P_0, \mathbf{P}'), \\ P_0 &= \sqrt{k_1^2 + m_1^2} + \sqrt{k_2^2 + m_2^2}, & P'_0 &= \sqrt{k_1'^2 + m_1^2} + \sqrt{k_2'^2 + m_2^2}, \\ \mathbf{P} &= \mathbf{k}_1 + \mathbf{k}_2, & \mathbf{P}' &= \mathbf{k}'_1 + \mathbf{k}'_2, \end{aligned} \quad (14)$$

There exist an arbitrariness in the definition of X and in fact we can put

$$\begin{aligned} X_\mu &= a_1 x_{\mu 1} + a_2 x_{\mu 2} \\ a_1 + a_2 &= 1, \end{aligned} \quad (13')$$

where a_1 is an arbitrary real number. But it is shown later that this arbitrariness is not essential in our problem and we use (13) hereafter. From equations (6), (7) and (8)

$$(k'_1, k'_2 | R | k_1, k_2) = \lim_{\substack{t_1 \rightarrow +\infty \\ t_2 \rightarrow +\infty}} \iint \chi_0^*(x'_1, x'_2) \{ \Psi(x_1, x_2) - \Psi_0(x_1, x_2) \} dx_1 dx_2 \quad (15)$$

By inserting (12) into the right side of (15) and performing integrations over dX we obtain

$$\begin{aligned} (k'_1, k'_2 | R | k_1, k_2) &= (2\pi)^3 \delta(\mathbf{P}' - \mathbf{P}) \lim_{T \rightarrow +\infty} \exp\{i(P'_0 - P_0)T\} \\ &\cdot \int \exp\left\{-i\left(k'_1 - \frac{\mathbf{P}'}{2}\right) \cdot x\right\} \cdot \chi_0^*(k'_1, \mathbf{P}' - k'_1) \{ \varphi(x)_{t=0} - \varphi_0(x)_{t=0} \} dx \quad (16) \end{aligned}$$

Making use of the identity

$$\lim_{\tau \rightarrow +\infty} \exp(i\tau x) \cdot \delta_+(x) = \delta(x) \quad (17)$$

$$\text{and} \quad (\mathbf{k}'_1, \mathbf{k}'_2 | R | \mathbf{k}_1, \mathbf{k}_2) = \delta(\mathbf{P}' - \mathbf{P}) \delta(P'_0 - P_0) (\mathbf{k}'_1, \mathbf{k}'_2 | R | \mathbf{k}_1, \mathbf{k}_2), \quad (18)$$

we obtain from (16)

$$\begin{aligned} \delta_+(P'_0 - P_0) (\mathbf{k}'_1, \mathbf{k}'_2 | R | \mathbf{k}_1, \mathbf{k}_2) &= (2\pi)^3 \int \exp \left\{ -i \left(\mathbf{k}'_1 - \frac{\mathbf{P}'}{2} \right) \cdot \mathbf{x} \right\} \chi_0^*(\mathbf{k}'_1, \mathbf{P}' - \mathbf{k}'_1) \\ &\quad \{ \varphi(x)_{t=0} - \varphi_0(x)_{t=0} \} d\mathbf{x} \end{aligned} \quad (19)$$

Now we want to know the asymptotic form of $\varphi(x)_{t=0}$ for large r from (19). Because of the complexity of the diagonalization of R in the case of spinor particles we investigate it in the below from the corresponding one for scalar particles;

$$\begin{aligned} \delta_+(P'_0 - P_0) (\mathbf{k}'_1, \mathbf{k}'_2 | R | \mathbf{k}'_1, \mathbf{k}'_2) &= (2\pi)^3 \int \exp \left\{ -i \left(\mathbf{k}'_1 - \frac{\mathbf{P}'}{2} \right) \cdot \mathbf{x} \right\} \\ &\quad \{ \varphi(x)_{t=0} - \varphi_0(x)_{t=0} \} d\mathbf{x} \end{aligned} \quad (19')$$

The essential point of the following arguments, however, is to remain also in the case of spinor particles. After Fourier transformation of (19') in the center of mass system,

$$\varphi(x)_{t=0} = \varphi_0(x)_{t=0} + \int \exp(i\mathbf{k}' \cdot \mathbf{x}') \cdot \delta_+(P'_0 - P_0) (\mathbf{k}', -\mathbf{k}' | R | \mathbf{k}, -\mathbf{k}) d\mathbf{k}' \quad (20)$$

On the other hand, from (4) and (12) $\varphi(x)$ satisfies an equation

$$\varphi(x) = \varphi_0(x) + \int L(x, x') \varphi(x') d\tau', \quad (21)$$

with known $L(x, x')$ and $\varphi_0(x)$:

$$\varphi_0(x) = \exp \left\{ i\mathbf{k} \cdot \mathbf{x} - \frac{i}{2} (\sqrt{\mathbf{k}^2 + m_1^2} - \sqrt{\mathbf{k}^2 + m_2^2}) t \right\} \quad (22)$$

We introduce polar coordinates for \mathbf{x} , \mathbf{k} and \mathbf{k}' as follows:

$$\left. \begin{aligned} \mathbf{x} &\equiv (r \sin \theta \cos \Phi, r \sin \theta \sin \Phi, r \cos \theta), \\ \mathbf{k} &\equiv (k \sin \theta \cos \phi, k \sin \theta \sin \phi, k \cos \theta), \\ \mathbf{k}' &\equiv (k' \sin \theta' \cos \phi', k' \sin \theta' \sin \phi', k' \cos \theta'), \end{aligned} \right\} \quad (23)$$

and expand $\varphi(x)$ and $\varphi_0(x)$ by spherical harmonics $Y_{l,m}(\cos \theta, \Phi)$:

$$\varphi(x) = \sum_{l=0}^{\infty} \sum_{-l \leq m \leq l} f_{l,m}(r, t) Y_{l,m}(\cos \theta, \Phi), \quad (24)$$

$$\begin{aligned} \varphi_0(x) &= 4\pi \sum_{l=0}^{\infty} \sum_{-l \leq m \leq l} (i)^l (-1)^m \sqrt{\frac{\pi}{2kr}} J_{l+\frac{1}{2}}(kr) Y_{l,m}(\cos \theta, \Phi) Y_{l,-m}(\cos \theta, \phi) \\ &\quad \exp \left\{ -\frac{i}{2} (\sqrt{k^2 + m_1^2} - \sqrt{k^2 + m_2^2}) t \right\}. \end{aligned} \quad (25)$$

Then $f_{l,m}(r, t)$ satisfies an equation

$$f_{l,m}(r, t) = f_{l,m}^0(r, t) + \int \mathfrak{L}_{l,m}(r, t; r', t') f_{l,m}(r', t') dr' dt', \quad (26)$$

with known $\mathfrak{L}_{l,m}(r, t; r', t')$ and $f_{l,m}^0(r, t)$:

$$f_{l,m}^0(r, t) = 4\pi(i)^l(-1)^m \sqrt{\frac{\pi}{2kr}} J_{l+\frac{1}{2}}(kr) \exp\left\{-\frac{i}{2}(\sqrt{k^2+m_1^2}-\sqrt{k^2+m_2^2})t\right\}$$

Now when we multiply the both sides of (20) by $Y_{l,m}(\cos \theta, \phi)$ and integrate over angles we obtain

$$\begin{aligned} f_{l,m}(r, o) = & 4\pi(i)^l(-1)^m Y_{l,-m}(\cos \theta, \phi) \left[\sqrt{\frac{\pi}{2kr}} J_{l+\frac{1}{2}}(kr) + \right. \\ & \left. + \int \sqrt{\frac{\pi}{2k'r}} J_{l+\frac{1}{2}}(k'r) \delta_+(\sqrt{k'^2+m_1^2} + \sqrt{k'^2+m_2^2} - \sqrt{k^2+m_1^2} - \sqrt{k^2+m_2^2}) \right. \\ & \left. \sqrt{\Delta(k')} \sqrt{\Delta(k)} (l, m | R | l, m) dk' \right] \end{aligned} \quad (28)$$

where $\sqrt{\Delta(k)}$ and $\sqrt{\Delta(k')}$ are what result from Jacobian for transformations of $(\mathbf{k}_1, \mathbf{k}_2)$ -to $(\mathbf{P}, P_0, \cos \theta, \phi)$ -representation,

$$\Delta(k) = \frac{\sqrt{k^2+m_1^2} + \sqrt{k^2+m_2^2}}{k \sqrt{k^2+m_1^2} \sqrt{k^2+m_2^2}} \quad (29)$$

Also the transformation functions

$$(\cos \theta', \phi' | l, m) = Y_{l,m}(\cos \theta', \phi') \quad (30)$$

was used.

Since for large r

$$J_{l+\frac{1}{2}}(kr) \sim \frac{1}{2} \sqrt{\frac{2}{\pi kr}} \left[\exp\left\{i\left(kr - \frac{l+1}{2}\pi\right)\right\} + \exp\left\{-i\left(kr - \frac{l+1}{2}\pi\right)\right\} \right], \quad (31)$$

and

$$\left. \begin{aligned} \int_0^\infty \exp(ik'r) \cdot g(k') \delta_+(A) dk' &\sim \frac{\exp(i\bar{k}'r) f(\bar{k}')}{\frac{\partial A(k')}{\partial k'}} \\ \int_0^\infty \exp(-ik'r) \cdot g(k') \delta_+(A) dk' &\sim 0, \quad A(\bar{k}') = 0. \end{aligned} \right\} \quad (32)$$

(28) gives the asymptotic form of $f_{l,m}(r, o)$ for large r ;

$$\begin{aligned} f_{l,m}(r, o) \sim & 2\pi(i)^l(-1)^m Y_{l,-m}(\cos \theta, \phi) (kr)^{-1} \left[\exp\left\{-i\left(kr - \frac{l+1}{2}\pi\right)\right\} \right. \\ & \left. + \exp\left\{i\left(kr - \frac{l+1}{2}\pi\right)\right\} (l, m | S | l, m) \right] \end{aligned} \quad (33)$$

Now we define new functions $f'_{l,m}(x)$ by

$$f'_{l,m}(x) \equiv f_{l,m}(x) (l, m | S | l, m)^{-1}, \quad (34)$$

which satisfies an equation

$$f'_{l,m}(r, t) = f_{l,m}^0(r, t) (l, m | S | l, m)^{-1} + \int \mathfrak{L}_{l,m}(r, t; r', t') f'_{l,m}(r', t') dr' dt' \quad (35)$$

Then from (32) the asymptotic form of $f'_{l,m}(r, o)$ is given by

$$f'_{l,m}(r, o) \sim 2\pi(i)^l (-1)^m Y_{l,-m}(\cos \theta, \phi) (kr)^{-1} \left[\exp \left\{ -i \left(kr - \frac{l+1}{2} \pi \right) \right\} \cdot (l, m | S | l, m)^{-1} + \exp \left\{ i \left(kr - \frac{l+1}{2} \pi \right) \right\} \right] \quad (36)$$

We assume that $f'_{l,m}(r, t)$ is an analytic function of k and extend the original interval of k ($0 \leq k < \infty$) to a complex k -plane, and consider the analytic continuation of $f'_{l,m}(r, t)$ at poles \underline{k} of $(l, m | S | l, m)$ on the upper half k -plane, which we write $\underline{f}'_{l,m}(r, t)$. Then $\underline{f}'_{l,m}(r, t)$ satisfies the homogeneous integral equation;

$$\underline{f}'_{l,m}(r, t) = \int \mathfrak{L}_{l,m}(r, t; r', t') \underline{f}'_{l,m}(r', t') dr' dt', \quad (37)$$

and the asymptotic form of $\underline{f}'_{l,m}(r, o)$ is given by

$$\underline{f}'_{l,m}(r, o) \sim 2\pi(i)^l (-1)^m Y_{l,-m}(\cos \theta, \phi) (\underline{k}r)^{-1} \exp(-\underline{k}r) \exp \left\{ i \left(\underline{k}r - \frac{l+1}{2} \pi \right) \right\}, \quad (38)$$

where

$$\underline{k} = \underline{k}'_l + i\underline{k}_l, \quad (\underline{k}_l > 0)$$

In the S -matrix theory $(l, m | S | l, m)$ is independent of quantum number m , and poles of $(l, m | S | l, m)$ on the positive imaginary axis of k -plane give the discrete energy values of bound states in the center of mass system by the relation

$$\underline{E} = \sqrt{\underline{k}_l^2 + m_1^2} + \sqrt{\underline{k}_l^2 + m_2^2}, \quad (39)$$

$$(\underline{k}'_l = 0).$$

So, equation (37) or

$$\Psi(x_1, x_2) = \iint G(x_1, x_2; x'_1, x'_2) \Psi(x'_1, x'_2) d\tau'_1 d\tau'_2 \quad (40)$$

determines the discrete energy values of bound systems. It is also clear from the above arguments that the energy values are determined independent of any choice of the definition (13') of X .

The complex numbers may exist among the eigen-values of E which are to be determined by solving (40). These circumstances will be investigated in future in connection with decay processes.

Of course the above arguments are only formal, because it contains divergencies inherent in all quantum field theories. But in quantum electrodynamics and in some other systems divergence-free S -matrices can be obtained by the so-called renormalization procedures after Dyson⁽⁵⁾. Therefore our corresponding formulation also can be made by replacing the

propagation functions of fields and γ -matrices by the modified ones respectively.

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On the Rate of Irreversible Production of Entropy*

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In the case of some gaseous substances (a simple gas, a gaseous mixture of reacting molecules, and free electrons in metals) the equation of entropy transport is derived from the Boltzmann's kinetic theory. The definition of entropy used in this derivation is the one given by the Boltzmann's H -functional.

The expression of the rate of irreversible entropy production appearing in the above-mentioned equation is essentially the same as that given by Tolman-Fine and others in their thermodynamical theories.

§ 1. Introduction

Eckart,¹⁾ Bridgman,²⁾ Tolman-Fine,³⁾ Meixner,⁴⁾ Sugita⁵⁾ and others have recently given various attempts to express the second law of thermodynamics in terms of an equality in order to obtain concrete conclusions on some irreversible processes. They gave due recognition to the irreversible entropy production associated with such processes on such fundamental assumption that every irreversible process consists of some elementary ones, each of which produces its own characteristic amount of entropy irrespective of the existence and of the nature of the other ones. As pointed out by Tolman-Fine and independently by Sugita, the principles of these theories lies in computing beforehand the characteristic amount of local entropy production for each elementary irreversible process regarded as quasi-static in its extended sense, i.e., at each stage of which each small part of the system is substantially in a state of metastable equilibrium.

In the case of gases their results are as follows. The local entropy per unit volume of the gas, S , satisfies the equation

$$\partial S / \partial t + \partial S_{total} / \partial x = (dS/dt)_{irr}, \quad (1.1)$$

where S_{total} is the entropy flow vector, which consists of two parts, one carried by the substance itself and the other by the heat flow:

$$S_{total} = Sv + S, \quad (1.2)$$

$$S = Q/T. \quad (1.3)$$

Here v denotes the flow velocity, Q the heat flow vector, and T the local temperature of the gas. $(dS/dt)_{irr}$ is the rate of irreversible production of entropy, i.e., the amount of entropy produced per unit time and per unit volume inside each small element of the

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gas. It is composed of the contributions from the elementary irreversible processes taking place in the gas. For example, the rate of irreversible production of entropy due to the irreversible heat flow is

$$(dS/dt)_{irr.} = \mathbf{Q} \cdot \partial/\partial \mathbf{x} (1/T). \quad (1.4)$$

And those due to the dissipation of mechanical energy through the action of viscosity or of electrical energy by the flow of current through a resistance are all of the form

$$(dS/dt)_{irr.} = 1/T (dF/dt)_{irr.}, \quad (1.5)$$

where $(dF/dt)_{irr.}$ is the rate at which work is done against viscous force, etc. It was interpreted by Sugita⁵⁾ as the heat supplied per unit time and per unit volume by virtual heat reservoirs. In the case of viscous flow we have

$$(dF/dt)_{irr.} = (p\mathbf{1} - \mathbf{P}) : \partial \mathbf{v} / \partial \mathbf{x}, \quad (1.6)$$

where p , $\mathbf{1}$ and \mathbf{P} denote the pressure, the unit tensor, and the pressure tensor respectively. When an electric current \mathbf{J} flows through a resistance R at temperature T , we obtain the expression

$$(dF/dt)_{irr.} = \mathbf{J}^2 R. \quad (1.7)$$

The equation (1.5) can also be used for chemical reactions. If we assume the reaction of the type $A + B \rightleftharpoons C + D$, then the expression

$$\left(\frac{d}{dt} F \right)_{irr.} = k \left\{ \log K - \log \frac{C_C C_D}{C_A C_B} \right\} \frac{dx}{dt} \quad (1.8)$$

will be valid. Here k is the Boltzmann's constant, K the equilibrium constant, and C^i are the number concentration of the molecules. dx/dt denotes the net rate of the reaction represented in terms of the numbers of molecules A , for example, used up in the reaction per unit time and per unit volume.

When there are various kinds of elementary irreversible processes taking place at the same time, the rate of irreversible entropy production for the over-all process can be obtained simply by summing up the rates associated with each of them. Accordingly the possible correlations among them are not taken into consideration except those through the variables \mathbf{Q} , \mathbf{v} , \mathbf{J} , T , etc. appearing in some of the summands.

The usual expression for the second law of thermodynamics is of the form

$$(dS/dt)_{irr.} \geq 0, \quad (1.9)$$

or of the form

$$ds/dt \geq -1/\rho \partial S / \partial \mathbf{x}, \quad (1.10)$$

s and ρ being the entropy per unit mass and the mass density of the gas respectively. Here we defined $s \equiv S/\rho$, and $d/dt \equiv \partial/\partial t + \mathbf{v} \cdot \partial/\partial \mathbf{x}$, and used the equation of continuity for the mass. This inequality (1.10) was changed into the equality (1.1) by virtue of the idea that every elementary irreversible process produces independently the entropy increase given by the formulae (1.4), (1.8), etc.

From the molecular standpoint, however, every irreversible process takes its rise in some forms of the irreversible molecular motion, thus it will be possible to prove the phenomenological thermodynamic relations from the molecular kinetic theories. Moreover such a proof will show that the validity of these phenomenological relations does not depend upon the details of the molecular mechanism, since they were derived from some general thermodynamical considerations. In the following sections we shall examine these anticipations by a few examples.

§ 2. Irreversible heat flow and viscous force

Firstly, we consider a simple gas. If we neglect the difference of molecular structures, each state of a molecule is specified by its position \mathbf{x} and momentum \mathbf{p} and each state of the gas as a whole by a distribution function $f(\mathbf{x}, \mathbf{p}, t)$ in the phase space, t being the time. According to Boltzmann, the distribution function for a sufficiently rarefied gas satisfies the equation of continuity

$$\left\{ \partial / \partial t + \mathbf{p} / m \cdot \partial / \partial \mathbf{x} + m \mathbf{F}(\mathbf{x}) \cdot \partial / \partial \mathbf{p} \right\} f = (df/dt)_{irr.}, \quad (2.1)$$

where m denotes the molecular mass, \mathbf{F} an external force acting on unit mass of the gas which is assumed to be dependent only on the position. $(df/dt)_{irr.}$ on the right-hand side is the term representing the irreversible change of f , say, due to molecular collisions, the explicit form of which will not be employed for the time being.

The macroscopic quantity per unit volume of the gas corresponding to an arbitrary dynamical variable per unit mass, $\hat{\epsilon}(\mathbf{x}, \mathbf{p}, t)$, is defined by

$$\Xi \equiv \int m \hat{\epsilon} f d\tau_{\mathbf{p}};$$

where $d\tau_{\mathbf{p}}$ denotes either the volume element of momentum space in classical statistics, or its quantum mechanically semi-corrected value $G d\mathbf{p} / h^3$, G being a weight factor due to the molecular structure and h the Planck's constant. The integration is to be taken over the whole momentum space. If we introduce the associated quantities

$$\Xi \equiv \int \mathbf{p} \hat{\epsilon} f d\tau_{\mathbf{p}},$$

$$F_{\Xi} \equiv \int m f \left\{ \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{x}} + m \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{p}} \right\} \hat{\epsilon} d\tau_{\mathbf{p}},$$

and

$$\left(\frac{d}{dt} \Xi \right)_{irr.} \equiv \int m \hat{\epsilon} \left(\frac{d}{dt} f \right)_{irr.} d\tau_{\mathbf{p}}, \quad (2.2)$$

we obtain an equation of continuity⁶⁾ for Ξ

$$\partial \Xi / \partial t + \partial \Xi / \partial \mathbf{x} = F_{\Xi} + (d\Xi/dt)_{irr.} \quad (2.2)$$

Especially, the equations of continuity of mass, of momentum, and of energy are given by

$$\partial \rho / \partial t + \rho \partial \mathbf{v} / \partial \mathbf{x} = (d\rho/dt)_{irr.},$$

$$\rho \, dv/dt + \partial \mathbf{P} / \partial \mathbf{x} = \rho \mathbf{F} + (d\mathbf{j}/dt)_{irr.}, \quad (2.3)$$

$$\rho \, du/dt + \mathbf{P} : \partial \mathbf{v} / \partial \mathbf{x} + \partial \mathbf{Q} / \partial \mathbf{x} = dE/dt_{irr.},$$

writing m , \mathbf{p} , and $\mathbf{p}^2/2m$ in turn for $m\hat{\epsilon}$. The expressions

$$\left(\frac{d}{dt} \rho \right)_{irr.} \equiv \int m \left(\frac{d}{dt} f \right)_{irr.} d\tau_{\mathbf{p}},$$

$$\left(\frac{d}{dt} \mathbf{j} \right)_{irr.} \equiv \int \mathbf{p} \left(\frac{d}{dt} f \right)_{irr.} d\tau_{\mathbf{p}},$$

and

$$\left(\frac{d}{dt} E \right)_{irr.} \equiv \int \frac{\mathbf{p}^2}{2m} \left(\frac{d}{dt} f \right)_{irr.} d\tau_{\mathbf{p}}$$

mean respectively the amount of mass, momentum, and energy produced per unit time and per unit volume inside the substance by the irreversible molecular mechanism which is represented by $(df/dt)_{irr.}$. The quantity $(dE/dt)_{irr.}$ should not be confused with the one in the expression (1.5), $(dF/dt)_{irr.}$. The following expressions are also defined:

$$\rho \equiv mn \equiv \int m f d\tau_{\mathbf{p}}, \quad \mathbf{j} \equiv \rho \mathbf{v} \equiv \int \mathbf{p} f d\tau_{\mathbf{p}},$$

$$U \equiv \rho u \equiv \int \frac{(\mathbf{p} - m\mathbf{v})^2}{2m} f d\tau_{\mathbf{p}}, \quad \mathbf{P} \equiv \int \frac{(\mathbf{p} - m\mathbf{v})(\mathbf{p} - m\mathbf{v})}{m} f d\tau_{\mathbf{p}},$$

$$\mathbf{Q} \equiv \int \frac{(\mathbf{p} - m\mathbf{v})^2}{2m} \left(\frac{\mathbf{p} - \mathbf{v}}{m} \right) f d\tau_{\mathbf{p}}.$$

Here u , say, is the internal energy (the energy of thermal agitation) per unit mass.

This method of derivation of equations (2.3) and the general anticipations stated in the last paragraph of § 1 suggest that the equation of continuity for the entropy (the equation of entropy transport) will also be deduced from the equation (2.2). For this purpose we put

$$m\hat{\epsilon} = -k \left\{ \log f - \frac{1+\theta f}{\theta f} \log(1+\theta f) \right\},$$

anticipating that the molecular interpretation of entropy at a state of equilibrium will be utilized also for a non-equilibrium state, as pointed out by Tolman and Fine, provided that it is not so much apart from equilibrium. In the preceding expression, θ should have the value $+1$ or -1 according to Bose or Fermi statistics respectively, and 0 in the case of classical statistics. Then the equation (2.2) becomes

$$\partial S / \partial t + \partial \mathbf{S}_{total} / \partial \mathbf{x} = F_s + (dS/dt)_{irr.}, \quad (2.4)$$

where

$$S \equiv -k \int \{ f \log f - \theta^{-1} (1 + \theta f) \log(1 + \theta f) \} d\tau_{\mathbf{p}},$$

$$\mathbf{S} \equiv \mathbf{S}_{total} - S\mathbf{v} \quad [\text{cf. Eq. (1.2)}]$$

$$\equiv -k \int \left(\frac{\mathbf{p} - \mathbf{v}}{m} \right) \{ f \log f - \theta^{-1} (1 + \theta f) \log(1 + \theta f) \} d\tau_{\mathbf{p}},$$

$$\begin{aligned}
 F_s &\equiv -k \int f \left\{ \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{x}} + m \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{p}} \right\} \{ \log f - (\theta f)^{-1} (1 + \theta f) \log (1 + \theta f) \} d\tau_{\mathbf{p}} \\
 &= -k \int \frac{\log (1 + \theta f)}{\theta f} \left(\frac{d}{dt} f \right)_{irr.} d\tau_{\mathbf{p}}, \\
 \left(\frac{d}{dt} S \right)_{irr.} + F_s &\equiv -k \int \left\{ \log \frac{f}{1 + \theta f} - \frac{\log (1 + \theta f)}{\theta f} \right\} \left(\frac{d}{dt} f \right)_{irr.} d\tau_{\mathbf{p}} + F_s \\
 &= -k \int \log \left(\frac{f}{1 + \theta f} \right) \left(\frac{d}{dt} f \right)_{irr.} d\tau_{\mathbf{p}}.
 \end{aligned}$$

As the terms $(d\rho/dt)_{irr.}$, $(d\mathbf{j}/dt)_{irr.}$, $(dE/dt)_{irr.}$, and F_s in the equations (2.3) and (2.4) are formally redundant in comparison with the ordinary hydrodynamical equations, we must evaluate their values. For this purpose, let us assume the distribution function to be of the form

$$\begin{aligned}
 f &\equiv 1 \left/ \exp \left\{ \frac{(\mathbf{p} - m\mathbf{v})^2 / 2m + W - \mu^0}{kT} \right\} - \theta \right. \\
 &\cong f^0 \left\{ 1 - (1 + \theta f^0) \frac{W}{kT} \right\}, \quad (|W| \ll kT).
 \end{aligned} \quad (2.5)$$

Here the distribution function corresponding to the state of local equilibrium, f^0 , is defined by

$$\begin{aligned}
 n &= \int f d\tau_{\mathbf{p}} = \int f^0 d\tau_{\mathbf{p}}, \\
 \rho \mathbf{v} &= \int \mathbf{p} f d\tau_{\mathbf{p}} = \int \mathbf{p} f^0 d\tau_{\mathbf{p}}, \\
 U &= \int \frac{(\mathbf{p} - m\mathbf{v})^2}{2m} f d\tau_{\mathbf{p}} = \int \frac{(\mathbf{p} - m\mathbf{v})^2}{2m} f^0 d\tau_{\mathbf{p}}.
 \end{aligned}$$

The unknown quantity W representing the deviation of f from f^0 is of course determined by the Boltzmann's equation (2.1). We may then use an approximate equation, in which the terms higher than the squares of $|W|/kT$ are all neglected,

$$\begin{aligned}
 \left(\frac{d}{dt} f \right)_{irr.} &\cong \left\{ \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{x}} + m \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{p}} \right\} f^0 \\
 &\cong \frac{f^0 (1 + \theta f^0)}{kT} \left\{ \left[\frac{(\mathbf{p} - m\mathbf{v})(\mathbf{p} - m\mathbf{v})}{m} - \frac{1}{3} \frac{(\mathbf{p} - m\mathbf{v})^2}{m} \mathbf{1} \right] : \frac{\partial}{\partial \mathbf{x}} \mathbf{v} \right. \\
 &\quad \left. + \left[\frac{(\mathbf{p} - m\mathbf{v})^2}{2m} - \frac{5}{2} \frac{\mathbf{p}}{n} \right] \left(\frac{\mathbf{p}}{m} - \mathbf{v} \right) \cdot \frac{\partial}{\partial \mathbf{x}} \log T \right\}.
 \end{aligned} \quad (2.6)^7$$

The assumptions (2.5) and (2.6) are the exact formulation of a quasi-static process in its extended sense stated in the first paragraph of § 1. Using the equation (2.6), we can easily show that

$$(d\rho/dt)_{irr.} \cong 0, \quad (d\mathbf{j}/dt)_{irr.} \cong 0, \quad (dE/dt)_{irr.} \cong 0. \quad (2.7)$$

In reality, the explicit structure of $(df/dt)_{irr.}$ must have the physical properties expressed

by the equations (2.7) (conservation laws) in order that the approximation (2.6) can be used. It is well known that this approximation gives good results for the case of sufficiently rarefied gases⁽⁸⁾. Using the explicit form of the distribution function (2.5), we obtain the expression

$$\begin{aligned} \left(\frac{d}{dt} S \right)_{irr.} + F_s &= \frac{1}{T} \int \left\{ \frac{(\mathbf{p} - m\mathbf{v})^2}{2m} + W - \mu^0 \right\} \left(\frac{d}{dt} f \right)_{irr.} d\tau_{\mathbf{p}} \\ &= \frac{1}{T} \left\{ \left(\frac{d}{dt} E \right)_{irr.} - \mu^0 \left(\frac{d}{dt} n \right)_{irr.} + \int W \left(\frac{d}{dt} f \right)_{irr.} d\tau_{\mathbf{p}} \right\} \\ &\cong \frac{1}{T} \int W \left(\frac{d}{dt} f \right)_{irr.} d\tau_{\mathbf{p}}, \end{aligned} \quad (2.8)$$

where $(dn/dt)_{irr.} \equiv (1/m)(d\rho/dt)_{irr.}$. The expression (2.8) is to be compared with the equation (1.5). Using further the approximate equation (2.6), we obtain the final result

$$\begin{aligned} \left(\frac{d}{dt} S \right)_{irr.} + F_s &\cong \frac{1}{T} \int f^0 (1 + \theta f^0) \frac{W}{kT} \left\{ \left[\frac{(\mathbf{p} - m\mathbf{v})(\mathbf{p} - m\mathbf{v})}{m} - \frac{1}{3} \frac{(\mathbf{p} - m\mathbf{v})^2}{m} \mathbf{1} \right] : \frac{\partial}{\partial \mathbf{x}} \mathbf{v} \right. \\ &\quad \left. + \left[\frac{(\mathbf{p} - m\mathbf{v})^2}{2m} - \frac{5}{2} \frac{p}{m} \right] \left(\frac{\mathbf{p}}{m - \mathbf{v}} \right) \cdot \frac{\partial}{\partial \mathbf{x}} \log T \right\} d\tau_{\mathbf{p}} \\ &\cong \frac{1}{T} \int (f^0 - f) \{ \dots \} d\tau_{\mathbf{p}} = \frac{1}{T} \left\{ (p^{\mathbf{1}-\mathbf{p}}) : \frac{\partial}{\partial \mathbf{x}} \mathbf{v} - \mathbf{Q} \cdot \frac{\partial}{\partial \mathbf{x}} \log T \right\}, \end{aligned} \quad (2.9)$$

where the equation $\mathbf{P}^0 = p\mathbf{1}$ and $\mathbf{Q}^0 = 0$ have been substituted. Except for the term F_s in the left hand side the equation (2.9) may be seen to be essentially the same as the expressions (1.4) and (1.6), if we compare the equation of continuity (2.4) with the equation (1.1). In the classical case, F_s becomes

$$-k \int \left(\frac{d}{dt} f \right)_{irr.} d\tau_{\mathbf{p}} = -\frac{k}{m} \left(\frac{d}{dt} \rho \right)_{irr.} \cong 0.$$

The equation (1.3) is now easily obtained, using the assumption (2.5):

$$\begin{aligned} \mathbf{S} &\cong k \int \left(\frac{\mathbf{p}}{m} - \mathbf{v} \right) f^0 (1 + \theta f^0) \frac{W}{kT} \log \left(\frac{f^0}{1 + \theta f^0} \right) d\tau_{\mathbf{p}} \\ &\cong k \int (f^0 - f) \left(\frac{\mathbf{p}}{m} - \mathbf{v} \right) \frac{\mu^0 - (\mathbf{p} - m\mathbf{v})^2 / 2m}{kT} d\tau_{\mathbf{p}} \\ &= \mathbf{Q} / T, \end{aligned} \quad (2.10)$$

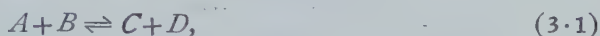
which, together with the equations (2.3), (2.4), (2.7), and (2.9), gives the relation between u and s

$$du/dt \cong T ds/dt - p d(1/\rho)/dt. \quad (2.11)$$

This relation is the one commonly used in thermodynamics, i.e., the Gibbs' relation.

§ 3. Chemical reaction and mutual friction

Now we consider a gaseous mixture consisting of, say, four kinds of molecules, $i=A, B, C$, and D . When there occurs a chemical reaction of the type



we must in general take into account energy levels of the internal motion of each molecule. Let us denote them by $\epsilon_{i\gamma}$, γ being a set of quantum numbers specifying each state of the internal motion. For the sake of simplicity, we shall restrict ourselves to the pure classical case. Denoting the distribution function for the i -th gas by $f_i(\mathbf{x}, \mathbf{p}, \gamma, t)$, the Boltzmann's equation of continuity becomes the simultaneous equations with respect to the functions f_A, f_B, f_C and f_D , in which the quantities m, \mathbf{F} , and f in the equation (2.1) are replaced by m_i, \mathbf{F}_i , and f_i respectively and

$$(df_i/dt)_{irr.} = (df_i/dt)_{coll.} + (df_i/dt)_{chem.} \quad (3.2)$$

is used. Here $(df_i/dt)_{coll.}$ denotes the term due to the non-reactive molecular collisions and $(df_i/dt)_{chem.}$ that due to those which cause the reaction. Each equation of continuity associated with a macroscopic quantity defined by

$$\Xi_i \equiv \sum_{\gamma} \int m_i \hat{\epsilon} f_i d\mathbf{p}$$

is obtained by the similar procedure as what was used to obtain the equation (2.2). Corresponding to (3.2), we have the following equations

$$\left(\frac{d}{dt} \Xi_i \right)_{irr.} = \left(\frac{d}{dt} \Xi_i \right)_{coll.} + \left(\frac{d}{dt} \Xi_i \right)_{chem.} \quad (3.3)$$

where $(d\Xi_i/dt)_{coll.}$, say, is defined by using $(df_i/dt)_{coll.}$.

Using the dynamical variables $m_i, \mathbf{p}, \mathbf{p}^2/2m_i + \epsilon_{i\gamma}$, and $-k(\log f_i - 1)$ in place of $m_i \hat{\epsilon}$, we obtain the equations of continuity for the partial density of the mass ρ_i , of the current \mathbf{j}_i , of the energy E_i , and of the entropy S_i respectively:

$$\left. \begin{aligned} \partial \rho_i / \partial t + \partial \mathbf{j}_i / \partial \mathbf{x} &= (d\rho_i/dt)_{irr.}, \\ \partial \mathbf{j}_i / \partial t + \partial \mathbf{P}_{total,i} / \partial \mathbf{x} &= \rho_i \mathbf{F}_i + (d\mathbf{j}_i/dt)_{irr.}, \\ \partial E_i / \partial t + \partial \mathbf{Q}_{total,i} / \partial \mathbf{x} &= \mathbf{j}_i \cdot \mathbf{F}_i + (dE_i/dt)_{irr.}, \\ \partial S_i / \partial t + \partial \mathbf{S}_{total,i} / \partial \mathbf{x} &= F_{S,i} + (dS_i/dt)_{irr.}, \end{aligned} \right\} \quad (3.4)$$

where

$$\begin{aligned} \rho_i &\equiv m_i n_i \equiv \sum_{\gamma} \int m_i f_i d\mathbf{p}, \quad \rho \equiv \sum_i \rho_i; \\ \mathbf{j}_i &\equiv \rho_i \mathbf{v}_i \equiv \sum_{\gamma} \int \mathbf{p} f_i d\mathbf{p}, \quad \mathbf{j} \equiv \rho \mathbf{v} \equiv \sum_i \mathbf{j}_i; \\ \mathbf{P}_{total,i} &= \rho_i (\mathbf{v}_i \mathbf{v} + \mathbf{v} \mathbf{v}_i - \mathbf{v} \mathbf{v}) + \mathbf{P}_i, \\ \mathbf{P}_i &\equiv \sum_{\gamma} \int \frac{(\mathbf{p} - m_i \mathbf{v})(\mathbf{p} - m_i \mathbf{v})}{m_i} f_i d\mathbf{p}; \end{aligned}$$

$$\begin{aligned}
 E_i &= \frac{1}{2} \rho_i (\mathbf{v}_i \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}_i - \mathbf{v} \cdot \mathbf{v}) + U_i, \\
 U &\equiv \sum_{\tau} \int \left\{ \frac{(\mathbf{p} - m_i \mathbf{v})^2}{2m_i} + \epsilon_{i\tau} \right\} f_i d\mathbf{p}, \\
 Q_{total,i} &= (E_i \mathbf{1} + \mathbf{P}_i) \cdot \mathbf{v} + \frac{1}{2} \rho_i v^2 (\mathbf{v}_i - \mathbf{v}) + Q_i, \\
 Q_i &\equiv \sum_{\tau} \int \left\{ \frac{(\mathbf{p} - m_i \mathbf{v})^2}{2m_i} + \epsilon_{i\tau} \right\} \left(\frac{\mathbf{p}}{m_i} - \mathbf{v} \right) f_i d\mathbf{p}; \\
 S_i &\equiv -k \sum_{\tau} \int f_i \{\log f_i - 1\} d\mathbf{p}, \\
 S_{total,i} &= S_i \mathbf{v} + S_i, \\
 S_i &\equiv -k \sum_{\tau} \int \left(\frac{\mathbf{p}}{m_i} - \mathbf{v} \right) f_i \{\log f_i - 1\} d\mathbf{p}, \\
 F_{S,i} &= -\frac{k}{m_i} \left(\frac{d}{dt} \rho_i \right)_{irr.} \equiv -k \left(\frac{d}{dt} n_i \right)_{irr.}
 \end{aligned}$$

In the next place we assume the explicit form of the distribution functions

$$\begin{aligned}
 f_i &\equiv \exp \frac{\mu_i^0 - (\mathbf{p} - m_i \mathbf{v})^2 / 2m_i - \epsilon_{i\tau} - W_i}{kT} \\
 &\equiv f_i^0 e^{-W_i/kT} \cong f_i^0 \left\{ 1 - \frac{W_i}{kT} \right\}, \quad (|W_i| \ll kT)
 \end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
 n_i &= \sum_{\tau} \int f_i d\mathbf{p} = \sum_{\tau} \int f_i^0 d\mathbf{p}, \\
 \rho \mathbf{v} &= \sum_{i,\tau} \int \mathbf{p} f_i d\mathbf{p} = \sum_{i,\tau} \int \mathbf{p} f_i^0 d\mathbf{p}, \\
 U &= \sum_{i,\tau} \int \left\{ \frac{(\mathbf{p} - m_i \mathbf{v})^2}{2m_i} + \epsilon_{i\tau} \right\} f_i d\mathbf{p} = \sum_{i,\tau} \int \left\{ \frac{(\mathbf{p} - m_i \mathbf{v})^2}{2m_i} + \epsilon_{i\tau} \right\} f_i^0 d\mathbf{p}.
 \end{aligned}$$

It must be noted here that, according to this definition of f_i^0 , the detailed balancing with respect to the reaction (3.1) does not generally hold. The approximate equations

$$\begin{aligned}
 \left(\frac{d}{dt} f_i \right)_{irr.} &\cong \left\{ \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m_i} \cdot \frac{\partial}{\partial \mathbf{x}} + m_i \mathbf{F}_i \cdot \frac{\partial}{\partial \mathbf{p}} \right\} f_i^0 \\
 &\cong f_i^0 \left\{ \left[\frac{(\mathbf{p} - m_i \mathbf{v})(\mathbf{p} - m_i \mathbf{v})}{m_i v^2} - 1 - \frac{(\mathbf{p} - m_i \mathbf{v})^2}{2m_i} + \Delta \epsilon_{i\tau} + \frac{3}{2} a k T \right] \cdot \frac{\partial}{\partial \mathbf{x}} \mathbf{v} \right. \\
 &\quad \left. + \left[\frac{n d_i}{n_i} + \frac{(\mathbf{p} - m_i \mathbf{v})^2}{2m_i} + \Delta \epsilon_{i\tau} - \frac{5}{2} k T \right] \frac{\partial \log T}{\partial \mathbf{x}} \right\} \cdot \left(\frac{\mathbf{p}}{m_i} - \mathbf{v} \right)
 \end{aligned}$$

$$+ \frac{1}{n_i} \left(\frac{d}{dt} n_i \right)_{chem.}^0 - \frac{\left(\frac{\mathbf{p} - m_i \mathbf{v}}{2m_i} \right)^2 + \Delta \epsilon_{iT} - \frac{3}{2} kT}{\frac{3}{2} n(1+u) (kT)^2} \sum_j \bar{\epsilon}_j \left(\frac{d}{dt} n_j \right)_{chem.}^0 \left. \right\} \quad (3.6)$$

are also employed in place of the original Boltzmann's equations. Here we have used the expressions

$$\begin{aligned} \mu_i^0 &\equiv kT \log n_i \Phi_i, \quad \Phi_i \equiv (2\pi m_i kT)^{3/2} \sum_T e^{-\epsilon_{iT}/kT} \Delta_{iT}, \\ \left(\begin{array}{l} \mu_i^0: \text{the chemical potential per molecule of the } i\text{-th kinds.} \\ \Delta_{iT}: \text{a statistical weight for the energy level } \epsilon_{iT}; \end{array} \right) \\ \Delta \epsilon_{iT} &\equiv \epsilon_{iT} - \bar{\epsilon}_i, \quad \bar{\epsilon}_i \equiv \sum_T \epsilon_{iT} e^{-\epsilon_{iT}/kT} \Delta_{iT} / \sum_T e^{-\epsilon_{iT}/kT} \Delta_{iT}; \\ u &\equiv \sum_i n_i (\bar{\epsilon}_i^2 - \bar{\epsilon}_i^2) / \left\{ \frac{3}{2} n (kT)^2 \right\}; \\ p &\equiv \sum_i p_i = \sum_i n_i kT \equiv n kT, \\ \mathbf{d}_i &\equiv \sum_j \mathbf{d}_{ij}, \quad \mathbf{d}_{ij} = -\mathbf{d}_{ji} \\ &\equiv 1/n^2 (n_j \partial n_i / \partial \mathbf{x} - n_i \partial n_j / \partial \mathbf{x} + (\mathbf{F}_j - \mathbf{F}_i) \\ &\quad \cdot \rho_i \rho_j / p \rho + (m_j - m_i) n_i n_j / n p \cdot \partial \log p / \partial \mathbf{x}) \end{aligned} \quad (3.7)$$

and the two assumptions that each energy level ϵ_{iT} is independent of ρ_i , \mathbf{v} , and T and that all the irreversible quantities for the local equilibrium distribution f_i^0 vanish except for $(dn_i/dt)_{chem.}^0$. Using the expressions (3.6), we obtain the results

$$\begin{aligned} \left(\frac{d}{dt} \rho_i \right) &\equiv \sum_T \int m_i \left(\frac{d}{dt} f_i \right)_{irr.} d\mathbf{p} \cong m_i \left(\frac{d}{dt} n_i \right)_{chem.}^0 = \left(\frac{d}{dt} \rho_i \right)_{chem.}^0, \\ \left(\frac{d}{dt} \mathbf{j}_i \right) &\equiv \sum_T \int \mathbf{p} \left(\frac{d}{dt} f_i \right)_{irr.} d\mathbf{p} \cong p \mathbf{d}_i + \mathbf{v} \left(\frac{d}{dt} \rho_i \right)_{irr.}, \\ \left(\frac{d}{dt} E_i \right) &\equiv \sum_T \int \left(\frac{\mathbf{p}^2}{2m_i} + \epsilon_{iT} \right) \left(\frac{d}{dt} f_i \right)_{irr.} d\mathbf{p} \\ &= \sum_T \int \left\{ \left(\frac{\mathbf{p} - m_i \mathbf{v}}{2m_i} \right)^2 + \epsilon_{iT} \right\} \left(\frac{d}{dt} f_i \right)_{irr.} d\mathbf{p} + \mathbf{v} \cdot \left(\frac{d}{dt} \mathbf{j}_i \right)_{irr.} - \frac{1}{2} \mathbf{v}^2 \left(\frac{d}{dt} \rho_i \right)_{irr.} \\ &\cong p \mathbf{d}_i \cdot \mathbf{v} + \left(\frac{1}{2} m_i \mathbf{v}^2 + \frac{3}{2} kT + \bar{\epsilon}_i \right) \left(\frac{d}{dt} n_i \right)_{chem.}^0 \\ &\quad + \frac{\sum_j n_i n_j \{ (\bar{\epsilon}_j^2 - \bar{\epsilon}_j^2) - (\bar{\epsilon}_i^2 - \bar{\epsilon}_i^2) \}}{\frac{3}{2} p(1+u)} \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v} - \frac{n_i}{n(1+u)} \left\{ 1 + \frac{\bar{\epsilon}_i^2 - \bar{\epsilon}_i^2}{\frac{3}{2} (kT)^2} \right\} \sum_j \bar{\epsilon}_j \left(\frac{d}{dt} n_j \right)_{chem.}^0, \end{aligned}$$

and further, remembering the explicit form (3.5), we get

$$\begin{aligned}
\left(\frac{d}{dt} S_i\right)_{irr.} + F_{S,i} &\equiv -k \sum_i \int \left(\frac{d}{dt} f_i\right)_{irr.} \log f_i d\mathbf{p} \\
&= \frac{1}{T} \sum_i \int \left\{ -\frac{(\mathbf{p} - m_i \mathbf{v})^2}{2m_i} + \epsilon_{i\tau} + W_i - \mu_i^0 \right\} \left(\frac{d}{dt} f_i\right)_{irr.} d\mathbf{p} \\
&= \frac{1}{T} \left\{ \left(\frac{d}{dt} E_i\right)_{irr.} + \frac{1}{2} \left(\frac{d}{dt} \rho_i\right)_{irr.} \mathbf{v} \cdot \mathbf{v} - \left(\frac{d}{dt} \mathbf{j}_i\right)_{irr.} \cdot \mu_i^0 \left(\frac{d}{dt} n_i\right)_{irr.} \right. \\
&\quad \left. + \sum_i \int W_i \left(\frac{d}{dt} f_i\right)_{irr.} d\mathbf{p} \right\}, \tag{3.8}
\end{aligned}$$

which should be compared with the expression (2.8). If we use the preceding results and the equation (3.6), the foregoing expression can be rewritten into the form

$$\begin{aligned}
&\left(\frac{d}{dt} S_i\right)_{irr.} - k \left(\frac{d}{dt} n_i\right)_{chem.}^0 \\
&\cong \frac{1}{T} \left\{ (\mathbf{P}_i^0 - \mathbf{P}_i) : \frac{\partial}{\partial \mathbf{x}} \mathbf{v} - \mathbf{v}_i \cdot \rho d_i - \mu_i^0 \left(\frac{d}{dt} n_i\right)_{chem.}^0 \right. \\
&\quad - [Q_i - H_i^0(\mathbf{v}_i - \mathbf{v})] \cdot \frac{\partial \log T}{\partial \mathbf{x}} + \left(\frac{d}{dt} E_i\right)_{irr.} - \frac{1}{2} \left(\frac{d}{dt} \rho_i\right)_{chem.}^0 \mathbf{v}^2 \\
&\quad \left. + \frac{U_i - U_i^0}{\frac{3}{2}(1+\alpha)} \left[\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v} + \frac{1}{\rho} \sum_j \bar{\epsilon}_j \left(\frac{d}{dt} n_j\right)_{chem.}^0 \right] \right\}, \tag{3.9}
\end{aligned}$$

where $H_i^0 \equiv n_i(\bar{\epsilon}_i + \frac{5}{2}kT)$ is the partial enthalpy per unit volume.

Corresponding to the type of chemical reaction (3.1), always hold the relations

$$m_A + m_B = m_C + m_D,$$

$$(dn_A/dt)_{chem.} = (dn_B/dt)_{chem.} = -(dn_C/dt)_{chem.} = -(dn_D/dt)_{chem.} \equiv -dx/dt. \tag{3.10}$$

Noting these and the definitions (3.7), we have

$$\left. \begin{aligned}
(d\rho/dt)_{irr.} &\equiv \sum_i (d\rho_i/dt)_{irr.} = (m_C + m_D - m_A - m_B) dx/dt = 0, \\
(d\mathbf{j}/dt)_{irr.} &\equiv \sum_i (d\mathbf{j}_i/dt)_{irr.} \cong 0, \\
(dE/dt)_{irr.} &\equiv \sum_i (dE_i/dt)_{irr.} \cong 0,
\end{aligned} \right\} \tag{3.11}$$

and hence, from the equation (3.9),

$$\begin{aligned}
\left(\frac{d}{dt} S\right)_{irr.} &\equiv \sum_i \left(\frac{d}{dt} S_i\right)_{irr.} \\
&\cong \frac{1}{T} \left\{ (\mathbf{P}^0 - \mathbf{P}) : \frac{\partial}{\partial \mathbf{x}} \mathbf{v} - \sum_i \rho d_i \cdot \mathbf{v}_i - \sum_i \mu_i^0 \left(\frac{d}{dt} n_i\right)_{chem.}^0 \right. \\
&\quad \left. - [Q - \sum_i H_i^0(\mathbf{v}_i - \mathbf{v})] \cdot \frac{\partial \log T}{\partial \mathbf{x}} \right\}.
\end{aligned}$$

If we use the relations

$$-\sum_i p d_i \cdot v_i = -\sum_{i,j} p d_{ij} \cdot v_i = -\frac{1}{2} \sum_{i,j} p d_{ij} \cdot (v_i - v_j),$$

$$-\sum_i \mu_i^0 \left(\frac{d}{dt} n_i \right)_{chem.}^0 = (\mu_C^0 + \mu_D^0 - \mu_A^0 - \mu_B^0) \left(\frac{dx}{dt} \right)^0,$$

which are obtained from the equations (3.7) and (3.10), we have the required result

$$\left(\frac{d}{dt} S \right)_{irr.} \cong \frac{1}{T} \left\{ (P^0 - P) : \frac{\partial}{\partial x} v + \frac{1}{2} \sum_{i,j} f_{ij} \cdot (v_i - v_j) \right\}$$

$$+ \{ Q - \sum_i H_i^0 (v_i - v) \} \cdot \frac{\partial}{\partial x} \left(\frac{1}{T} \right) + k \left\{ \log K - \log \frac{C_C C_D}{C_A C_B} \right\} \left(\frac{dx}{dt} \right)^0, \quad (3.12)$$

where

$$P \equiv \sum_i P_i, \quad Q \equiv \sum_i Q_i; \quad f_{ij} \equiv -p d_{ij} = -f_{ji}, \quad (\because f_{ii} = 0);$$

$$\mu_C^0 + \mu_D^0 - \mu_A^0 - \mu_B^0 \equiv kT \left\{ \log K - \log \frac{C_C C_D}{C_A C_B} \right\}, \quad C_i \equiv n_i / n.$$

The preceding results, (3.11) and (3.12), correspond to those obtained in § 2, (2.7) and (2.9). In the expression (3.12) besides the term due to the chemical reaction, (1.8), there appear additional terms containing the vectors f_{ij} , which may be interpreted as the mutual friction forces between the gases of the i -th and the j -th kinds, for they appear also in the second equation of (3.4) and irreversibly produce the entropy in proportion to the relative velocities $v_i - v_j$. Thus we can add the rate of energy dissipation through the action of mutual frictions

$$(dF/dt)_{irr.} = \frac{1}{2} \sum_{i,j} f_{ij} \cdot (v_i - v_j) \quad (3.13)$$

to the list of § 1.

The form of the equations obtained by adding the equations (3.4) over all i is different from the equations (2.3) and (1.1) only in the fact that they have the terms $\sum_i \rho_i F_i$ (in place of ρF in the equation of motion) and $\sum_i \rho_i F_i \cdot (v_i - v)$ (in the energy equation) on the right-hand sides. But S and $(dS/dt)_{irr.}$ take different forms. According to the approximation (3.5), we obtain

$$S_i = -k \sum_{\tau} \int \left(\frac{p}{m_i} - v \right) f_i \left\{ \frac{\mu_i^0 - \frac{(p - m_i v)^2}{2m_i}}{kT} - \epsilon_{i\tau} - W_i \right\} dp$$

$$\cong \frac{Q_i - n_i \mu_i^0 (v_i - v)}{T}$$

and hence

$$S \equiv \sum_i S_i \cong \{ Q - \sum_i n_i \mu_i^0 (v_i - v) \} / T. \quad (3.14)$$

It must be noted here that in the expressions (3.12) and (3.14) there appear the correction terms to the heat flow vector⁰⁾, $\sum_i H_i^0(\mathbf{v}_i - \mathbf{v})$ and $\sum_i n_i^0 \mu_i^0(\mathbf{v}_i - \mathbf{v})$. [cf. Eq. (1.3).] Using the expressions (3.12) and (3.14) and the definitions

$$u \equiv U/\rho, \quad U \equiv \sum_i U_i;$$

$$s \equiv S/\rho, \quad S \equiv \sum_i S_i;$$

and

$$d/dt \equiv \partial/\partial t + \mathbf{v} \cdot \partial/\partial \mathbf{x},$$

we can derive the equation corresponding to (2.11),

$$\begin{aligned} du/dt \cong & T ds/dt - p d/dt(1/\rho) + \sum_i \mu_i^0 d/dt(n_i/\rho) \\ & + 1/\rho \sum_i \{ \rho_i \mathbf{F}_i - \mathbf{f}_i - S_i^0 \partial T/\partial \mathbf{x} - n_i \partial \mu_i^0/\partial \mathbf{x} \} \cdot (\mathbf{v}_i - \mathbf{v}), \end{aligned}$$

where we have defined $\mathbf{f}_i \equiv \sum_j \mathbf{f}_{ij}$ and employed the relation $n_i \mu_i^0 = H_i^0 - T S_i^0$. S_i^0 is the partial entropy per unit volume, and it holds $S_i \cong S_i^0 + (U_i - U_i^0)/T$ in our approximation (3.5). In the foregoing equation, the last sum turns out to vanish.* In fact, according to the definitions (3.7), we have

$$\begin{aligned} \mathbf{f}_i &= -p \sum_j \mathbf{d}_{ij} \\ &= -p \left\{ \frac{\partial}{\partial \mathbf{x}} \left(\frac{n_i}{n} \right) + \left(\frac{n_i}{n} - \frac{\rho_i}{\rho} \right) \frac{\partial \log p}{\partial \mathbf{x}} + \frac{\rho_i}{\rho^2} \left(\sum_j \rho_j \mathbf{F}_j - p \mathbf{F}_i \right) \right. \\ &= \left(p_i \mathbf{F}_i - \frac{\partial p_i}{\partial \mathbf{x}} \right) - \frac{\rho_i}{\rho} \left(\sum_j \rho_j \mathbf{F}_j - \frac{\partial p}{\partial \mathbf{x}} \right), \end{aligned} \quad (3.15)$$

$$\begin{aligned} \sum_i \mathbf{f}_i \cdot (\mathbf{v}_i - \mathbf{v}) &= \sum_i \rho_i \mathbf{F}_i \cdot (\mathbf{v}_i - \mathbf{v}) - \sum_i \partial p_i/\partial \mathbf{x} \cdot (\mathbf{v}_i - \mathbf{v}), \\ &[\text{remembering the identity } \sum_i \rho_i(\mathbf{v}_i - \mathbf{v}) \equiv 0], \end{aligned}$$

and hence, using the relation $n_i \mu_i^0 = H_i^0 - T S_i^0$,

$$\begin{aligned} \sum_i \{ \rho_i \mathbf{F}_i - \mathbf{f}_i - S_i^0 \partial T/\partial \mathbf{x} - n_i \partial \mu_i^0/\partial \mathbf{x} \} \cdot (\mathbf{v}_i - \mathbf{v}) \\ = - \sum_i n_i \{ \partial/\partial \mathbf{x} (H_i^0/n_i) - T \partial/\partial \mathbf{x} (S_i^0/n_i) - 1/n_i \partial p_i/\partial \mathbf{x} \}, \end{aligned}$$

the last member of which vanishes by virtue of the values of H_i^0 , S_i^0 , and p_i ;

$$\partial/\partial \mathbf{x} (H_i^0/n_i) = T \partial/\partial \mathbf{x} (S_i^0/n_i) + 1/n_i (\partial p_i/\partial \mathbf{x}). \quad (3.16)$$

Thus the Gibbs' relation

$$du/dt \cong T ds/dt - p d/dt(1/\rho) + \sum_i \mu_i^0 d/dt(n_i/\rho) \quad (3.17)$$

is valid in our approximation too. Here $1/\rho$ means the specific volume of the mixture,

* I am much indebted to Messrs. M. Schimoi and S. Nakajima on this point.

and hence n_i/ρ the number of molecules of the i -th kind per unit mass of the mixture.

We should note here that the preceding method used to prove the vanishing of the terms due to mutual diffusion is the one used by Curtiss-Hirschfelder⁹⁾. But they started from the equation (3.17) and deduced the expression for $(dS/dt)_{irr}$ contrary to our course.

§ 4. Steady electric current

In the Lorentz-Sommerfeld free electron model of metals¹⁰⁾, the problem of the electric conductivity can be treated by the semi-classical method developed in § 2. The electron gas obeys the Fermi statistics ($\theta = -1$), and has the internal degrees of freedom due to spin ($G=2$). For the sake of simplicity, let us assume that the external force acting on each electron is only an electro-static field \mathbf{E} . Therefore we have $m\mathbf{F} = e\mathbf{E}$, $e (< 0)$ being the electronic charge and m the electronic mass. If we neglect the mass motion of the crystal lattice of the metal, the velocity itself of each electron must be regarded as its thermal velocity. Since there is no necessity for separating the local mass motion, the expression of the distribution function is obtained by putting formally $\mathbf{v} \equiv 0$ in the expression (2.5):

$$f \equiv 1 \left/ \left\{ \exp \frac{\mathbf{p}^2/2m + W - \mu^0}{kT} + 1 \right\} \right. \simeq f^0 \left\{ 1 - (1 - f^0) W/kT \right\}. \quad (4.1)$$

In this section the difference such as between S_{total} and S in the equation (1.2) will not appear.

Since in the present example the primary cause of irreversibility lies in the interaction with the environment, i.e., the collisions with quanta of the thermal vibration of the crystal lattice, the approximation such as used in the last member of the equation (2.6) will be inadequate because of the structure of $(df/dt)_{irr}$. In fact, the results obtained by putting formally $\mathbf{v} \equiv 0$ in the above-mentioned expression (2.6) are as follows:

$$(d\rho/dt)_{irr} \simeq 0, \quad (d\mathbf{j}/dt)_{irr} \simeq 0, \quad (dE/dt)_{irr} \simeq 0,$$

$$(dS/dt)_{irr} \simeq \{ \mathbf{Q} - (5/2)(p/\rho)\mathbf{j} \} \cdot \partial/\partial \mathbf{x} (1/T),$$

and the resistance to the electronic current $\mathbf{J} \equiv \frac{e}{m} \mathbf{j}$ vanishes.

As we have not yet any approximal method permissible for non-stationary cases in general, we confine ourselves here to the case of steady state, for which, as is well-known, the result is

$$\begin{aligned} (df/dt)_{irr} &\simeq \{ \partial/\partial t + \mathbf{p}/m \cdot \partial/\partial \mathbf{x} + e\mathbf{E} \cdot \partial/\partial \mathbf{p} \} f^0 \\ &= \frac{f^0(1-f^0)}{kT} \left\{ \left(\frac{\mathbf{p}^2}{2m} - \mu^0 \right) \frac{\partial \log T}{\partial \mathbf{x}} + \frac{\partial \mu^0}{\partial \mathbf{x}} - e\mathbf{E} \right\} \cdot \frac{\mathbf{p}}{m}. \end{aligned} \quad (4.2)$$

Here T and μ^0 are considered to be determined by the equilibrium state of the crystal lattice. According to this approximation, we have the expressions

$$\left(\frac{d}{dt} \rho \right)_{irr} \equiv \int m \left(\frac{d}{dt} f \right)_{irr} \frac{2d\mathbf{p}}{h^3} \simeq 0,$$

$$\left(\frac{d}{dt}\mathbf{j}\right)_{irr.} \equiv \int \mathbf{p} \left(\frac{d}{dt}f\right)_{irr.} \frac{2d\mathbf{p}}{h^3} \\ \cong n \left\{ \left(\frac{5}{2} \frac{p}{n} - \mu^0\right) \frac{\partial \log T}{\partial x} + \frac{\partial \mu^0}{\partial x} - e\mathbf{E} \right\}, \quad (4.3)$$

$$\left(\frac{d}{dt}E\right)_{irr.} \equiv \int \frac{\mathbf{p}^2}{2m} \left(\frac{d}{dt}f\right)_{irr.} \frac{2d\mathbf{p}}{h^3} \cong 0,$$

$$\left(\frac{d}{dt}S\right)_{irr.} + F_s \equiv -k \int \log\left(\frac{f}{1-f}\right) \left(\frac{d}{dt}f\right)_{irr.} \frac{2d\mathbf{p}}{h^3} \cong \frac{1}{T} \int W \left(\frac{d}{dt}f\right)_{irr.} \frac{2d\mathbf{p}}{h^3} \\ \cong 1/T \{ e\mathbf{E} + \mu^0 \partial \log T / \partial x - \partial \mu^0 / \partial x \} \cdot \mathbf{j} / m + \mathbf{Q} \cdot \partial / \partial x (1/T).$$

As is expected, there appear the resistance to the electronic current and the irreversible entropy production associated with it. The last expression can be rewritten into a more obvious form. Using the result (4.3), the equation of motion for the steady current becomes

$$\frac{\partial}{\partial x} \cdot \mathbf{P} \cong n \left\{ \left(\frac{5}{2} \frac{p}{n} - \mu^0\right) \frac{\partial \log T}{\partial x} + \frac{\partial \mu^0}{\partial x} \right\};$$

hence

$$\left(\frac{d}{dt}S\right)_{irr.} + F_s \cong \frac{1}{T} \left\{ \mathbf{E} - \frac{1}{\sigma} \left(\frac{\partial}{\partial x} \cdot \mathbf{P}\right) \right\} \cdot \mathbf{J} + \left\{ \mathbf{Q} - \frac{5}{2} \frac{p}{\sigma} \mathbf{J} \right\} \cdot \frac{\partial}{\partial x} \left(\frac{1}{T}\right), \quad (4.4)$$

$\sigma \equiv ev/m$ being the electronic charge density. If we notice that $\mathbf{v} \equiv \mathbf{J}/\sigma$ represents the average velocity of electron, it will be seen that the term due to irreversible electronic heat flow in the last term of the expression (4.4) has just the same form as the one in the expression (3.12). This analogy holds also in the case of the electronic entropy flow [cf. Eq. (3.14)]:

$$S \equiv -k \int \frac{\mathbf{p}}{m} \{ f \log f + (1-f) \log(1-f) \} \frac{2d\mathbf{p}}{h^3} \\ \cong \frac{1}{T} \int (f^0 - f) \left(\mu^0 - \frac{\mathbf{p}^2}{2m} \right) \frac{\mathbf{p}}{m} \frac{2d\mathbf{p}}{h^3} \\ = \frac{\mathbf{Q} - (\mu^0/e)\mathbf{J}}{T} = \frac{\mathbf{Q} - n\mu^0\mathbf{v}}{T}, \quad (4.5)$$

where we have used the approximation (4.1). The reason why the term of the energy dissipation due to viscous forces, such as the expression (1.6), does not appear in the equation (4.4) is that we have not separated the local mass motion.

In order to obtain the form corresponding to the expression (1.7), we need the solution f of the equation (4.2) for the case $\partial T / \partial x = \partial \mu^0 / \partial x = 0$. As is well-known, in this case we have⁽¹⁾

$$\mathbf{J} \cong R\mathbf{E}, \\ (dS/dt)_{irr.} + F_s \cong \mathbf{E} \cdot \mathbf{J} / T.$$

The general expression (4.4) will contain the terms due to the Thomson effect, etc.

§ 5. Concluding remarks

By virtue of the examples given in §§ 2, 3, and 4 we laid the molecular theoretic foundation of the thermodynamical theories developed by Tolman-Fine and others, and clarified the difference between the basic ideas of the molecular and the thermodynamical theories.

It will then also become clear that the essential points of our method lie in the two assumptions (approximations), i.e., the explicit form of the distribution function (Eqs. (2.5), (3.5), and (4.1)) and the approximation used in the expressions (2.6), (3.6), and (4.2). The justification of these approximations are naturally to be sought for in the irreversible molecular mechanism itself, i.e., the explicit structure of the term $(df/dt)_{irr.}$. But we may also use our method to clarify the thermodynamic natures of any process by an appropriate choice of our assumptions provided that some general features are known on the irreversibility of that process. Moreover, if we proceed to approximations of the higher order, we shall be able to deduce correlation terms between the elementary irreversible processes, which were neglected in the thermodynamical theories, and correction terms to the Gibbs' relation.

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Note added in proof: (1) After sending my manuscript, I could read Professor Prigogin's lecture in *Colloque de thermodynamique* (Bruxelles, Janvier 1948), in which he stated the results similar to mine. His original paper is, however, not yet available to me. (2) In §§ 3 and 4 the quantity \mathcal{Q} was wrongly termed as the "heat" flow vector. Since the latter concept is characteristic for thermodynamics, it must be defined consistently with thermodynamics, esp., with the well-known expression of the second law. From this view-point, the quantity $\mathcal{Q}^* \equiv \mathcal{Q} - \sum \mu_i^0 n_i (v_i - v)$, or $\equiv \mathcal{Q} - \mu^0 n v$ must be identified with the heat flow vector (see (1.10), (3.14), and (4.5)). Thus we see that the quantity \mathcal{Q} contains the "mass-action" terms.

Statistical Mechanics of Cooperative Phenomena

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The method of attack developed by Kirkwood and Born-Green, whose application was almost exclusively confined to the theory of liquids, is applied to the statistical treatment of cooperative phenomena, by which we mean the order-disorder problem, its related phenomena and the hindered rotation in molecular crystals. Using Kirkwood's coupling parameter, we find an integral equation, which is simplified by a method of expansion. By a further simplification, we reach two kinds of integral equations, corresponding to Bragg-Williams' and Bethe's approximations respectively. In the case of nearest neighbour systems the latter equation is derived from a variational principle, which corresponds to the maximum condition of the partition function per interaction bond expressed in terms of the distribution function. According to our formulation Bethe's method of internal field and Fowler-Guggenheim's quasi-chemical method are derived from the same fundamental equation. This fact serves to clear understanding of the equivalence of these two points of view, although this equivalence is demonstrated by Fowler-Guggenheim by use of the grand canonical ensemble. It is further noted that from our variational formula we can attain to the extension of the variation method of Kramers-Wannier to the three-dimensional lattices. Thus, Kramers-Wannier's approximation proves to be only a metamorphosis of Bethe's approximation.

Introduction

As early as in 1935 J. G. Kirkwood¹⁾ developed one unique method in the theory of liquids, in which the thermodynamical quantities of the system are expressed in terms of the so-called molecular distribution functions, accordingly an attempt plays an important role to obtain the molecular distribution function. For this purpose he invented an ingenious method to formulate an integral equation, whose solution should give the molecular distribution function. Recently M. Born and H. S. Green²⁾ formulated an alternative method, which is more convenient than Kirkwood's formulation because of the non-appearance of the integral of unknown function with respect to the parameter contained explicitly in it.

The theory of the hindered molecular rotation by Kirkwood³⁾ kept out, however, these papers were exclusively confined to the field of the theory of liquids in applications of their methods. Hence it would be desirable to extend the field of their applications, since they also prove to be most powerful in the so-called cooperative phenomena which are characterized by the prominent correlation among particles.

Now we can cite Bragg-Williams⁴⁾ and Bethe's^{5), 6), 7)} (strictly speaking, Bethe-Fowler-Guggenheim-Takagi's), approximations as the typical ones in the theory of binary alloys. Following the analytical method after Kirkwood, we shall attempt to formulate the integral equations, which correspond to the approximations cited above. It seems to be worth while to do this from the following reasons: In the first place, we may attain to the generaliza-

tion of Bethe's approximation to the case of continuous variables which was inaugurated by R. Kubo in our country. In the second place, we may see from our formulation what assumptions lie under these approximations, although it was somewhat vague until now.

We set up in § 1 the integral equation of Kirkwood type. This equation being unwieldy, a method of expansion is proposed in order to avoid this difficulty (§ 2). Starting from this procedure we find how to simplify the still formidable equation, where due regard is paid to the systematic method of approximation by Kirkwood¹⁾ and Mayer⁸⁾ (§ 3). Thus we derive two kinds of integral equations, the one corresponding to Bragg-Williams' and the other to Bethe's approximations. From § 4 we confine our considerations to nearest neighbour systems. We then derive an integral equation in Bethe's approximation (§ 4), which is written in the variational form. Writing down this integral equation in terms of the theory of binary alloys, we are lead to the quasi-chemical method of Fowler-Guggenheim⁶⁾ (§ 5), while the variational formula derived leads to Bethe's formulation⁵⁾ in Bethe's approximation (§ 6). In § 7 we extend Bethe's approximation to be able to apply it to the system composed of molecules, which contain a number of particles, and derive the extended formula of Kramers-Wannier's variation method.⁹⁾

We introduce some notations used through-out the paper in concluding this introduction. Following Mayer-Montroll¹⁰⁾, the state of the particle, i , is denoted by (i) , which is either continuous or discrete and either scalar or vector. Although the description runs as if the state variable was continuous, the continuity is not essential in the deductions made below, hence the obtained results are valid, if we replace the integral by summation signs, in the case of discrete variable. The set of state variables $(1), (2), \dots, (N)$ is denoted by $\{N\}$, and the set of $(n+1), \dots, (N)$ by $\{N-n\}$. We denote the integral of an arbitrary function of (i) , $\varphi(i)$, by $\int \varphi(i) d(i)$, and define ω by $\int 1 \cdot d(i) = \omega$. Conforming to the above conversions, $d(1)d(2)\cdots d(N)$ is denoted by $d\{N\}$, and $d(n+1)\cdots d(N)$ by $d\{N-n\}$. The multiple integral of an arbitrary function concerning $\{N\}$ or $\{N-n\}$ is denoted in the same way as above.

§ 1. The integral equation between the distribution functions

Let us consider the system composed of N identical particles, whose potential energy, $U\{N\}$, is given by

$$U\{N\} = \sum_{N \geq j \geq i \geq 1} u(i, j), \quad (1.1)$$

$u(i, j)$ being the mutual potential energy determined by the states (i) , (j) of two particles i, j .

From the point of view of the classical statistical mechanics, the equilibrium properties of the system are derived from the free energy A_0 defined by

$$\exp(-\beta A_0) = \int \int \cdots \int \exp(-\beta U\{N\}) d\{N\}, \quad \beta = 1/kT, \quad (1.2)$$

k being the Boltzmann constant, T the absolute temperature. By using the above expression and the local free energy, $A_\lambda\{\lambda\}$,

$$\exp(-\beta A_\lambda\{\lambda\}) = \omega^\lambda \iint \dots \int \exp(-\beta U\{N\}) d\{N-\lambda\}, \quad (1.3)$$

which was introduced by Kirkwood, the distribution function of the λ -th order $g_\lambda\{\lambda\}$ is defined by

$$g_\lambda\{\lambda\} = \exp(-\beta W_\lambda\{\lambda\}); \quad W_\lambda\{\lambda\} = A_\lambda\{\lambda\} - A_0, \quad (1.4)$$

where $W_\lambda\{\lambda\}$ means the potential of average force. $g_\lambda\{\lambda\}$ is normalized as

$$\omega^{-(\lambda-\mu)} \iint \dots \int g_\lambda\{\lambda\} d\{\lambda-\mu\} = g_\mu\{\mu\}, \quad g_0=1, \quad (1.5)$$

in order to give the probability that the state of λ particles are found in the configurational elements $d\{\lambda\}$ at $\{\lambda\}$.

Now, following Kirkwood, we consider a virtual system, whose potential energy is defined by

$$U(\{N\}; \xi) = \xi U\{\lambda\} + \xi \sum_{\substack{\lambda \geq \mu \geq 1 \\ N \geq k \geq \lambda}} u(\mu, k) + U\{N-\lambda\}; \quad 1 \geq \xi \geq 0. \quad (1.6)$$

$\xi=1$ corresponds to the real system, with which we are concerned, and $\xi=0$ to the real system composed of $N-\lambda$ particles. The expressions for the free energy, $A_0(\xi)$, the local free energy, $A_\lambda(\{\lambda\}; \xi)$, the distribution function, $g_\lambda(\{\lambda\}; \xi)$, and the potential of average force, $W_\lambda(\{\lambda\}; \xi)$, of this virtual system are introduced in the same way as before.

If we differentiate $W_\lambda(\{\lambda\}; \xi)$ with respect to ξ , the following equations are obtained from (1.2), (1.3), (1.4) and (1.5), in which $U\{N\}$ is replaced by $U(\{N\}; \xi)$,

$$\partial W_\lambda(\{\lambda\}; \xi) / \partial \xi = \partial A_\lambda(\{\lambda\}; \xi) / \partial \xi - \partial A_0(\xi) / \partial \xi, \quad (1.7a)$$

$$\frac{\partial A_\lambda(\{\lambda\}; \xi)}{\partial \xi} = U\{\lambda\} + \omega^{-1} \sum_{\substack{\lambda \geq \mu \geq 1 \\ N \geq k \geq \lambda}} \int u(\mu, k) \frac{g_{\lambda+1}(\{\lambda\}, (k); \xi)}{g_\lambda(\{\lambda\}; \xi)} d(k), \quad (1.7b)$$

$$\frac{\partial A_0(\xi)}{\partial \xi} = -\omega^{-1} \sum_{\substack{\lambda \geq i \geq 1 \\ N \geq j \geq \lambda}} \iint u(i, j) g_2((i), (j); \xi) d(i) d(j). \quad (1.7c)$$

After integrating (1.7) with respect to ξ , and regarding $\beta W_\lambda(\{\lambda\}; 1) = -\ln g_\lambda\{\lambda\}$, $\beta W_\lambda(\{\lambda\}; 0) = 0$, we have

$$\ln A g_\lambda\{\lambda\} = -\beta U\{\lambda\} - \beta \omega^{-1} \sum_{\substack{\lambda \geq \mu \geq 1 \\ N \geq k \geq \lambda}} \int u(\mu, k) \int_0^1 \frac{g_{\lambda+1}(\{\lambda\}, (k); \xi)}{g_\lambda(\{\lambda\}; \xi)} d\xi d(k), \quad (1.8)$$

$$\text{in which} \quad -\beta^{-1} \ln A = A_0(1) - A_0(0), \quad (1.9)$$

or

$$-\beta^{-1} \ln A = -\omega^{-2} \sum_{N \geq j \geq i \geq 1} \iint u(i, j) \int_0^1 g_2((i), (j); \xi) d\xi d(i) d(j). \quad (1.10)$$

Alternative expressions for A obtained from the normalization condition of $g_\lambda\{\lambda\}$, i. e. (1.5), are as follows: Multiplying (1.8) by $g_\lambda\{\lambda\}$ and integrating with respect to $\{\lambda\}$, we have the third expression for A :

$$-\beta^{-1} \ln A = \beta^{-1} \omega^{-\lambda} \iint \dots \int g_\lambda\{\lambda\} \ln g_\lambda\{\lambda\} d\{\lambda\}$$

$$\begin{aligned}
& + \omega^{-\lambda} \iint \dots \int U\{\lambda\} g_{\lambda}\{\lambda\} d\{\lambda\} \\
& + \omega^{-(\lambda+1)} \sum_{\substack{\lambda \geq \mu \geq 1 \\ N \geq k \geq \lambda}} \iint \dots \int u(\mu, k) g_{\lambda}\{\lambda\} \int_0^1 \frac{g_{\lambda+1}(\{\lambda\}, (k); \hat{\xi})}{g_{\lambda}(\{\lambda\}; \hat{\xi})} d\hat{\xi} d(k) d\{\lambda\}. \quad (1.11)
\end{aligned}$$

The fourth expression for A :

$$A = \omega^{-\lambda} \iint \dots \int \exp \left\{ -\beta U\{\lambda\} - \beta \omega^{-1} \sum_{\substack{\lambda \geq \mu \geq 1 \\ N \geq k \geq \lambda}} \int u(\mu, k) \int_0^1 \frac{g_{\lambda+1}(\{\lambda\}, (k); \hat{\xi})}{g_{\lambda}(\{\lambda\}; \hat{\xi})} d\hat{\xi} d(k) \right\} d\{\lambda\} \quad (1.12)$$

is obtained by integrating the exponent of (1.8) with respect to $\{\lambda\}$.

According to the expression (1.9), $-\beta^{-1} \ln A$ gives the free energy per λ particles when $N \gg \lambda$. While the expression (1.10) is not so useful, the expressions (1.11) and (1.12) serve to give some insights into the physical situation. The former expression gives the expression of A in the form of free energy and the latter in the form of partition function.

§ 2. Method of expansion

In integral equation (1.8) we cannot see the explicit dependence of $g_{\lambda}(\{\lambda\}; \hat{\xi})$ on $\hat{\xi}$ directly, this fact offering a serious obstacle before us in solving this equation. In order to avoid this difficulty, giving up the closed form, we expand $g_{\lambda+1}(\{\lambda+1\}; \hat{\xi})/g_{\lambda}(\{\lambda\}; \hat{\xi})$ in a Taylor series with respect to $\hat{\xi}$ at $\hat{\xi}=1$ to obtain

$$\begin{aligned}
\int_0^1 g_{\lambda+1}(\{\lambda+1\}; \hat{\xi})/g_{\lambda}(\{\lambda\}; \hat{\xi}) d\hat{\xi} &= g_{\lambda+1}(\{\lambda+1\})/g_{\lambda}(\{\lambda\}) \\
&+ 1/2 \cdot [\partial/\partial \hat{\xi} \cdot g_{\lambda+1}(\{\lambda+1\}; \hat{\xi})/g_{\lambda}(\{\lambda\}; \hat{\xi})]_{\hat{\xi}=1} \\
&+ 1/6 \cdot [\partial^2/\partial \hat{\xi}^2 \cdot g_{\lambda+1}(\{\lambda+1\}; \hat{\xi})/g_{\lambda}(\{\lambda\}; \hat{\xi})]_{\hat{\xi}=1} + \dots \quad (2.1)
\end{aligned}$$

The right-hand side of this expression can be obtained from equations concerning the potential of average force of higher order than λ .

Let us find an expression of the second term of the right-hand side of (2.1). Making use of (1.7) and the equation for $\partial W_{\lambda+1}(\{\lambda+1\}; \hat{\xi})/\partial \hat{\xi}$, which is obtained in the same way as in (1.7), we have

$$\begin{aligned}
\frac{\partial}{\partial \hat{\xi}} \left\{ \frac{g_{\lambda+1}(\{\lambda+1\}; \hat{\xi})}{g_{\lambda}(\{\lambda\}; \hat{\xi})} \right\} &= -\beta \omega^{-1} \sum_{\substack{\lambda \geq \nu \geq 1 \\ N \geq l \geq \lambda}} \int u(\nu, l) \left\{ \frac{g_{\lambda+2}(\{\lambda+1\}, (l); \hat{\xi})}{g_{\lambda}(\{\lambda\}; \hat{\xi})} \right. \\
&\quad \left. - \frac{g_{\lambda+1}(\{\lambda+1\}; \hat{\xi}) g_{\lambda+1}(\{\lambda\}, (l); \hat{\xi})}{[g_{\lambda}(\{\lambda\}; \hat{\xi})]^2} \right\} d(l). \quad (2.2)
\end{aligned}$$

The first and second terms of the right-hand side of (2.1) correspond to the terms of the zero-th and first power to β/ω respectively, and we see that the n -th term of the right-hand side of (2.1) is of the $(n-1)$ -th power to β/ω .

As a consequence, (1.8) is written as

$$\begin{aligned}
\ln A_{g_\lambda\{\lambda\}} = & -\beta U\{\lambda\} - \beta\omega^{-1} \sum_{\substack{\lambda \geq \mu \geq 1 \\ N \geq k > \lambda}} \iint u(\mu, k) \frac{g_{\lambda+1}(\{\lambda\}, (k))}{g_\lambda\{\lambda\}} d(k) \\
& + 1/2 \cdot (\beta\omega^{-1})^2 \sum_{\substack{\lambda \geq \mu \geq 1 \\ N \geq k > \lambda}} \sum_{\substack{\lambda \geq \nu \geq 1 \\ N \geq l > \lambda}} \iint u(\mu, k) u(\nu, l) \left\{ \frac{g_{\lambda+2}(\{\lambda\}, (k), (l))}{g_\lambda\{\lambda\}} \right. \\
& \left. - \frac{g_{\lambda+1}(\{\lambda\}, (k)) g_{\lambda+1}(\{\lambda\}, (l))}{[g_\lambda\{\lambda\}]^2} \right\} d(k) d(l) + \dots \quad (2.3)
\end{aligned}$$

The nature of the above expansion is just what one would get if the potential of average force was expanded in terms of the average potential and the corrective terms coming from the correlations among particles of higher order.

§ 3. Simplification

Generalizing Kirkwood's idea, Mayer³⁾ attempted to divide the potential of average force, $W_\lambda\{\lambda\}$, in its component potentials, $W_\lambda\{\lambda\}$, as

$$W_\lambda\{\lambda\} = (\sum \{l\}_\lambda) w_l\{l\}_\lambda, \quad (3.1)$$

of which the inversion formula is given by

$$w_\lambda\{\lambda\} = (\sum \{l\}_\lambda) (-)^{\lambda-l} W_l\{l\}_\lambda, \quad (3.2)$$

where by $\{l\}_\lambda$ we mean the set of l particles, all of which are members of the set $\{\lambda\}$, and by $(\sum \{l\}_\lambda)$ summation over all possible sets of subsets. We have generalized Kirkwood approximation after putting $w_\lambda=0$, whose gist consists in approximating the potential of average force, $W_\lambda\{\lambda\}$, by the superposition of that of lower order.

The most rough approximation is given by

$$w_l=0, \quad l \geq 2 \quad \text{i.e.} \quad W_N\{N\} = \sum_{N \geq i \geq 1} W_1(i) \quad (3.3a)$$

which leads, regarding (1.4), to the Hartree-like approximation

$$g_N\{N\} = \prod_{i=1}^N g_1(i). \quad (3.3b)$$

A more good approximation is given by

$$w_l=0, \quad l \geq 3. \quad (3.4a)$$

Writing W_3 at the stage of this approximation, we have

$$\begin{aligned}
W((i), (j), (k)) = & W_2((i), (j)) + W_2((j), (k)) + W_2((k), (i)) \\
& - W_1(i) - W_1(j) - W_1(k), \quad (3.4b)
\end{aligned}$$

which leads to the Kirkwood approximation

$$g_3((i), (j), (k)) = g_2((i), (j)) g_2((j), (k)) g_2((k), (i)) / g_1(i) g_1(j) g_1(k). \quad (3.4c)$$

There exist however the intermediate approximations between these two ones. The one is given by the further substitution in (3.4b):

$$w_2((i), (k)) = 0 \quad (3.5a)$$

assuming that the i -th and j -th particles are close together and the k -th particle is close to the j -th but not to the i -th, which reduces the approximation (3.4c) to

$$g_3((i), (j), (k)) = g_2((i), (j))g_2((j), (k))/g_1(j). \quad (3.5b)$$

Although the approximation (3.4) is used in the theory of liquid, in which there exists no long range order, we shall make use of the more rough approximations (3.3) and (3.5), since we have interest in the transition from the ordered state to the disordered and hence in the integral equation containing g_1 , where it seems to be difficult to solve these equations under the approximation (3.4).

Let us take the approximation (3.3) first. Writing down (2.3) for $\lambda=1$ under this approximation, we have the most simplified equation

$$\ln A'g_1(i) = -\beta\omega^{-1} \sum'_{N \geq k \geq 1} \int u(i, k)g_1(k)d(k), \quad (3.6)$$

being the equation at the stage of Bragg-Williams' approximation.³⁾

Secondly, on substitution of the approximation (3.5b) for an arbitrary ξ in the right-hand side of (2.3) for $\lambda=1$, the terms of higher powers of β/ω than zero-th vanish, since this approximation results in

$$\partial/\partial\xi \cdot \{g_2((i), (j); \xi)/g_1((i); \xi)\} = 0$$

by virtue of (2.2). Then we have

$$\ln A'g_1(i) = -\beta\omega^{-1} \sum'_{N \geq k \geq 1} \int u(i, k)g_2^{(1)}((i), (k))d(k), \quad (3.7)$$

$$g_2^{(1)}((i), (k)) = g_2((i), (k))/g_1(i). \quad (3.7a)$$

Thirdly, in simplifying (2.3) for $\lambda=2$, if we introduce the further approximation

$$w_2((k), (l); \xi) = 0 \quad \text{between the third and fourth fixed particles,} \quad (3.8)$$

we obtain

$$\ln A''g_2((i), (j)) = -\beta u(i, j) - \beta\omega^{-1} \sum_{i \neq j} \sum'_{N \geq k \geq 1} \int u(i, k)g_2^{(1)}((i), (k))d(k) \quad (3.9)$$

by use of the approximations (3.4) and (3.5) in the same way as in (3.7).

It should be noticed that the physical meaning of A changes at the stage of our approximation. For example, the value of $-\beta^{-1} \ln A'$ determined by (3.7) does not give the free energy per particle, but the free energy of a particle embedded in the system, which has too much energy of interaction. Neglecting the correlation of higher order yields a large error of the value of the integral in (1.8), but the integral equation remains, although approximately, valid by the adjustment of the value of A .

Clearly, the approximation (3.4a) with (3.5a) and with (3.8) is not incompatible with (3.4a) together with (3.5a), and hence we see that (3.7) and (3.9) stand at the same stage of approximation. Then if we substitute (3.7) and the equation, which is obtained from replacing i by j in (3.7), into (3.9), the equation

$$\ln A'' g_2((i), (j)) = -\beta u(i, j) + \ln [A'^2 g_1(i) g_1(j)] \\ + \beta \omega^{-1} \int u(i, j) g_2^{(1)}((i), (j)) d(j) + \beta \omega^{-1} \int u(j, i) g_2^{(1)}((j), (i)) d(i) \quad (3.10)$$

is obtained.

The equation (3.10) leads to Bethe's approximation in crystal statistics. Hence we see that the essential feature of Beth's approximation consists in the assumption (3.4) together with (3.5) and with (3.8), which is exactly justified in the one dimensional case with the interactions between nearest neighbours only but by no means in the liquid or the face-centered lattice etc. where the direct interactions among nearest neighbours exist as is discussed by Ono and Murakami⁽¹¹⁾. The correctness of (3.5b) in one dimensional case may be proved by using the following identity:

$$\begin{aligned} & \int \cdots \int_{k=1}^N \prod \exp[-\beta u(k, k+1)] d\{i-2\} d\{N-i-1\} \\ &= \frac{\int \cdots \int_{k=1}^N \prod \exp[-\beta u(k, k+1)] d\{i-2\} d\{N-i\}}{\int \cdots \int_{k=1}^N \prod \exp[-\beta u(k, k+1)] d\{i-1\} d\{N-i\}} \\ & \quad \times \frac{\int \cdots \int_{k=1}^N \prod \exp[-\beta u(k, k+1)] d\{i-1\} d\{N-i\}}{1} \end{aligned} \quad (3.11)$$

where $d\{i-1\} = d(1) \cdots d(i-2)$ and $d\{N-i-1\} = d(i+2) \cdots d(N)$ etc., and (3.5 b) for an arbitray $\hat{\varepsilon}$ is proved by the same way and further (3.8) too.

§ 4. The general forms in Bethe's approximation

Let us consider the crystal lattice, in which only interactions with the nearest neighbours are taken into account. Let z be the number of nearest neighbours, assumed to be equivalent. In this case (3.7) reduces to

$$\ln A' g_1(i) = -\beta z \omega^{-1} \int u(i, i') g_2^{(1)}((i), (i')) d(i'). \quad (4.1)$$

Then, for the two neighbouring particles, i and j , the equation

$$\ln \lambda g_2((i), (j)) = -\beta u(i, j) + (z-1)/z \cdot \ln[g_1(i) g_1(j)], \quad (4.2)$$

$$\lambda = A'' / (A')^{2(z-1)/z} \quad (4.3)$$

is obtained by substitution of (4.1) and the equation, which is obtained by replacing i by j in (4.1), into (3.10). Integrating the exponent of (4.1) with respect to (j) and regarding the normalization condition (1.5), we have

$$\int K((i), (j)) \psi(j) d(j) = \lambda [\psi(i)]^{1/(z-1)}, \quad (4.4)$$

in which

$$K((i), (j)) = \omega^{-1} \exp(-\beta u(i, j)), \quad (4.5)$$

$$\phi(i) = [g_1(i)]^{(z-1)/z} \quad (4.6)$$

ϕ is subject to the condition

$$\omega^{-1} \int [\phi(i)]^{z/(z-1)} d(i) = 1 \quad (4.7)$$

in view of the normalization condition of g_1 . Integral equation (4.4) is now derived from the following variation principle: To make the expression

$$D[\phi] = \iint K((i), (j)) \phi(i) \phi(j) d(i) d(j) - 2(z-1)/z \cdot \lambda \int [\phi(i)]^{z/(z-1)} d(i) \quad (4.8)$$

stationary with respect to the variations of ϕ .

In the special one dimensional case the generally non-linear equation (4.4) reduces to the linear homogeneous equation

$$\int K((i), (j)) \phi(j) d(j) = \lambda \phi(i). \quad (4.9)$$

This equation is in accordance with the fundamental equation of matrix method, including the fact that λ gives the partition function per particle (cf. next paragraph) and the eigenfunction ϕ is given by $[g_1]^{1/2}$. However it is impossible as it stands, to give a reason for selecting one of eigen-values obtained from this eigen-value problem.

What is the physical meaning of λ ? As was described in § 3, $-\beta^{-1} \ln A'$ gives the free energy of a particle and then contains the entropy per particle and the average energy per z interaction bonds, accordingly $-2\beta^{-1} (z-1)/z \cdot \ln A'$ containing the entropy per $2(z-1)/z$ particles and the average energy per $2(z-1)$ interaction bonds. Similarly we see that $-\beta^{-1} \ln A''$ contains the entropy per two particles and the average energy per $2z-1$ interaction bonds. Hence, from (4.3),

$$-\beta^{-1} \ln \lambda = -\beta^{-1} \ln A'' + 2\beta^{-1} (z-1)/z \cdot \ln A' \quad (4.10)$$

contains the entropy per $2/z$ particles and the average energy per interaction bond. Finally we conclude that $-\beta^{-1} \ln \lambda$ gives the free energy per interaction bond, since the entropy per $2/z$ particles is equal to that per interaction bond.

The free energy and the partition function per interaction bond are thus given by

$$\begin{aligned} -\beta^{-1} \ln \lambda = & \beta^{-1} \omega^{-2} \iint g_2((i), (j)) \ln g_2((i), (j)) d(i) d(j) \\ & + \omega^{-2} \iint u(i, j) g_2((i), (j)) d(i) d(j) - \beta^{-1} (z-1)/z \cdot \omega^{-2} \\ & \times \iint g_2((i), (j)) \ln [g_1(i) g_1(j)] d(i) d(j). \end{aligned} \quad (4.11)$$

and

$$\lambda = \omega^{-2} \iint K((i), (j)) \phi(i) \phi(j) d(i) d(j) \quad (4.12)$$

from (4.2), by the same procedure as in (1.11) and (1.12) respectively.

We can now understand the variational formula (4.8) clearly, which corresponds to

$$\delta\lambda[\phi]/\delta\phi=0 \quad (4.13)$$

with the restricting condition (4.7). Since ϕ is a functional of g_1 , and $g_2^{(1)}$ would also be so (cf. (4.2)), (4.13) may then be rewritten $\delta\lambda/\delta g_2^{(1)}=0$ under the normalization conditions. Therefore, using (4.3), we have an alternative variational formula

$$\delta\lambda[g_2^{(1)}]/\delta g_2^{(1)}=0 \quad (4.14)$$

with the normalization condition of $g_2^{(1)}$, in which

$$\lambda[g_2^{(1)}]=A''[g_2^{(1)}]/\{A'[g_2^{(1)}]\}^{2(z-1)/z}, \quad (4.15)$$

$$A''[g_2^{(1)}]=\omega^{-2}\iint\exp\left\{-\beta u(i,j)-\beta(z-1)\omega^{-1}\sum_{i=i,j}u(i,i')g_2^{(1)}((i),(i'))d(i')\right\}d(i)d(j), \quad (4.16)$$

$$A'[g_2^{(1)}]=\omega^{-1}\int\exp\left\{-\beta z\omega^{-1}\int u(i,i')g_2^{(1)}((i),(i'))d(i')\right\}d(i), \quad (4.17)$$

where we used the expressions for A'' and A' derived from the same procedure as in (1.12). We shall return to this formula in § 6.

§ 5. Quasi-chemical method

It will be shown that (4.2) leads to the quasi-chemical method by Fowler-Guggenheim⁽⁶⁾.

Our formulation may be applied, *mutatis mutandis*, to the case of binary alloys. although we proceeded on as if the system was composed of identical particles. For this purpose we have only to re-define the index i so that i may denote the i -th lattice point instead of the i -th particle, thus the concept of the state of a particle should be replaced by that of the state of a lattice point, which is determined by the kind of particle occupying it.

If we consider the system composed of N lattice points to be occupied by particles of the A and B types, supposing their numbers to be equal for simplicity, and divide these lattice points into two kinds of sites, referred to as the a and b so that an a site may be surrounded by b sites and vice versa, then (4.2) remains valid for the i -th and the j -th lattice points, which are nearest neighbours with each other.

Making use of Fowler-Guggenheim-Takagi's notation, by $[A/a]$ is denoted the probability with which an A is found on an a site etc., by $[A/a \cdot B/b]$ the probability with which, concerning the both sites of the pair, an A on an a site and a B on a b site are found at the same time respectively etc., and by $u(A, B)$'s $u(i, j)$'s. For the convenience' sake we shall put $\omega=1$.

Equation (4.2) is then written

$$\ln\lambda\left[\frac{A}{a},\frac{B}{b}\right]=-\beta u(A,B)+\frac{z-1}{z}\ln\left\{\left[\frac{A}{a}\right]\cdot\left[\frac{B}{b}\right]\right\} \quad (5.1)$$

and three similar equations for $[A/a \cdot A/b]$, $[B/a \cdot B/b]$ and $[B/a \cdot A/b]$.

Using equations (5.1), the fundamental equation in the quasi-chemical method

$$\left[\frac{A}{a} \cdot \frac{B}{b}\right] \cdot \left[\frac{B}{a} \cdot \frac{A}{b}\right] / \left[\frac{A}{a} \cdot \frac{A}{b}\right] \cdot \left[\frac{B}{a} \cdot \frac{B}{b}\right] = \exp 2\beta\chi \quad (5.2)$$

follows at once, in which

$$u(A, A) + u(B, B) - 2u(A, B) \equiv 2\chi. \quad (5.3)$$

Let us find an equation to determine the degree of long range order s defined by

$$[A/a] = [B/b] = (1+s)/2, \quad [B/a] = [A/b] = (1-s)/2. \quad (5.4)$$

Introducing a parameter η defined by

$$[A/a \cdot A/b] = [B/a \cdot B/b] \equiv \eta \quad (5.5)$$

we have

$$[A/a \cdot B/b] = (1+s-2\eta)/2, \quad [B/a \cdot A/b] = (1-s-2\eta)/2, \quad (5.6)$$

where in (5.5) the first term proves to be equal to the second term by using (5.4) and the normalization condition $[A/a \cdot A/b] + [A/a \cdot B/b] = [A/a]$ etc. We then substitute (5.5) and (5.6) into (5.2) to obtain

$$\eta = (a-1)/2(\exp 2\beta\chi - 1), \quad u = [1 + (1-s^2)(\exp 2\beta\chi - 1)]^{1/2}. \quad (5.7)$$

By substituting (5.5), (5.6) and (5.7) into equation

$$\ln \left\{ \left[\frac{A}{a} \cdot \frac{B}{b}\right] \cdot \left[\frac{B}{a} \cdot \frac{A}{b}\right] \right\} = 2^{\frac{z-1}{z}} \ln \frac{1+s}{1-s} \quad (5.8)$$

derived from (5.1), we obtain

$$(1+s)/(1-s) = ((u+s)/(u-s))^{z/(z-2)} \quad (5.9)$$

in agreement with Fowler-Guggenheim's expression⁶⁾.

The free energy per interaction bond may be calculated from (4.11), whose result is as follows:

$$\begin{aligned} -\beta^{-1} \ln \lambda &= \beta^{-1}/z \cdot [(1+s) \ln(1+s) + (1-s) \ln(1-s) - 2 \ln 2] \\ &+ \beta^{-1} \cdot \frac{1+s}{2} \ln \frac{u+s}{1+s} + \beta^{-1} \frac{1-s}{2} \ln \frac{u-s}{1-s} - \ln(u+1) + \ln 2 + u(A, B) \end{aligned} \quad (5.10)$$

in agreement with Fowler-Guggenheim's expression again.

We append Bragg-Williams' formula⁴⁾ for determining the long range order herewith, which is derived from (4.1) as

$$s = \tanh(\beta V_{zs}/2) \quad (5.11)$$

with the approximation $u(B, B) - u(A, B) = u(A, A) - (A, B) \equiv V$.

§ 6. Bethe's method in Bethe's approximation⁵⁾

It will be interesting to see that our formulation embraces Bethe's method. The equivalence of Bethe's method to the quasi-chemical method by Fowler-Guggenheim may be

understood clearly from our point of view.

Let us rewrite (4.14) with (4.15), (4.16) and (4.17) in terms of the Ising model, by which it is meant that the state of a particle is specified by the scalar spin variable $\mu = \pm 1$ and the interaction energy between two neighbouring particles is $-J/2$ or $J/2$ according as the mutual orientation is parallel or anti-parallel to each other and we put $H = \beta J/2$. Expanding $g_2^{(1)}(\mu, \mu')$ in terms of the orthogonal set $\{1, \mu, \mu', \mu\mu'\}$ as

$$g_2^{(1)}(\mu, \mu') = 1 + a\mu' + b\mu\mu', \quad (6.1)$$

and substituting this in (4.16) and (4.17), for (4.15) we have

$$\lambda(a) = \frac{2^{-2} \sum_{\mu, \mu' = \pm 1} \exp [H' \mu \mu' + aH(z-1)(\mu + \mu')]}{\left[2^{-1} \sum_{\mu = \pm 1} \exp (Hza\mu) \right]^{2(z-1)/z}}; \quad H' = H, \quad (6.2)$$

then, using the distribution function, $g_2(\mu, \mu')$, and (3.10), we find

$$\sigma = \langle \mu \mu' \rangle_{AV} = (\partial \ln \lambda / \partial H')_{H'=H}, \quad (6.3)$$

$$s = 1/2 \cdot \langle \mu + \mu' \rangle_{AV} = 1/2 \cdot (\partial \ln \lambda / \partial [H(z-1)a])_{H=H}, \quad (6.4)$$

σ giving the the short range order and s the long range order. The calculation of (6.2) gives

$$\lambda(a) = \frac{2^{-1} [e^H \cosh 2H(z-1)a + e^{-H}]}{[\cosh Hza]^{2(z-1)/z}}. \quad (6.5)$$

The variational formula (4.14) is now written

$$\partial \ln \lambda(a) / \partial a = 0, \quad (6.6)$$

or

$$\frac{e^{H'} \sinh 2H(z-1)a}{e^{H'} \cosh 2(z-1)a + e^{-H'}} = \tanh Hza. \quad (6.7)$$

With the introduction of a parameter δ defined by $H\delta = \delta$, (6.7) becomes

$$e^{-2H'} = \sinh(z-2)\delta / z\delta; \quad H = H', \quad (6.8)$$

that is Bethe's expression, which determines the parameter a in (6.5).

The application of (6.3) to (6.5) gives

$$\sigma = \frac{\cosh 2(z-1)\delta - e^{-2H}}{\cosh 2(z-1)\delta + e^{2H}}$$

thus, using (6.8), we have

$$1 - \sigma = \frac{2 \sinh(z-2)\delta}{\sinh(z-2)\delta \cdot \cosh z\delta} \quad (6.9)$$

in complete accordance with Bethe's expression. Similarly, if we apply (6.4) to (6.5), with the help of (6.7), Bethe's expression for determining the long range order s is obtained as

$$s = \tanh z\delta. \quad (6.10)$$

While Bethe derived (6.8) from the self-consistent condition, we found the same from the minimum condition of the free energy. It is, of course, possible to derive (6.8) from the self-consistent condition in the present formulation. To see this situation it may be noticed that the two expressions for the average value of μ , $\langle \mu \rangle$, derived from A' and A'' respectively should be equal to each other. Hence the self-consistent solution makes the free energy of the system minimum at a certain degree of approximation.

§ 7. A metamorphosis of Bethe's approximation and a generalization of Kramers-Wannier's method

Hitherto, our considerations are confined to the system whose potential energy is given by the mutual potential between particles. We may take off this restriction by replacing expression (1.1) for $U\{N\}$ by

$$U\{N\} = \sum_{N \geq k \geq 1} u(k) + \sum_{N \geq k > l \geq 1} u(k, l). \quad (7.1)$$

In this system a particle has the potential energy determined by its state itself, thus the object for our application is extended to the system under the external field on the one hand, and to that composed of identical units which contain a number of particles on the other.

Now we have an interest in the latter case, then the particle may be more naturally called the *molecule*. Equation (4.1) and (3.10) are replaced by

$$\ln A' g_1(k) = -\beta u(k) - \beta \varepsilon \omega^{-1} \int u(k, k') g_2^{(1)}((k), (k')) d(k'), \quad (7.2)$$

$$\begin{aligned} \ln A'' g_2((k), (l)) = & -\beta \{u(k) + u(k, l) + u(l)\} \\ & -\beta (\varepsilon - 1) \omega^{-1} \sum_{k=k, l} \int u(k, k') g_2^{(1)}((k), (k')) d(k') \end{aligned} \quad (7.3)$$

adapted to the present case, respectively.

Equation (4.2) is replaced by

$$\ln \lambda g_2((k), (l)) = -\beta \{1/\varepsilon \cdot u(k) + u(k, l) + 1/\varepsilon \cdot u(l)\} + (\varepsilon - 1)/\varepsilon \cdot \ln[g_1(k) g_1(l)]. \quad (7.4)$$

The integral equation (4.4) remains valid if its kernel $K((k), (l))$ is replaced by

$$K((k), (l)) = \omega^{-1} \exp \{-\beta (1/\varepsilon \cdot u(k) + u(k, l) + 1/\varepsilon \cdot u(l))\}. \quad (7.5)$$

Let us consider Bethe's approximation of one linear chain surrounded by ε linear chains in the case of the simple cubic lattice, where a linear chain contains n particles. We have now two kinds of means to attack this problem. The one is given by (4.4) with the kernel (7.5), in which a linear chain is taken as a molecule. The other, in which we have an interest now, is given by the variational formula (4.14), where (4.16) and (4.17) are replaced by

$$A''[g_2^{(1)}] = \iint \exp \left[-\beta \{u(k) + u(k, l) + u(l)\} - \beta(z-1) \sum_{k=k,l} \int u(k, k') g_2^{(1)}((k), (k')) d(k') \right] d(k) d(l), \quad (7.6)$$

$$A'[g_2^{(1)}] = \int \exp \left[-\beta u(k) - \beta z \int u(k, k') g_2^{(1)}((k), (k')) d(k') \right] d(k) \quad (7.7)$$

from (7.3) and (7.1) respectively, where we put $\omega=1$ again.

Let us write down (4.14) with (7.6) and with (7.7) in terms of the Ising model again. Since the state of a molecule k is specified by the set of spin variables: $\mu_1, \mu_2, \dots, \mu_n = \{\mu_i\}$, $u(k)$ and $u(k, l)$ are written

$$u(k) = -J/2 \cdot \sum_{i=1}^n \mu_i \mu_{i+1}; n+1 \equiv 1, \quad u(k, l) = -J/2 \cdot \sum_{i=1}^n \mu_i \mu'_i, \quad (7.8)$$

$\{\mu'_i\}$ being the set of state variables of the l -th molecule.

If one considers the translational symmetry and the normalization condition of $g_2^{(1)}$, $g_2^{(1)}(\{\mu_i\}, \{\mu'_i\})$ may be expanded in the following series with respect to the orthogonal set $\{\mu_i \mu_j \dots \mu_k \mu'_l \dots \mu'_k\}$:

$$g_2^{(1)}(\{\mu_i\}, \{\mu'_i\}) = a_0 + a_1 \sum_i \mu'_i + a_2 \sum_i \mu_i \mu'_i + \dots \quad (7.9)$$

Then we expand as

$$\sum_{\{\mu_i, \mu'_i\}=\pm 1} \left(\sum_{i=1}^n \mu_i \mu'_i \right) g_2^{(1)}(\{\mu_i\}, \{\mu'_i\}) = b_0 + b_1 \sum_i \mu_i + b_2 \sum_i \mu_i \mu_{i+1} + \dots, \quad (7.10)$$

since $\sum_i \mu_i \mu'_i$ does not spoil the translational symmetry of (7.9).

The variational formula (4.14) is now written in the form

$$\lambda = \text{Max}_{\{b_n\}} A''(\{b_n\}) / [A'(\{b_n\})]^{2(z-1)/z}, \quad (7.11)$$

where we expected the stationary value of λ leading to the maximum and

$$A''(\{b_n\}) = \sum_{\{\mu_i\}=\pm 1} \sum_{\{\mu'_i\}=\pm 1} \exp \left[H \left(\sum_i \mu_i \mu_{i+1} + \sum_i \mu_i \mu'_i + \sum_i \mu'_i \mu'_{i+1} \right) + 2H(z-1)b_0 \right. \\ \left. + H(z-1)b_1 \left(\sum_i \mu_i + \sum_i \mu'_i \right) + H(z-1)b_2 \left(\sum_i \mu_i \mu_{i+1} + \sum_i \mu'_i \mu'_{i+1} \right) + \dots \right], \quad (7.12)$$

$$A'(\{b_n\}) = \sum_{\{\mu_i\}=\pm 1} \exp \left[H \sum_i \mu_i \mu_{i+1} + Hzb_0 + Hzb_1 \sum_i \mu_i + Hzb_2 \sum_i \mu_i \mu_{i+1} + \dots \right]. \quad (7.13)$$

It should be noticed that the variational formula (7.11) is subject to no restricting condition, since the normalization condition of $g_2^{(1)}$ is satisfied automatically by the appropriate constant term in the expansion (7.9) and this term vanishes in (7.10).

Our problem thus reduces to that for obtaining the analytical expressions of A' and A'' , which contain the set of variational parameters $\{b_n\}$. Since it is prohibitively difficult to solve this problem, we neglect all parameters but b_0 , b_1 and b_2 after Kramers-Wannier⁹⁾, and obtain

$$\lambda = \underset{A, B}{\text{Max}} \frac{\sum_{\{\mu_i\}=\pm 1} \sum_{\{\mu'_i\}=\pm 1} \exp [H \sum_i \mu_i \mu_{i+1} + (H/z + (z-1)A) (\sum_i \mu_i \mu_{i+z} + \sum_i \mu'_i \mu'_{i+1})]}{[\sum_{\{\mu_i\}=\pm 1} \exp \{z(A \sum_i \mu_i \mu_{i+1} + B \sum_i \mu_i)\}]^{2(n-1)/z} + (z-1)B(\sum_i \mu_i + \sum_i \mu'_i)]} \quad (7.14)$$

where

$$H(1+zb_2)=zA, \quad Hb_1=B. \quad (7.15)$$

The right-hand side of (7.14) may be obtained analytically, in which the numerator and the denominator are obtained from the partition function of two stripes and that of a linear chain respectively, as was adopted by Kramers-Wannier. In the case: $z=2$, n -th root of (7.14) of course reduces to Kramers-Wannier's formula.¹²⁾

In conclusion the author wishes to express his sincere thanks to Assist. Profs. S. Ono and T. Tanaka for their reading through the manuscripts.

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Referee's note: An undue time has been spent in inspecting this paper by external circumstances. The referee deeply appreciates his accountability to the author. K. Husimi

On Yukawa's Theory of Non-local Field, I

— The Case of Free Field —

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The relation of Yukawa's non-local field theory¹⁾ to the ordinary local field theory is investigated. In § 3, it is shown that the equations of motion and the commutation relations of the non-local field are derived from those of the local field by a canonical transformation, which means that both fields are equivalent at least in the case of no interaction. § 4 is devoted to some remarks on the feature of the non-local field as a mixed field composed of those with various spins. The considerations in this paper are all restricted to the free field.

§ 1. Introduction and summary

Recently, Yukawa discussed a generalization of the field concept by introducing a non-local field, which is free from the restriction that a field is determined as a point function in the ordinary coordinate space. This attempt is noteworthy at present, when the difficulty to remove various divergences in the present field theory within the framework of the local field theory has become clearer, as giving a new scope to the future theories. In his theory a finite radius of the elementary particle can be introduced without contradiction to the condition of Lorentz-invariance. In view of this promising nature of the non-local field, it would be interesting to investigate to what extent the theory of the non-local field will succeed in resolving the difficulties of the present field theory and in elucidating the structure of the elementary particles. In trying such a generalization, however, we have no guiding principle generally acceptable, and the Yukawa theory which was proposed under the guiding principle of Lorentz-invariance and reciprocity, could not avoid to include some speculative nature. Then, in order to make the character of the non-local field clearer, it may be desirable to investigate its relation to the local field. This may also serve to give a preparation for a further development of the non-local field theory.

The first step toward this line was made by Fierz²⁾. He anticipated the equivalence of the Yukawa model to the local field, at least in the case of no interaction, by pointing out that the Yukawa model could be regarded as a superposition of local fields of various spins. We investigate this situation furthermore by using the method of canonical transformation. In § 2, the equations of motion of the non-local field are derived by a canonical transformation from those of the local field. In the course of the procedure, a representation, in which K , L and M in (2.3) are diagonal, is used to make the correspondence

clearer. A quantized non-local field is discussed in § 3. In quantizing the field, we can start from any form of c -number equations of motion which can be transformed one another by canonical transformations. The correspondence of the non-local field to the local field as a quantized system is then given with respect to this arbitrariness in the c -number theory. A more detailed explanation concerning this point is given in the appendix. The commutation relations of the non-local field are also derived from those of the local field. In § 4, the internal freedom of the non-local field, which is explained as spin, is discussed. It is shown that the appearance of the higher spin part of the non-local field is related to whether the field is irreducible or not by the spatial rotation in the rest system of the center of mass, rather than whether the field is non-local or local.

From these analyses, we may conclude that the non-local field is equivalent to the local field at least in the case of no interaction, in the sense that a irreducible non-local field just corresponds to a irreducible local field. But a different manner to describe a field may have a possibility to take in new types of interaction more naturally than to modify the interaction type in the local field. The investigation of the interaction between non-local fields will form the subject of the succeeding paper.

§ 2. The relation of the non-local to the local field, I

(The c -number theory)

The equations of motion of the non-local scalar field* were given by Yukawa as follows**.

$$\begin{aligned}(\partial^2/\partial X_\mu \partial X^\mu - \kappa^2)U(X_\mu, r_\mu) &= 0, \\(r_\mu r^\mu - \lambda^2)U(X_\mu, r_\mu) &= 0, \quad r_\mu \partial/\partial X_\mu \cdot U(X_\mu, r_\mu) = 0.\end{aligned}\quad (2.1)$$

Let us consider the field $V(X_\mu, r)$, which satisfies the equations of motion to be obtained from (2.1) by annulling λ , as a local field.

$$\begin{aligned}(\partial^2/\partial X_\mu \partial X^\mu - \kappa^2)V(X_\mu, r) &= 0, \\r_\mu r^\mu V(X_\mu, r_\mu) &= 0, \\r_\mu \partial/\partial X_\mu \cdot V(X_\mu, r_\mu) &= 0.\end{aligned}\quad (2.2)$$

We shall show that we can find a canonical transformation which transform (2.2) into (2.1). For this purpose it is convenient to transform (2.1) and (2.2) into a representation, in which the following three operators are diagonal.

$$\mathbf{K} = \partial^2/\partial X_\mu \partial X^\mu, \quad \mathbf{L} = r_\mu r^\mu, \quad \mathbf{M} = (1/i)r_\mu \partial/\partial X_\mu. \quad (2.3)$$

\mathbf{K} , \mathbf{L} , and \mathbf{M} are Lorentz invariant Hermite operators and are mutually commutative. We write the simultaneous eigenfunction of \mathbf{K} , \mathbf{L} , and \mathbf{M} which belongs to the eigenvalues K , L , and M as $W(X_\mu, r_\mu, K, L, M, \dots)$. (As eigenfunctions specified by K , L , M

* For simplicity, we consider only the scalar field; the situation is essentially the same in other cases.

** Our notations are in accordance with those of Yukawa.

are in general degenerate, we need extra eigenvalues to remove the degeneracy. ...written in $W(X_\mu, r_\mu, K, L, M, \dots)$ means these eigenvalues). $W(X_\mu, r_\mu, K, L, M, \dots)$ satisfies the equations,

$$\begin{aligned} K \cdot W(X_\mu, r_\mu, K, L, M, \dots) &= K \cdot W(X_\mu, r_\mu, K, L, M, \dots), \\ L \cdot W(X_\mu, r_\mu, K, L, M, \dots) &= L \cdot W(X_\mu, r_\mu, K, L, M, \dots), \\ M \cdot W(X_\mu, r_\mu, K, L, M, \dots) &= M \cdot W(X_\mu, r_\mu, K, L, M, \dots), \end{aligned} \quad (2.4)$$

and the ortho-normalization condition and the completeness relation.

$$\begin{aligned} \int W^*(X_\mu, r_\mu, K, L, M, \dots) W(X_\mu, r_\mu, K', L', M', \dots) (dX) (dr) \\ = \delta(K - K') \delta(L - L') \delta(M - M') \dots, \end{aligned} \quad (2.5)$$

$$\begin{aligned} \int W^*(X_\mu, r_\mu, K, L, M, \dots) W(X'_\mu, r'_\mu, K, L, M, \dots) dK dL dM \dots \\ = \delta(X_\mu - X'_\mu) \delta(r_\mu - r'_\mu). \end{aligned} \quad (2.6)$$

The transformation to the new representation is readily performed with $W(X_\mu, r_\mu, K, L, M, \dots)$ as a transformation function. If we expand $U(X_\mu, r_\mu)$ and $V(X_\mu, r_\mu)$ in $W(X_\mu, r_\mu, K, L, M, \dots)$ as

$$U(X_\mu, r_\mu) = \int u(K, L, M, \dots) W(X_\mu, r_\mu, K, L, M, \dots) dK dL dM, \quad (2.7)$$

$$V(X_\mu, r_\mu) = \int v(K, L, M, \dots) W(X_\mu, r_\mu, K, L, M, \dots) dK dL dM. \quad (2.8)$$

$u(K, L, M, \dots)$ and $v(K, L, M, \dots)$ appear as wave functions in the new representation, and (2.1) and (2.2) take the following forms respectively.

$$\begin{aligned} (K - x^2) u(K, L, M, \dots) &= 0, \\ (L - \lambda^2) u(K, L, M, \dots) &= 0, \quad M \cdot u(K, L, M, \dots) = 0; \end{aligned} \quad (2.1')$$

$$\begin{aligned} (K - x^2) v(K, L, M, \dots) &= 0, \\ L \cdot v(K, L, M, \dots) &= 0, \quad M \cdot v(K, L, M, \dots) = 0. \end{aligned} \quad (2.2')$$

The differential operator in $K L M \dots$ space $i\partial/\partial L$ satisfies the commutation relations,

$$[K, i\partial/\partial L] = 0, \quad [L, i\partial/\partial L] = -i, \quad [M, i\partial/\partial L] = 0. \quad (2.9)$$

A Hermite operator*

$$P = [(i\partial/\partial L) + (i\partial/\partial L)^*]/2$$

where $(i\partial/\partial L)^*$ is the Hermite conjugate of $(i\partial/\partial L)$, satisfies the same commutation

* We must be careful in using $(i\partial/\partial L)^*$, since the domain of L depends on K and M , and does not extend from $-\infty$ to $+\infty$. For example, it runs from 0 to $+\infty$ when $K > 0$ and $M = 0$, and

$$(i\partial/\partial L)^* = i(\partial/\partial L + \delta(L)).$$

Singular character of $\delta(L)$ may cause difficulty in some cases, but we do not enter any detail here. We are grateful to Dr. Utiyama for his valuable discussion concerning this point.

relations as (2.9), i.e.,

$$[K, P]=0, \quad [L, P]=-i, \quad [M, P]=0.$$

Therefore an operator T defined by

$$T=e^{i\lambda^2 F} \quad (2.11)$$

is unitary and satisfies

$$[K, T]=0, \quad [L, T]=\lambda^2 T, \quad [M, T]=0. \quad (2.12)$$

Performing a canonical transformation with T as a transformation function, (2.2') is transformed into

$$\begin{aligned} (K-x^2)\bar{v}(K, L, M, \dots) &= 0, \\ (L-\lambda^2)\bar{v}(K, L, M, \dots) &= 0, \\ M\bar{v}(K, L, M, \dots) &= 0, \end{aligned} \quad (2.13)$$

where

$$\bar{v}=Tv. \quad (2.14)$$

(2.13) is of the same form as (2.1'), and \bar{v} can be regarded as a non-local field. This fact shows that the equations of motion of the non-local field can be derived from those of the local field by a canonical transformation. Transforming back to the original representation, (2.14) gives

$$\bar{V}(X_\mu, r_\mu) = \int T(X_\mu, r_\mu | X'_\mu, r'_\mu) V(X'_\mu, r'_\mu) (dX') (dr') = T^\dagger V(X_\mu, r_\mu), \quad (2.15)$$

where

$$\bar{V}(X_\mu, r_\mu) = \int \bar{v}(K, L, M, \dots) W(X_\mu, r_\mu, K, L, M, \dots) dK dL dM, \dots, \quad (2.16)$$

$$\begin{aligned} T(X_\mu, r_\mu | X'_\mu, r'_\mu) &= \int W(X_\mu, r_\mu, K, L, M, \dots) T(K, L, M, \dots | K', L', M', \dots) \\ &\times W^*(X'_\mu, r'_\mu, K', L', M', \dots) dK dL dM \dots dK' dL' dM' \dots, \end{aligned} \quad (2.17)$$

and $T(K, L, M, \dots | K', L', M', \dots)$, $T(X_\mu, r_\mu | X'_\mu, r'_\mu)$ are the matrix element of T in their respective representations. It is evident from (2.12) that the operator T in the $X_\mu r_\mu$ space defined by (2.17) is unitary and satisfies

$$\begin{aligned} T\partial^2/\partial X_\mu \partial X^\mu T^{-1} &= \partial^2/\partial X_\mu \partial X^\mu, \\ Tr_\mu r^\mu T^{-1} &= r_\mu r^\mu - \lambda^2, \\ Tr_\mu \partial/\partial X_\mu T^{-1} &= r_\mu \partial/\partial X_\mu. \end{aligned} \quad (2.18)$$

§ 3. The relation of the non-local to the local field, II

(The q -number theory)

When the field is quantized according to the procedure of the second quantization, $U(X_\mu, r_\mu)$ and $V(X_\mu, r_\mu)$ are regarded operators that satisfy, besides the equations of

motion (2.1) and (2.2), the commutation relations

$$\begin{aligned}
 & [U(X_\mu, r_\mu), U^*(X'_\mu, r'_\mu)] \\
 &= \int \frac{k_4}{|k_4|} \delta(k_\mu k^\mu + x^2) \delta(r_\mu r^\mu - \lambda^2) \delta(k_\mu r^\mu) \delta(r_\mu - r'_\mu) e^{i k_\mu (X^\mu - X'^\mu)} (dk), \\
 & [V(X_\mu, r_\mu), V^*(X'_\mu, r'_\mu)] \\
 &= \int \frac{k_4}{|k_4|} \delta(k_\mu k^\mu + x^2) \delta(r_\mu r^\mu - \lambda^2) \delta(k_\mu r^\mu) \delta(r_\mu - r'_\mu) e^{i k_\mu (X^\mu - X'^\mu)} (dk).
 \end{aligned} \tag{3.2}$$

Expanding the right hand of (3.1) in $W(X_\mu, r_\mu, K, L, M, \dots)$

$$\begin{aligned}
 & [U(X_\mu, r_\mu), U^*(X'_\mu, r'_\mu)] \\
 &= \int \alpha(K, L, M, \dots K', L', M', \dots) W(X_\mu, r_\mu, K, L, M, \dots) W^*(X'_\mu, r'_\mu, K', L', M', \dots) \\
 & \quad \times dK dL dM \dots dK' dL' dM' \dots,
 \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
 & \alpha(K, L, M, \dots K', L', M', \dots) \\
 &= \int \frac{k_4}{|k_4|} \delta(k_\mu k^\mu + x^2) \delta(r_\mu r^\mu - \lambda^2) \delta(k_\mu r^\mu) \delta(r_\mu - r'_\mu) e^{i k_\mu (X^\mu - X'^\mu)} \\
 & \quad \times W^*(X_\mu, r_\mu, K, L, M, \dots) W(X'_\mu, r'_\mu, K', L', M', \dots) (dk) (dX) (dX') (dr) (dr').
 \end{aligned}$$

In order to perform this integration, we first rewrite it in the form

$$\begin{aligned}
 & \int \frac{k_4}{|k_4|} \delta(\partial^2/\partial X_\mu \partial X^\mu - x^2) \delta(r_\mu r^\mu - \lambda^2) \delta\left(\frac{1}{i} r_\mu \partial/\partial X^\mu\right) \delta(r_\mu - r'_\mu) e^{i k_\mu (X^\mu - X'^\mu)} \\
 & \quad \times W^*(X_\mu, r_\mu, K, L, M, \dots) W(X'_\mu, r'_\mu, K', L', M', \dots) (dk) (dX) (dX') (dr) (dr').
 \end{aligned}$$

Integrating by part and making use of (2.4), we have*

$$\begin{aligned}
 & \delta(K - x^2) \delta(L - \lambda^2) \delta(M) \int \delta(r_\mu - r'_\mu) e^{i k_\mu (X^\mu - X'^\mu)} \\
 & \quad \times W^*(X_\mu, r_\mu, K, L, M, \dots) W(X'_\mu, r'_\mu, K', L', M', \dots) (dk) (dX) (dX') (dr) (dr').
 \end{aligned}$$

The integration over k_μ gives $(2\pi)^4 \delta(x_\mu - x'_\mu)$, and we have after the integration over X'_μ and r'_μ

$$\begin{aligned}
 & (2\pi)^4 \delta(K - x^2) \delta(L - \lambda^2) \delta(M) \int W^*(X_\mu, r_\mu, K, L, M, \dots) W(X_\mu, r_\mu, K', L', M', \dots) \\
 & \quad (dX) (dr).
 \end{aligned}$$

The last integral is nothing but (2.5), and we find finally

$$(2\pi)^4 \delta(K - x^2) \delta(L - \lambda^2) \delta(M) \delta(K - K') \delta(L - L') \delta(M - M') \dots \tag{3.4}$$

* Here we dropped $k_4/|k_4|$ in the integrand to simplify the calculation. This term is taken into account correctly if we supplement (2.3) with $\theta = \frac{1}{i} \partial/\partial X_4 / \left| \frac{1}{i} \partial/\partial X_4 \right|$, which is a invariant operator if $\frac{1}{i} \partial/\partial X_4$ is a time like vector, and use the simultaneous eigenfunction of K, L, M , and θ , $W(X_\mu, r_\mu, K, L, M, \theta, \dots)$ instead of $W(X_\mu, r_\mu, K, L, M, \dots)$.

Using this result, (3.3) becomes,

$$[U(X_\mu, r_\mu), U^*(X'_\mu, r'_\mu)] \\ = (2\pi)^4 \int W(X_\mu, r_\mu, x^2, \lambda^2, 0, \dots) W^*(X'_\mu, r'_\mu, x^2, \lambda^2, 0, \dots) d\ldots, \quad (3.5)$$

By exactly the same way, (3.2) is written in the form,

$$[V(X_\mu, r_\mu), V^*(X'_\mu, r'_\mu)] \\ = (2\pi)^4 \int W(X_\mu, r_\mu, x^2, 0, 0, \dots) W^*(X'_\mu, r'_\mu, x^2, 0, 0, \dots) d\ldots. \quad (3.6)$$

Performing a canonical transformation (see the Appendix)

$$\bar{V}(X_\mu, r_\mu) = TV(X_\mu, r_\mu),$$

where T is given by (2.17). We obtain, after using (2.18),

$$(\partial^2/\partial X_\mu \partial X^\mu - x^2) \bar{V}(X_\mu, r_\mu) = 0. \\ (r_\mu r^\mu - \lambda^2) \bar{V}(X_\mu, r_\mu) = 0, \quad r_\mu \partial/\partial X_\mu \bar{V}(X_\mu, r_\mu) = 0; \quad (3.7) \\ [\bar{V}(X_\mu, r_\mu), \bar{V}^*(X'_\mu, r'_\mu)] \\ = (2\pi)^4 \int W(X_\mu, r_\mu, x^2, \lambda^2, 0, \dots) W^*(X'_\mu, r'_\mu, x^2, \lambda^2, 0, \dots) d\ldots \quad (3.8)$$

(3.7) and (3.8) are the same forms as (2.1) and (3.5) respectively. $V(X_\mu, r_\mu)$ can then be regarded as a non-local field and it is concluded that, when the field is quantized, the equations of motion and the commutation relations of the non-local field can be derived from those of the local field by a canonical transformation.

§ 4. The problem of higher spin

In § 2 and § 3 we verified the equivalence of the non-local field to the local field. But the non-local field have apparent new degrees of freedom r_μ , and they were shown to correspond to various spin states. We consider in this section, what a counter part have these new degrees of freedom and higher spin states in the local field. The non-local scalar field with the momentum k_μ of the center of mass that satisfies (2.1) can be written, in the rest system of k_μ , in the following form

$$U(X_\mu, r_\mu) = \sum_{l,m} A_{lm}(X_\mu, r_\mu, x, \lambda) P_l^m(\theta, \varphi), \quad (4.1)$$

where

$$A_{lm}(X_\mu, r_\mu, x, \lambda) = C_{lm} \delta(r_\mu r^\mu - \lambda^2) \delta(xr^4) e^{i\mathbf{x} \cdot \mathbf{X}^4}. \quad (4.2)$$

Obviously A_{lm} satisfies Klein-Gordon's equation

$$(\square - x^2) A_{lm}(X_\mu, r_\mu, x, \lambda) = 0, \quad (4.3)$$

and transforms as D_l by the spatial rotation. This term can be interpreted, as first pointed out by Fierz, as a local field of mass x and spin l .

From this analysis it is found that (omitting trivial propotional factors) the new degrees

of freedom θ, φ serve only to construct unit vectors $P_l^m(\theta, \varphi)$ in the representation space of D_l , and do not express the new degrees of the freedom of the field.

As for the appearance of the higher spin field, we must notice that (4.1) is not irreducible by the spatial rotation. In general, the solution of Klein-Gordon equation with momentum k_μ has the following form in the rest system of k_μ if the condition is not imposed that it should be irreducible by the spatial rotation.

$$U(X_\mu) = \sum_{l,m} A'_{lm}(X_\mu, \mathbf{x}) e_{lm}, \quad (4.4)$$

$$(\square - \mathbf{x}^2) A'_{lm}(X_\mu, \mathbf{x}) = 0. \quad (4.5)$$

Here A'_{lm} transform as D_l , and e_{lm} 's are unit vectors in the representation space of D_l , which can be taken, to be $P_l^m(\theta, \varphi)$. A'_{lm} of (4.4) express the field of higher spin, exactly as A_{lm} of (4.1). Conversely, if the field is irreducible by the spatial rotation, the terms other than the first, therefore the terms of higher spins vanishes in (4.4) and in (4.1).

Therefore the appearance of higher spin parts in the non-local field is ascribed to the reducible character of the field. The characteristic feature of the non-local field should rather be found in having replaced the less intuitive representation space by the rotation of a rigid sphere.

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Appendix

It is well known that in the quantized field equation such as

$$(\square - \mathbf{x}^2) U(x_\mu) = 0, \quad (A.1)$$

the expression of the field quantity $U(x_\mu)$ is not uniquely determined, but has a certain arbitrariness. This originates from an arbitrariness in choosing the basic coordinate system to represent $U(x_\mu)$, and expressions which are transformed one another by unitary transformations are regarded as equivalent. There is, however, another kind of arbitrariness. When we quantize the field, it is rather evident that we can start from any c -number equation which can be transformed one another by canonical transformations, and we can expect that this situation survives in the q -number theory. Roughly speaking, c -number equation is a Schrodinger equation to determine the behaviour of a particle, the assembly of which is described by the q -number equation, and it is natural that the arbitrariness pertaining to the expression of the former reflects upon that of the latter.

In order to make the situation clearer, we consider the Klein-Gordon equation (A.1) in more detail. Separating the time depending term as

$$U(x_\mu) = T(t) V(x_1, x_2, x_3),$$

we obtain

$$\ddot{T} = -E^2 T, \quad (\Delta - \mathbf{x}^2) V = -E^2 V. \quad (A.2)$$

Using the eigenfunction of (A.2), $U(x_\mu)$ can be expanded as

$$U(x_\mu) = \sum \frac{1}{\sqrt{E_n}} a_n v_n(x) e^{i E_n t}. \quad (A.3)$$

When the field is quantized, $U(X_\mu)$ is regarded an operator that satisfies the following commutation relation.

$$i[U(X_\mu), U(X_{\mu'})] = D(X_\mu - X_{\mu'}),$$

which is equivalent, using (A.3), to $[a_n, a_n^*] = \delta_n$.

Then $N_n = a_n^* a_n$ has eigenvalues 0, 1, 2, ..., and the energy is given by

$$E = \frac{1}{2} \int [U_+(-\Delta + x^2)U_- + U_-(-\Delta + x^2)U_+] dV = \sum E_n \left(N_n + \frac{1}{2} \right). \quad (\text{A.4})$$

Thus we can consider that $U(X_\mu)$ represents the assembly of quanta with energy E_n ($n=0, 1, 2, \dots$).

On the other hand, we can start from an other c -number equation which is transformed from (A.1) by an unitary transformation

$$(g\Delta S^{-1} - x^2 - \partial^2/\partial t^2)\bar{U} = 0, \quad \bar{U} = SU, \quad (\text{A.5})$$

where S is an unitary operator independent of t . Performing the same procedure described above, one obtains instead of (A.2)

$$S(\Delta - x^2)S^{-1}\bar{V} = -E^2\bar{V}, \quad \bar{V} = SV, \quad (\text{A.6})$$

and the energy becomes

$$E = \frac{1}{2} \int [\bar{U}_+ S(-\Delta + x^2)S^{-1}\bar{U}_- + \bar{U}_- S(-\Delta + x^2)S^{-1}\bar{U}_+] dV = \sum E_n \left(N_n + \frac{1}{2} \right),$$

which is same as (A.4).

Both (A.1) and (A.5) describes an assembly of scalar particles of energy E_n , and physically they are regarded as equivalent. Thus we can conclude that the q -number field equations have, besides the ordinary one, the following arbitrariness

$$U \rightarrow S^{-1}U, \quad \square - x^2 \rightarrow S^{-1}(\square - x^2)S. \quad (\text{A.7})$$

(A.7) is in marked contrast to the ordinary unitary transformation in that S operates upon the coordinate variable X_μ of $U(X_\mu)$, which are regarded in the q -number theory as a parameter to specify the degree of freedom of the field.

In the case when the transformation function depends on time, the following equation appears instead of (A.5).

$$[S(\Delta - x^2)S^{-1} - S\partial^2/\partial t^2 S^{-1}]\bar{V} = -E_n^2\bar{V}.$$

The eigenvalue E_n' does not in general agree with E_n . Although our T defined by (2.11) commutes with $\partial/\partial X_\mu$, (X_μ is the coordinate of the center of mass), if we try to annihilate x (from (2.1)) in the same way, the transformation function becomes uncommutable with $\partial/\partial x_\mu$. Therefore x can not be annihilated by a mere unitary transformation. This fact shows a different character of mass and radius in the non-local field theory.

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Letters to the Editor

Microwave Resonance in Ferrimagnetic Substance

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February 15, 1951

Many interesting measurements of resonance absorption in various kinds of ferrites have been reported, the magnetism of which was interpreted by Néel¹⁾ with his theory of "Ferrimagnetism." The magnetism in ferrimagnetic substances is caused by the partially compensated magnetic moments of ions on the two crystallographically different lattice sites, in such a way that the total magnetic moment of one sublattice is directed to the direction of external magnetic field whereas that of the other is directed antiparallelly to the former, by the strong *negative* exchange interaction. In this communication, a theory concerning the resonance phenomena in a ferrimagnetic substance is proposed.

We consider a ferrite of inversed spinel type $\text{Fe}(\text{MFe})\text{O}_4$, as an example of the ferrimagnetic substances, in this substance, the ferric ions Fe^{+++} are considered to be situated on the lattice points both of sublattices f (tetrahedral) and c (octahedral), whereas other kinds of magnetic ions M^{++} are on the sublattice c . The Hamiltonian of the system in this substance is given by

$$\begin{aligned} \mathcal{H} = & \sum_i^f g_3 \beta H S_{zi} + \sum_j^c g_3 \beta H S_{zj} \\ & + \sum_k^c g_2 \beta H S_{zk} + \sum_{j>i}^{f,f} J_{ij} (s_i, s_j) \\ & + \sum_{i,j}^{f,c} J_{ij} (s_i, s_j) + \sum_{i,k}^{f,c} J_{ik} (s_i, s_k) \end{aligned}$$

$$\begin{aligned} & + \cdots + \sum_{k,j}^{c,c} J_{ik} (s_i, s_k) \\ & + \sum_{i>j}^{f,f} g_3^2 \beta^2 r_{ij}^{-3} [(s_i, s_j) \\ & - 3r_{ij}^{-2} (s_i, r_{ij}) (s_j, r_{ij})] + \cdots \\ & + \sum_{j>k}^{c,c} g_3 g_2 \beta^2 r_{jk}^{-3} [(s_j, s_k) \\ & - 3r_{jk}^{-2} (s_j, r_{jk}) (s_k, r_{jk})]. \quad (1) \end{aligned}$$

In this expression g_3 ($=2.00$) and g_2 are the Landé-factors corresponding to trivalent ions Fe^{+++} (6S) and divalent magnetic ions M^{++} respectively, H the external magnetic field, β the Bohr magneton, J_{ij} the exchange integral between i -th and j -th ions and \sum^f means the summation over the sublattice of f , and $\sum^{f,c}$ over both of sublattices f and c . In the above Hamiltonian, the subscripts i and j mean the trivalent ions on the sublattices f and c respectively, and k the divalent ions on the sublattice c . The equations of motion of the total spin moment of ions of the same type on each sublattice become

$$\begin{aligned} i\hbar \dot{S}_{f,c} &= \mathcal{H} S_{f,c} - S_{f,c} \mathcal{H}, \\ i\hbar \dot{S}_{fy} &= \mathcal{H} S_{fy} - S_{fy} \mathcal{H}, \\ i\hbar \dot{S}_{cx} &= \mathcal{H} S_{cx} - S_{cx} \mathcal{H}, \cdots \\ \cdots i\hbar \dot{S}_{2y} &= \mathcal{H} S_{2y} - S_{2y} \mathcal{H}, \quad (2) \end{aligned}$$

where $\mathbf{s}_f = \sum_i^f \mathbf{s}_i$, $\mathbf{s}_c = \sum_j^c \mathbf{s}_j$
and $\mathbf{s}_2 = \sum_k^c \mathbf{s}_k$. (3)

In the following calculation, Bragg and Williams' approximation was used, in which some of the spin operators were replaced by their average values. Strictly speaking this treatment is not correct, but it is supposed that the approximation is not so wrong as in the case of antiferromagnetic.²⁾ Putting $J_{ii} = J_{ff}$; $J_{ij} = J_{ik} = J_{jc}$

and $J_{jj} = J_{kk} = J_{jk} = J_{cc}$, and assuming $\mathbf{s}_f = i\omega \mathbf{s}_f$ etc., where ω being the circular frequency of the microwave, then the above equations were transformed into simultaneous equations containing six unknowns S_{fx} , S_{fy} , ..., S_{zy} . To solve these equations, it should be noted that the roots of the determinantal equation with the argument ω , which give the resonance conditions, should not contain any exchange integral $O(J)^*$ but such terms $O(g\beta H J_{cc}/J_{fc})$, since the energy differences between the levels containing exchange energies do not contribute to the microwave resonance absorption and the probabilities of such transitions may be small. In this way, the following resonance condition is obtained:

$$h\nu = g\beta [\{ H + (N_y - N_z) M_z \} \times \\ \times \{ H + (N_x - N_z) M_z \}]^{1/2}, \quad (4)$$

$$\text{with } M_z = g_3\beta(\langle S_{fx} \rangle + \langle S_{cz} \rangle) + g_2\beta\langle S_{2z} \rangle. \quad (5)$$

From the above condition, it can be seen that the effect of the dipolar interaction is converted into the form of demagnetization fields, which coincides exactly with the usual ferromagnetic case.³⁾ The effective g -values, \bar{g} , became approximately ($J_{cc}/J_{fc} \ll 1$)

$$\bar{g} = (g_3 G_3 + g_2 G_2) / (G_3 + G_2),$$

$$\text{with } G_3 = \langle S_{fx} \rangle + \langle S_{cz} \rangle,$$

$$G_2 = \langle S_{2z} \rangle, \quad (6)$$

or exactly

$$G_3 = \langle S_{fx} \rangle + \langle S_{cz} \rangle + \{ \langle S_{2z} \rangle + \langle S_{cz} \rangle \langle \langle S_{fx} \rangle \rangle \\ + \langle S_{cz} \rangle + \langle S_{2z} \rangle \} \langle S_{fx} \rangle^{-1} (J_{cc}/J_{fc}), \\ G_2 = \langle S_{2z} \rangle [1 + \{ \langle \langle S_{cz} \rangle \rangle + \langle S_{2z} \rangle \} / \langle S_{fx} \rangle \} \\ \times (J_{cc}/J_{fc})]. \quad (7)$$

The physical meaning of the approximate formula obtained will be quite simple. Based on the above formula, the variation of g -valued in the solid solution of Zn-ferrite and Ni-ferrite with the change of contents x of the former was calculated by the data proposed by E. W. Gorter⁴⁾ and by the interaction coefficient J_{cc}/J_{fc} by C. Guillaud,⁵⁾ and the result is given in Fig. 1.

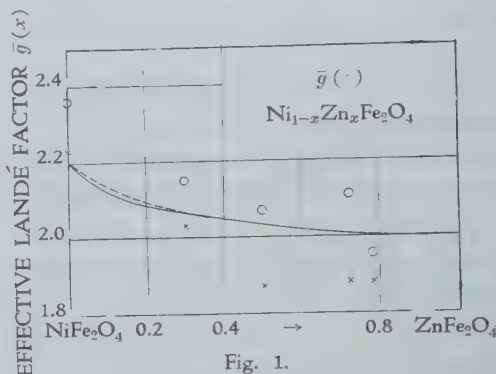


Fig. 1.

g -values of Ni-Zn-ferrite calculated by Eq. (5) (dotted line) and by Eq. (6) (full line) assuming $g_3 = 2.20$, and those obtained experimentally. Circles are referred to the case of 2 mm in diameter and crossed to that of 4 mm.

The measured points by Beljers and Polder⁶⁾ are also given in the figure. From this figure, a quite satisfactory agreement of the theory with experiments will be seen, though there are other data showing some discrepancies^{7), 8)}, the reasons of which are not still made clear.[†]

The author wishes to express his thanks to Prof. T. Hirone for his valuable discussions.

* $O(\)$ means of the order of.

† The anomalous g -values may probably come from an apparent cause.

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Transformation Function in Quantum Electrodynamics

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February 29, 1952

Recently Schwinger¹⁾ proposed the new treatment of the so-called Green function in quantumelectrodynamics. His method will afford the one way to the final solution of quantumelectrodynamics. However there are many unclear points for us. It is desirable to be published his detailed theory as soon as possible.

In this paper, we treat the transformation function in the interaction representation from the similar but little different point of view and then obtain the same results.

The transformation function in quantum-electrodynamics is given by

$$U(\sigma, \sigma_0) = P \exp \left[i \int_{\sigma_0}^{\sigma} dx j_{\mu} A_{\mu} \right],$$

$$j_{\mu} = e \psi^{\dagger} \gamma_{\mu} \psi \quad (1)$$

where P is the Dyson's chronological operator. Hereafter, we drop this P -symbol and then understand the exponentials to obey the Feynman's ordered calculus.²⁾

Following Wick³⁾ we decompose this function to the "well-ordered." Here, it is convenient to introduce the following operators:

$$\left[\frac{\delta}{\delta J_{\mu}}, \langle A'_{\nu} \rangle \right] = \delta_{\mu\nu} D_{+}(x, x'),$$

$$\left\{ \frac{\delta}{\delta \eta}, \langle \phi' \rangle \right\} = \left\{ \frac{\delta}{\delta \eta'^+}, \langle \phi^+ \rangle \right\}$$

$$= S_{+}(x, x'), \quad (2)$$

$$\left\{ \frac{\delta}{\delta \eta'^+}, \langle \phi' \rangle \right\} = \left\{ \frac{\delta}{\delta \eta'}, \langle \phi^+ \rangle \right\} = 0$$

and other commutators are zero. The symbol $\langle \rangle$ means the well-ordered factor.

Using these operators, we obtain the well-ordered forms of the transformation function. For instance, if we want the well-ordered form for photon field, we have

$$\langle U(\sigma, \sigma_0) \rangle_P$$

$$= \exp \left[i \int_{\sigma_0}^{\sigma} dx j_{\mu} \left(\langle A_{\mu} \rangle - i \frac{\delta}{\delta J_{\mu}} \right) \right]$$

$$= \exp \left[i \int_{\sigma_0}^{\sigma} dx j_{\mu} \langle A_{\mu} \rangle + \frac{i}{2} \times \right.$$

$$\left. \int \int_{\sigma_0}^{\sigma} dx dx' j_{\mu}(x) D_{+}(x, x') j_{\mu}(x') \right]. \quad (3)$$

This is the same result deduced by Glauber.⁴⁾ For electron field only, we obtain the following well-ordered form after the elemental calculations:

$$\langle U(\sigma, \sigma_0) \rangle_E = \exp \left[ie \int_{\sigma_0}^{\sigma} dx A_{\mu} \left(\langle \phi^+ \rangle \right. \right.$$

$$\left. \left. + i \frac{\delta}{\delta \eta} \right) \gamma_{\mu} \left(\langle \phi \rangle - i \frac{\delta}{\delta \eta'^+} \right) \right]$$

$$= e^{-L} \exp \left[ie \int_{\sigma_0}^{\sigma} dx \langle \phi^+ \rangle \gamma_{\mu} A_{\mu} \times \right.$$

$$\left. \times \left(1 - e \int_{\sigma_0}^{\sigma} dx' \gamma_{\mu} A_{\mu}' \langle \phi' \rangle \frac{\delta}{\delta \eta'} \right)^{-1} \langle \phi \rangle \right]$$

$$= e^{-L} \exp \left[ie \int_{\sigma_0}^{\sigma} dx \langle \phi^+ \rangle \gamma_{\mu} A_{\mu} \times \right.$$

$$\left. \times (1 - e S_{+} \gamma_{\mu} A_{\mu})^{-1} \langle \phi \rangle \right], \quad (4)$$

$$L = -S_p \log [1 - e S_{+} \gamma_{\mu} A_{\mu}]. \quad (5)$$

This result agrees the one obtained by many authors.⁵⁾

Next, we try to obtain the well-ordered form for both the electron and the photon fields. For this purpose, we may start from (3) or (4) and then take the above mentioned operations for electron or photon fields respectively. In practice, it is uninteresting to obtain such complicated form. Then, we restrict the case of one electron system, that is, we take from (4)

$$\begin{aligned} R_1 C = & -i \int_{\sigma_0}^{\sigma} dx \langle \psi^+ \rangle L_0 \left[\frac{1}{L_0} - \frac{1}{L_0 - eA} \right] \\ & \times L_0 \langle \psi \rangle \quad (6) \\ & \times \exp \left[-S_p \log \left(1 + eA \frac{1}{L_0 - eA} \right) \right], \end{aligned}$$

where $A = \gamma_\mu A_\mu$ and $S_+ = L_0^{-1}$. Following the above procedures, we obtain

$$\begin{aligned} \langle R_1 C \rangle_P = & -i \int_{\sigma_0}^{\sigma} dx \langle \psi^+ \rangle L_0 \left[\frac{1}{L_0} - G_+ \right] \\ & \times L_0 \langle \psi \rangle \cdot \exp \left[-S_p \right. \\ & \times \log \left(1 + e \left(\langle A \rangle - i \frac{\partial}{\partial J} \right) G_+ \right) \Big], \quad (7) \end{aligned}$$

where $G_+ = (L_0 - e(\langle A \rangle - i \partial / \partial J))^{-1}$. In this expression, the operation $\partial / \partial J = \gamma_\mu \cdot \partial / \partial J_\mu$ means to operate from left to right. For instance, the operator $\partial / \partial J$ in the first G_+ operates not only to $\langle A \rangle$ in the same G_+ , but also to the ones in the following other G_+ 's.

Considering this operation order, we calculate G_+ in the first bracket, i.e. the transition kernel of the single electron. Using Feynman's ordered operator calculus, we have

$$G_+ \cdot \exp \left[-S_p \left(1 + e \left(\langle A \rangle - i \frac{\partial}{\partial J} \right) G_+ \right) \right]$$

$$\begin{aligned} = & \int_0^\infty dS e^{-i \int_0^S ds' L_0 s' } e^{+ie \int_0^S ds' (\langle A s' \rangle - i \frac{\partial}{\partial J s'})} \times \\ & \times \exp \left[-S_p \left(1 + e \left(\langle A \rangle - i \frac{\partial}{\partial J} \right) G_+ \right) \right] \\ = & \left[\int_0^\infty dS e^{-i \int_0^S ds' L_0 s' } e^{ie \int_0^S ds' \langle A s' \rangle} \times \right. \\ & \times e^{ie^2 \int_0^S ds' \int_0^{s'} ds'' \frac{\partial \langle A s'' \rangle}{\partial J s''}} \left(e^e \int_0^S ds' \frac{\partial}{\partial J s'} - 1 \right) \times \\ & \times \exp \left[-S_p \left(1 + e \left(\langle A \rangle - i \frac{\partial}{\partial J} \right) \right. \right. \\ & \times \left. \left. G_+ \right) \right] \Big] e^{-I} \\ = & \left[\int_0^\infty dS e^{-i \int_0^S ds' L_0 s' } e^{ie \int_0^S ds' \exp \left(e \int_0^{s'} ds'' \frac{\partial}{\partial J s''} \right)} \times \right. \\ & \times \left. \left(\langle A s' \rangle + ie \frac{\partial}{\partial J s'} \int_{\sigma_0}^{\sigma} dx D_+(x_s, x) S_p \tau_\mu G_+(x, x) \right) \right] e^{-I} \end{aligned}$$

and consequently,

$$\begin{aligned} G_+ = & \int_0^\infty dS e^{-i \int_0^S ds' L_0 s' } e^{ie \int_0^S ds' (\langle A s' \rangle + ie \frac{\partial}{\partial J s'})} \times \\ & \times \int_{\sigma_0}^{\sigma} dx D_+(x_s, x) S_p \tau_\mu G_+(x, x) e^{-i \frac{\partial}{\partial J s'}}. \quad (8) \end{aligned}$$

If we introduce the new field variable a_μ by next definition

$$\begin{aligned} \langle a_\mu \rangle = & \langle A_\mu \rangle + ie \int_{\sigma_0}^{\sigma} dx' D_+(x, x') \times \\ & \times S_p \gamma_\mu G_+(x', x'), \quad (9) \end{aligned}$$

we have

$$G_+ = \int_0^\infty dS e^{-i \int_0^S ds' L_0 s' } e^{ie \int_0^S ds' (L_0 s' - e \tau_{s'} \langle a_{s'} \rangle + ie \tau_{s'} \frac{\partial}{\partial J s'})} \quad (10)$$

and

$$\langle R_1 \rangle_P = -i \int_{\sigma_0}^{\sigma} dx \langle \psi^+ \rangle L_0 \left[\frac{1}{L_0} - G_+ \right] L_0 \langle \psi \rangle. \quad (11)$$

From (9) and (10), we get Schwinger's equations

$$-\partial^2 \langle a_\mu \rangle = J_\mu + ie S_p \gamma_\mu G_+(x, x),$$

$$\left(L_0 - c\gamma_\mu \langle \alpha_\mu \rangle + ie\gamma_\mu \frac{\partial}{\partial J_\mu} \right) G_+ = 1 \quad (12)$$

and then

$$\langle R_1 \rangle_F = i \int_{\sigma_0}^{\sigma} dx \langle \psi^+ \rangle \left[c\gamma_\mu \langle \alpha_\mu \rangle G_+ - ie\gamma_\mu \frac{\partial G_+}{\partial J_\mu} \right] L_1 \langle \psi \rangle. \quad (13)$$

In conclusions, the author thanks to Prof. M. Kobayasi and Prof. K. Husimi for

their kind guidances and encouragements. He also wishes to express his gratitude for the financial aids from the Yukawa Fellowship of Osaka University.

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Errata

Neutral-Meson Production by Gamma-Ray

Shigeo MINAMI (Prog. Theor. Phys. **7** (1952), 69)

p. 70, last line,	for $H(x)$	read $H(x)$
p. 71, 6 line,	for $S_\mu^0(ig^0/\mu)\bar{\psi}\gamma_5\gamma_\mu\tau\psi$	read $S_\mu^0 = (ig^0/\mu)\bar{\psi}\gamma_5\gamma_\mu\tau\psi$
p. 75, 9 line,	for Y_0	read Y^0
p. 79, 16 line,	for $G_A'^2$	read $G_A'^2$
p. 81, (3) in Fig. 4,	for e^2f^5	read e^2f^6
p. 85, first line,	for with	read which
p. 86, 6 line,	for $\lambda_{IV}(a)$	read $\lambda_{VI}(a)$
p. 88, footnote,	for ...are equal...	read ...are not equal...
p. 89, Table 6,	for $(10^{30} \text{ cm}^2/\text{sterad.})$	read $(10^{-30} \text{ cm}^2/\text{sterad.})$

Meson Reactions in Deuterium and Meson-Nucleon Scattering

Yoshio YAMAGUCHI (Prog. Theor. Phys. **7** (1952), 93)

The ordinates in Fig. 2 (p. 98) should be multiplied by 2.

Nuclear Cross Sections for Fast Neutrons and Interaction between High Energy Nucleons

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(Received January 7, 1952)

Recent observation of proton-proton scattering shows that the agreement with existing phenomenological theories based on static potential is rather poor, and considerations of velocity dependent forces are required. It will be shown, however, that the only familiar velocity dependent force, spin-orbit coupling introduced by Case and Pais, are not acceptable from the consideration of high energy neutron-nucleus scattering.

§ 1. Introduction

On scattering of fast neutrons by nuclei numerous experiments¹⁾ have been performed about various nuclei. If only a ordinary finite interaction exists between nucleons, it is expected that the cross sections will decrease continuously with increasing energies of incoming neutrons. Actually it is found, however, that the cross sections are constant at high energies, above 150 Mev. and that the drop to these high energy levels occurs very sharply at intermediate energies, between 90 Mev and 150 Mev. These remarkable results show the presence of a short range singular force between nucleons at high energies. It is already accepted that a certain singular interaction must be taken into consideration to explain the high energy proton-proton scattering.^{2),3),4)} Christian and Noyes strongly deny the charge independence and introduce a singular tensor interaction, but on the other hand, preserving the assumption of the charge independence, Case and Pais explain the situation with spin-orbit coupling qualitatively and Jastrow with hard core model respectively. Recently newly observed results⁵⁾ on proton-proton scattering at high energies are published, and it is shown that agreement with existing phenomenological theories based on static potential is rather poor and considerations of velocity dependent forces are desired. The only familiar velocity dependent force is spin-orbit coupling and it seems desirable to inquire into its detail. Under the assumption of the charge independence we have hard core model and spin-orbit interaction, and have not a decisive criterion between the two. In these situations about nucleon-nucleon interaction it is expected that anomaly of the cross sections of neutron-nucleus scattering may give a clear interpretation for these points. Already Jastrow⁶⁾ have shown that with his hard core model anomaly of neutron-nucleus cross section is explained to some extent. Although satisfactory agreement with experimental results is not attained, it is certain the very cause explaining proton-proton scattering anomaly is also effective in this case. Contrary to this feature it will be found that in the velocity dependent spin-orbit interaction the very cause which can explain the

proton-proton scattering prevents from accounting for the neutron-nucleus anomaly, and as to the velocity dependent force in high energy regions another approach is required.

§ 2. Optical treatment in neutron-nucleus scattering

To treat the scattering of high energy neutrons by nuclei it is appropriate to calculate the problem based on the notion of transparent nucleus proposed by Serber.⁷⁾ According to this assumption nuclear matter is described by specifying its refraction index n and absorption coefficient K analogous to optics. Fernbach et al⁽⁸⁾ calculated this refraction index n in the following way. They adopted Fermi gas model, replace the nuclear potential V by Fermi energy plus 8 Mev, take the energies of neutron in the nucleus as $E + V$, E being the energy of incoming neutron, and determine the refraction index n (or k : difference of wave vector of neutron in and outside the nucleus which is immediately related to n) by taking the ratio of neutron energy in and outside the nucleus. It is, however, not a accurate recipe, because V depends on the incident energy of neutron, and further, to examine the effect of singular potential it seems meaningless to take nuclear potential as Fermi energy plus 8 Mev. We, then, compute the refraction index with the following formula;

$$n = 1 + \frac{2\pi\rho}{k_0^2} \left\{ \frac{Zf_{np}(0)}{A} + \frac{(A-Z)f_{nn}(0)}{A} \right\}, \quad (1)$$

$$\rho = \frac{3A}{4\pi R^3} : \text{nucleon density } (R; \text{nuclear radius}),$$

$$A : \text{mass number of nucleus},$$

$$Z : \text{number of protons},$$

$$k : \text{wave number of incoming neutron}.$$

We investigate $f_{na}(0)$, $f_{np}(0)$ only, forward scattering amplitude of neutron-neutron and neutron-proton in the laboratory system, as forward scattering alone contributes in the diffraction scattering. The forward scattering amplitudes are not uniquely determined by experiments even on the supposition of identical neutron-neutron and proton-proton forces, but must be computed for the particular interaction of the nucleon-nucleon system.

Absorption coefficient is shown in the following expression:

$$K = \rho \times \frac{Z\sigma_{np} + (A-Z)\sigma_{nn}}{A} \quad (2)$$

σ_{np} is replaced by the cross section for free neutron-proton scattering. Assuming the equality of neutron-neutron and proton-proton force, the observed proton-proton scattering cross section is inserted for σ_{nn} . The consideration of the effect of binding of target nucleons in the nucleus is desirable. For example, Goldberger⁷⁾ took into consideration it by multiplying $\sigma_{np(\text{free})}$ by a factor 4/5. But as we need the comparison with Jastrow's results and he does not regard the effect of nuclear binding, we neglect it here. Further,

us we wish to show the inadequacy of spin-orbit coupling, if it is shown that the discrepancy obtained is so large as not to be remedied with introducing the nuclear binding, it may not be a serious error to assert the conclusion derived without considering it.

The portion of the wave of incident neutron which strikes the nucleus described by a sphere at a distance ρ from a line through the center of the sphere emerges after traveling a distance $2t$, with $t^2 = R^2 - \rho^2$. We neglect refraction at the surface of the sphere, since it gives only a smaller correction. Its amplitude on emerging is $a = \exp(-K + 2ik_1)$ where k_1 is a difference of a wave vector in and outside the nucleus,

$$k_1 + k_0 = nk_0. \quad (3)$$

Hereafter the influence of refraction index n is represented through k_1 . The absorption cross section σ_a is

$$\begin{aligned} \sigma_a &= \int (1 - |a|^2) dV = 2\pi \int_0^R (1 - e^{-2Kt}) \rho d\rho \\ &= 2\pi \int_0^R (1 - e^{-2Kt}) t dt. \\ &= \pi R^2 [1 - \{1 - (1 + 2KR)e^{-2KR}\} / 2K^2 R^2]. \end{aligned} \quad (4)$$

The diffraction cross section can be derived from the consideration that behind the disk the wave differs from a plane wave by an amplitude $1 - a$ which shows a scattered wave. The corresponding cross section is

$$\begin{aligned} \sigma_d &= \int |1 - a|^2 dV = 2\pi \int_0^R |1 - e^{(-K + 2ik_1)t}|^2 \rho d\rho \\ &= \pi R^2 [1 + (1/2K^2 R^2) \{1 - (1 + 2KR)e^{-2KR}\} \\ &\quad - (1/(\frac{1}{4}K^2 + k_1^2) R^2) \{(\frac{1}{4}K^2 - k_1^2) \\ &\quad + e^{-KR} [2k_1 R (\frac{1}{4}K^2 + k_1^2) + k_1 K] \sin k_1 R \\ &\quad - e^{-KR} [(\frac{1}{4}K^2 - k_1^2) + KR (\frac{1}{4}K^2 + k_1^2)] \cos^2 k_1 R \}]. \end{aligned} \quad (5)$$

Then, the total cross section of the neutron-nucleus scattering σ_t is the sum of these two

$$\sigma_t = \sigma_a + \sigma_d.$$

§ 3. Features of spin-orbit interaction and hard core model

We, here, review how the spin-orbit coupling and hard core model can explain the behavior of the high energy proton-proton scattering. Examining the expression of the cross section for proton-proton scattering which is derived by partial wave method, we will find it readily that successive even Legendre polynomials alternate in sign at 90° , so the interference terms having successive even angular momentum, e.g. S - D term, are negative at 90° , positive at 0° , and further the contributions from the states of the odd angular momenta are zero at 90° and rise to maximum at 0° . Accordingly, in the case of the central force the forward scattering predominates. In the hard core model, however, S

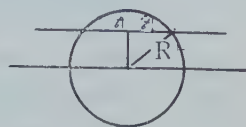


Fig. 1.

wave is more influenced by the core than the outer potential above the certain critical energy, and S phase shift changes from positive to negative. Waves of higher angular momenta are still influenced by the outer potential, and signs of phase shifts remain as in the cases of the lower energies. Then S - D interference term become positive at 90° and negative at 0° , so forward scattering is reduced, and a certain contribution at 90° is assured. Namely, above a certain critical energy the effect which makes the contribution at 90° is tightly related to the effect which reduces the forward scattering. Reduction of the forward scattering have then the consequence that the refraction index is influenced by it and total cross section will be decreased. In the case of the spin-orbit interaction the situation has a different feature.

Scattering amplitude in the center of gravity system reads as follows :

$$F(\theta) = -\frac{g}{4\pi} \sum_{\text{spin}} \int \chi_f^* \exp(i\mathbf{P}'\mathbf{X}) (\alpha P_x + \beta) \phi(x) \times \mathbf{L} \cdot \mathbf{S} \chi_i \exp(i\mathbf{P}\mathbf{X}) d\mathbf{X}, \quad (6)$$

P_x : exchange operator,

$\phi(x)$: function which shows radial dependence of spin-orbit interaction, e.g. singular function such as $\frac{1}{\lambda x} \frac{d}{d(\lambda x)} \frac{e^{-\lambda x}}{\lambda x}$,

g : coupling constant of spin-orbit interaction.

We consider here the neutron-proton case, for we wish to examine only the property of the matrix element. In the proton-proton collision we have only to antisymmetrize the wave function. Calculating this matrix element we obtain

$$F(\theta) = -\sqrt{2} g P^2 \sin \theta [aI(M_1) - \beta I(M_2)],$$

$$I(M) = \frac{1}{M^3} \int_0^\infty r \phi(r) [\sin Mr - Mr \cos Mr] dr, \quad (7)$$

$$M_1 = 2P \cos \frac{\theta}{2}, \quad M_2 = 2P \sin \frac{\theta}{2},$$

θ : scattering angle.

The limiting case $\theta \rightarrow 0$ of $F(\theta)$ gives the forward scattering amplitude. In this limit $I(M)$ being finite, the matrix element becomes proportional to $\sin \theta$, so it vanishes entirely as $\theta \rightarrow 0$. Conversely this behavior of $\sin \theta$ assures the actual value of the cross section at 90° . Namely, the very effect which makes contribution to the cross section at 90° have nothing to do with the forward scattering. It seems that with this property of spin-orbit coupling we can not understand the anomaly of neutron-nucleus scattering. As above mentioned, only the central force being effective in the forward direction, it is expected that the cross section will be decreased continuously with increasing energies of incident neutron, which is clearly contradict with the observed sharp drop between the intermediate energies. Even in the case of proton-proton collision, we could expect the difference of the features of the forward scattering between these two potentials, which can not actually be detected in experiments because of predominant Coulomb scattering.

§ 4. Numerical calculations and conclusion

Now it will be shown in the following that the above mentioned argument is surely justified quantitatively. We neglect the tensor force, it scarcely contributing to the forward scattering. The forward scattering amplitudes for neutron-proton and proton-proton system read as follows in the laboratory system,

$$\begin{aligned} f_{np}(0) &= \frac{m}{4\hbar^2} (3C_3 + C_1) \left\{ \frac{1}{x^2} + \frac{1}{x^2 + k_0^2} \right\}, \\ f_{nn}(0) = f_{pp}(0) &= \frac{m}{4\hbar^2} C_1 \left\{ \frac{1}{x^2} + \frac{1}{x^2 + k_0^2} \right\}, \end{aligned} \quad (8)$$

m : nucleon mass,

$1/x = 1.2 \times 10^{-13}$ cm,

k_0 : wave number of incoming neutron,

$$\frac{C_3}{\hbar c} = 0.404, \quad \frac{C_1}{\hbar c} = 0.280.$$

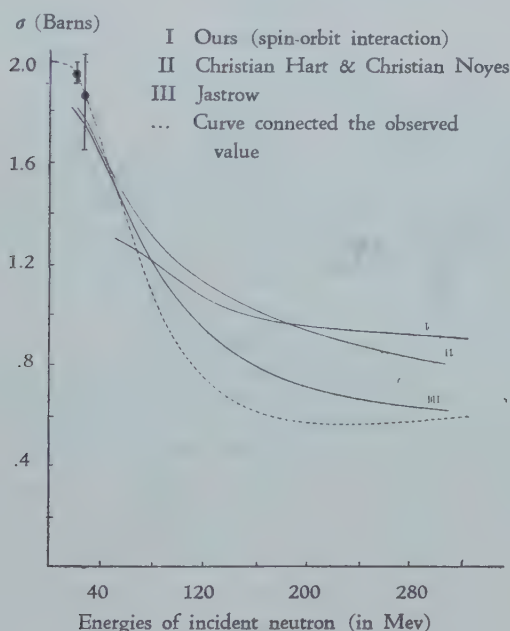


Fig. 2.

We will find it readily from these expressions that $f_{np}(0)$, $f_{nn}(0)$ diminish slowly with the increasing energies of incident neutron. With these and the expression (1) we can compute the refraction index. To acquire the absorption coefficient we use the observed cross sections of neutron-proton and proton-proton collision. For the free proton-proton scattering only the differential cross sections^{(10), (11), (12)} can be observed, since forward nuclear scattering is masked by Coulomb scattering. So making the assumption of the isotropic scattering we take the total cross section as the differential multiplied by 4π . As to the

energy dependence we assume the $1/E$ law between 120 Mev and 30 Mev, which is supposed correct to some extent, and insert a constant value above 120 Mev. Equality of neutron-neutron and proton-proton forces is adopted again. With regard to the total cross section of neutron-proton scattering the observed values are obtained at 40 Mev,^{13), 14)} 90 Mev,^{13), 15), 16)} and 260 Mev.¹⁷⁾ It seems that we can assume $1/E$ dependence also in this case within the experimental errors. With these observed values and expression (2) the absorption coefficient can be computed immediately. Using the refraction index and the absorption coefficient thus obtained we calculate the diffraction cross section σ_d and the absorption cross section σ_a and finally attain the total cross section σ_t . The calculation is performed for Al ($Z=13$, $A=27$). The results are plotted against incoming neutron energies. It seems evident that with spin-orbit interaction anomaly of neutron-nucleus scattering can not be understood satisfactorily as is expected. So that it will not be a grave error to assert that if we need a velocity dependent force between high energy nucleons another approach will be more desirable.

In conclusion the present author expresses his heartiest thanks to Messrs. S. Tani and Y. Fujimoto for their valuable discussions.

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On the Theory of the Dielectric, Piezoelectric, and Elastic Properties of $\text{NH}_4\text{H}_2\text{PO}_4$

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The theory of KH_2PO_4 developed by Yomosa and the present writer can be applied to the upper phase of $\text{NH}_4\text{H}_2\text{PO}_4$ without any alteration. All the experimental data can be represented by the formulas of this theory. It is shown that this theory has a more general character than it appears as based upon Slater's model of KH_2PO_4 , and, when applied to Rochelle salt, it is equivalent to Mueller's interaction theory. Some discussions on the phase change in $\text{NH}_4\text{H}_2\text{PO}_4$ are given.

§ 1. Introduction

The crystalline $\text{NH}_4\text{H}_2\text{PO}_4$ shows a phase transition at 148°K accompanied by a very sharp peak of the specific heat curve.¹⁾ The narrowness of the transition range seems to indicate that the transition is of the nature of the first kind, in spite of its apparent character of the second kind. The nature of the transition has not yet been made clear.

The properties of this substance are not very similar to those of KH_2PO_4 . Its dielectric constant^{2),3)} in the direction of the tetragonal axis, ϵ_{\parallel} , does not show so pronounced an anomaly at the transition point as that of KH_2PO_4 , and the dielectric constant perpendicular to the tetragonal axis, ϵ_{\perp} , has greater values than ϵ_{\parallel} , contrary to the case of KH_2PO_4 . The crystal, which breaks into pieces on passing the transition temperature from above, shows below it small values of the dielectric constants and low values of the dielectric loss⁴⁾, indicating that the lower modification probably does not have a spontaneous polarization, unlike in KH_2PO_4 . There are, however, similar properties between these substances, namely, that the piezoelectric constant d_{33} takes anomalously high values and the elastic compliance s_{66} shows large temperature variations above the transition temperature.³⁾

The crystal structure of the upper phase was studied in detail by Ueda⁵⁾ in Japan, and also by Pepinsky and Fraser⁶⁾ in Pennsylvania. Their results show that the O-O distance of the hydrogen bonds connecting the neighboring PO_4 groups is 2.49\AA (Ueda) or 2.51\AA (Fraser), smaller than the corresponding value for KH_2PO_4 , 2.54\AA . Each nitrogen is surrounded by four oxygen atoms in a form of a flattened tetrahedron, and the N-O distance is 2.87\AA (Ueda) or 2.84\AA (Fraser). (The next nearest N-O distance is 3.09\AA). It was inferred from this small value of the N-O distance that the formation of the $\text{N-H}\cdots\text{O}$ hydrogen bonds may play some important rôle for the phase transition. It must be noticed, however, that the sum of the ionic radii of NH_4^+ and O^{--} is nearly equal to the observed N-O distance, namely, $1.48 + 1.40 = 2.88\text{\AA}$, and this indicates a relatively unimportant rôle played by such bonds. Moreover, it will be shown in the following that

the anomalous properties of $\text{NH}_4\text{H}_2\text{PO}_4$ in its upper phase can be accounted for by the same theory as that of KH_2PO_4 where K^+ is spherically symmetric.

It is reported⁷ that the tetragonal modification of RbH_2PO_4 has similar properties as KH_2PO_4 . Since the ionic radius of Rb^+ is almost equal to that of NH_4^+ , this fact implies that the ionic radius of the constituent cation is not a sole determinate factor, but that there are other factors which must be taken into consideration, such as the polarizability, the steric form of the cation, the electrostatic interaction among cations or between cations and anions (since the ammonium ion has a strong octapole moment), and so on. An interesting investigation was carried out by Merz⁸ on the mixture of ammonium and thallium phosphates in search for the effect of the cation on the properties of the crystal. Kiriya⁹ also carried out an experiment with the mixture of ammonium and potassium salts. No satisfactory theoretical interpretation of their results is yet put forward.

The present paper concerns mainly with the properties of $\text{NH}_4\text{H}_2\text{PO}_4$ in its upper modification. It will be shown that ϵ_c , d_{30} , and s_{66} are all consistently represented by the formulas of the theory developed by Yomosa and the present writer¹⁰ for the case of KH_2PO_4 . Some discussions about the phase transition will then be given, although no definite conclusion can be attained. Finally, the general character of our theory is explained and, on applying our theory to Rochelle salt, Mueller's interaction theory is shown to be identical with ours.

§ 2. Properties of $\text{NH}_4\text{H}_2\text{PO}_4$ above the transition point

Extensive measurements of the dielectric, piezoelectric, and elastic properties of this substance have been carried out by Mason.³ His results above the transition point can be summarized by the following formulas:

$$\epsilon_c^F = 1 + 4\pi\chi_c^F = 7 + \frac{2690}{T+17}, \quad \chi_c^F = 0.48 + \frac{214}{T+17},$$

$$\epsilon_c^C = 1 + 4\pi\chi_c^C = 6.8 + \frac{2510}{T+67}, \quad \chi_c^C = 0.46 + \frac{200}{T+67},$$

$$d_{30} = \left(0.5 + \frac{303}{T+17}\right) \times 10^{-6} \text{ cgs. esu},$$

$$s_{66} - \frac{(d_{30} - 0.5 \times 10^{-6})^2}{\chi_c^F - 0.48} = 1.47 \times 10^{-11} (1 + 0.0008t) \text{ cgs. esu},$$

(t : degree in $^{\circ}\text{C}$)

$$\epsilon_a^F = 1 + 4\pi\chi_a^F = 7 + \frac{18000}{T+55}, \quad \chi_a^F = 0.48 + \frac{1430}{T+55}.$$

Here the upper indices F and C of the dielectric constant ϵ_0 and the susceptibility χ_c mean that the crystal is subjected to no constraint (free) or it is not strained (clamped), respectively. These results are shown in Fig. 1 and 2.

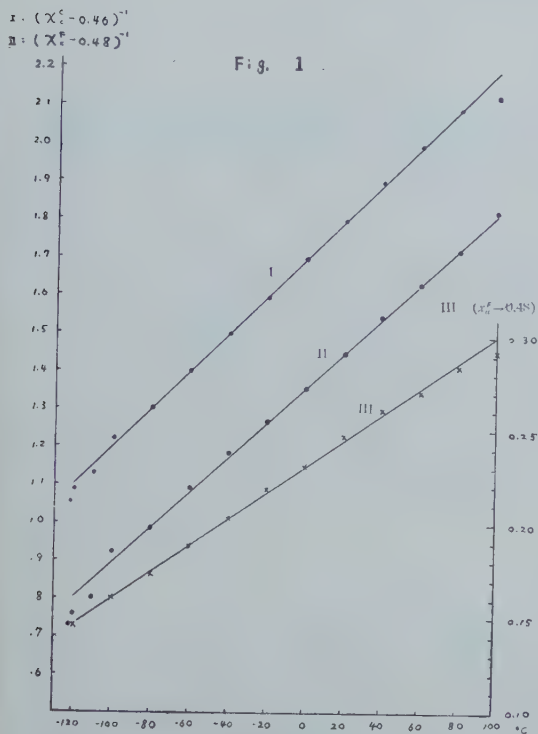


Fig. 1. Measured points for $(\chi_c^C - 0.46)^{-1}$, $(\chi_c^P - 0.48)^{-1}$ and $(\chi_a^P - 0.48)^{-1}$.

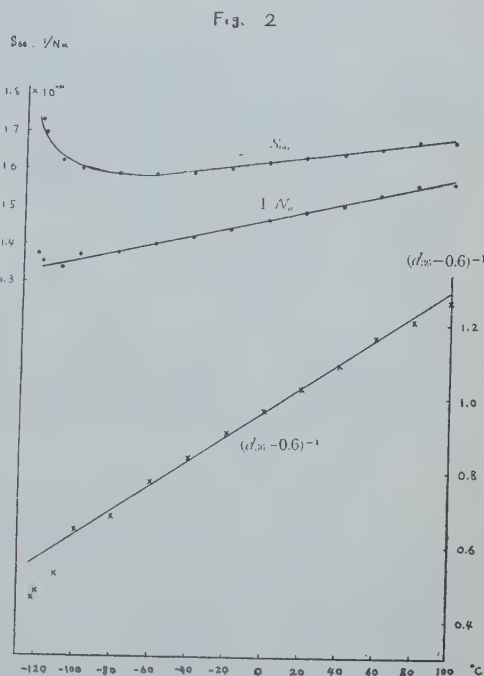


Fig. 2. Curves for s_{66} , $1/Na$, and d_{36} . The former two are measured in units of 10^{-11} cgs, and the latter in units of 10^{-6} cgs. esu.

Yomosa and the writer¹⁰⁾ developed a theory for the quantities given above in the case of KH_2PO_4 except for the dielectric constant in the direction of the axis a . They are expressible in the form

$$\chi_c^P = \frac{N\mu'^2}{D - \beta^2/u} + \chi_1 + \frac{\zeta_1^2}{Nu}, \quad \chi_c^C = \frac{N\mu^2}{D} + \chi_1,$$

$$d_{36} = \frac{\beta}{a} \frac{\mu'}{D - \beta^2/u} + \frac{\zeta_1}{Nu},$$

$$s_{66} = \frac{1}{Nu} + \frac{\beta^2}{Na^2} \frac{1}{D - \beta^2/a} = \frac{1}{Na} + \frac{(d_{36} - \zeta_1/Na)^2}{(\chi_c^P - \chi_1 - \zeta_1^2/Na)},$$

where

$$D = kT(e^{-\varepsilon_0/kT} - 1), \quad \mu' = \mu + \frac{\zeta_1\beta}{Nu}.$$

Here μ is the dipole moment of an H_2PO_4 group directed to + or - c -axis, β the coupling constant between the strain x_y and the dipolar polarization to the c -axis, u the normal elastic constant per molecule, ε_0 the energy parameter introduced by Slater (his ε), and χ_1 and ζ_1 are constants. For temperatures higher than the transition temperature, D

can be written approximately as

$$D = kT - 2\varepsilon_0.$$

Now, the direct comparison between these empirical and theoretical formulas yields the following numerical relations:

$$\chi_1 = 0.46, \quad \frac{\xi_1^2}{Na} = 0.02,$$

$$\mu' = 1.77 \times 10^{-18}, \quad \mu = 1.71 \times 10^{-18}, \quad \therefore \frac{\xi_1 \beta}{Na} = 0.06 \times 10^{-18},$$

$$-\frac{2\varepsilon_0}{k} = 67^\circ, \quad \frac{\beta^2}{ak} = 50^\circ,$$

$$\frac{\xi_1}{Na} = 0.5 \times 10^{-6}, \quad \frac{\beta \mu'}{ak} = 303 \times 10^{-6},$$

$$\frac{1}{Na} = 1.47 \times 10^{-11} (1 + 0.0008 t).$$

Here the elastic constant $1/Na$ alone was obtained as temperature dependent but its temperature coefficient is of quite a normal magnitude. Other constants may also be more or less temperature dependent, but it is not possible to determine their dependencies in the present comparison, since they can be taken care of by adjusting numerical parameters in the empirical formulas.

There are six independent parameters ε_0 , μ , a , β , χ_1 , ξ_1 in the above nine independent numerical relations. It can be shown that these relations are satisfied by taking a suitable set of values of the independent parameters, as follows. (1) From X-ray measurement we have for the number of molecules in one cubic centimeter

$$N = 0.951 \times 10^{22}.$$

This value, combined with the value of $1/Na$ at 0°C , yields

$$a = 7.15 \times 10^{-12}.$$

From this and the value for β^2/ak , we have $\beta/a = 0.031$. On the other hand, combination of the values for $\beta \mu'/ak$ and μ' gives $\beta/a = 0.024$. The agreement is not perfect. However, if we take into consideration the temperature dependency of a in the formulas for d_{30} and χ_c^F , a simple consideration will show that a value about 20% higher must be assigned to $\beta \mu'/ak$ and a value about 9% lower to β^2/ak . We then have from the first combination $\beta/a = 0.028$ and from the second $\beta/a = 0.029$, in satisfactory agreement. We have therefore

$$\beta/a = 0.028, \quad \beta = 0.2 \times 10^{-12}.$$

(2) Next, from the values for ξ_1/Na and Na we have

$$\zeta_1 = 0.34 \times 10^5, \text{ from which } \frac{\zeta_1^2}{Nu} = 0.017.$$

The last value is in agreement with that obtained by direct comparison, 0.02. But this agreement is rather apparent, because any small value of this order can be assigned to the latter by adjusting numerical values in the empirical formula for the clamped susceptibility without seriously affecting the agreement with observations. (3) Furthermore, from the values for ζ_1/Nu and the value for β obtained above, we have $\zeta_1\beta/Nu = 0.1 \times 10^{-18}$, in agreement with the value obtained by direct comparison, 0.06×10^{-18} . If we assumed a value 2% higher for μ' , which seems a little outside the experimental errors, the agreement would be perfect.

We may therefore conclude that the theory of KH_2PO_4 is applicable to the 'upper phase of $\text{NH}_4\text{H}_2\text{PO}_4$ without any alteration in its formalism. An essential difference in these two substances is that the 'energy parameter ε_0 is negative for the former while it is positive for the latter. This means that the dipolar groups H_2PO_4 in $\text{NH}_4\text{H}_2\text{PO}_4$ have a lower energy when it is perpendicular to the c -axis than it is parallel, contrary to the case of KH_2PO_4 .

Another interesting difference in these two substances is that β/u of $\text{NH}_4\text{H}_2\text{PO}_4$ is 3.7 times as large as that of KH_2PO_4 . This quantity is the angle of spontaneous shearing strain x_y which the crystal would undergo if it were polarized spontaneously to the direction of c . It was calculated to be $26'$ for KH_2PO_4 (in agreement with Ubbelohde's observation), and is thus $1^\circ 36'$ for the crystal of $\text{NH}_4\text{H}_2\text{PO}_4$. β/u is also the ratio between the shearing strain x_y and the degree of polarization for a free crystal at any temperature and at any field strength.

In our previous paper, it was pointed out that a 'phase transition occurs even if ε_0 is negative, so long as $|\beta^2/u| > 2|\varepsilon_0|$. In the present case, $|\beta^2/uk| = 41^\circ\text{K}$ and $2|\varepsilon_0|/k = 67^\circ\text{K}$ (if the dependency of $T\mu'^2$ upon temperature is neglected), so that the relation does not hold. In any event, it seems scarcely possible to account for the transition in $\text{NH}_4\text{H}_2\text{PO}_4$ by the same mechanism as that in KH_2PO_4 .

§ 3. Discussion about the phase transition

Sugawara¹¹⁾ at Tohoku University observed no change in the X-ray patterns in passing through the transition point. The structural change would be therefore very minute, if it could exist. If the crystal were polarized along the c -axis, it would be strained through an angle of $1^\circ 36'$ according to the calculation given above, and this would not have escaped from Sugawara's observation. A possible atomic arrangement which does not give rise to spontaneous polarization (but which can be piezoelectric) is shown in Fig. 3. We saw in the foregoing paragraph that the energy parameter ε_0 is negative, that is, the H_2PO_4 groups are more stable when it is perpendicular to the c -axis than when it is parallel to it, and this is realized by this arrangement. The figure is the projection of successive layers of H_2PO_4 groups upon (001) plane. They are forming a net-work of hydrogen linkages connecting neighboring phosphate groups. Each pair of the nearest hydrogen bonds are

antiparallel, so that the electrostatic interactions among the hydrogen bonds take the lowest value. (The same situation is realized when the crystal is polarized along the c -axis.) Furthermore, the dipole moments of the PO_4 groups, induced by the two H atoms attached to them, form a lattice of antiparallel arrangement as shown in Fig. 4, which is also one of the most stable configurations with respect to its electrostatic energy.

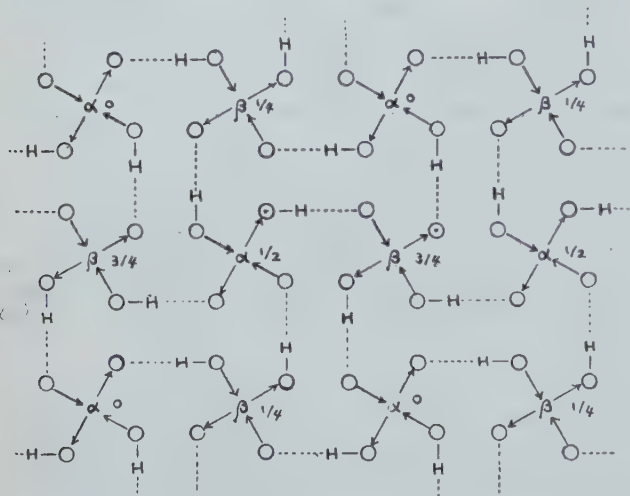


Fig. 3. The projection of a possible atomic arrangement in the lower modification of $\text{NH}_4\text{H}_2\text{PO}_4$ on (001) plane. α and β are P atoms, the attached figures are the heights of the corresponding P atoms, and arrows connecting P and O are inclined to the plane of the paper with their tops directed upwards.

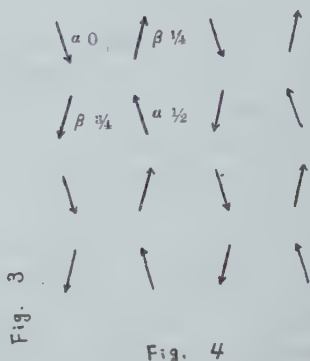


Fig. 4. The arrangement of the induced dipole moments of PO_4 groups corresponding to Fig. 3.

Other structures, such as in which the acid protons lie midway between each two oxygen atoms, can be conceivable. In any case there must be a regular arrangement of the protons in the low temperature modification of $\text{NH}_4\text{H}_2\text{PO}_4$, because this shows no anomalous dielectric behavior. The change in entropy due to the disordering of the acid protons in passing through the transition temperature from below is given by the formula

$$S = -R[(1-\nu_0)\ln 2(1-\nu_0) + \nu_0\ln \nu_0],$$

where

$$\nu_0 = 2[\exp(\epsilon_0/kT) + 2]^{-1}$$

(see Eqs. (14) and (15) of the reference 10). Using the data obtained in the preceding paragraph, we have $\nu_0 = 0.715$ and $S = 0.80$ cal/deg. mole, while the observed value of the entropy change is 1.05 cal/deg. mole.¹⁾ The difference must be ascribed to a certain configurational change in the system of ammonium radicals.

The small value of this difference excludes the possibility of a perfect disordering of the ammonium radicals in the upper modification. The free rotation of these radicals is also in disaccordance with the nuclear magnetic resonance experiment of Newman.¹²⁾ But there must be a partial disordering of these radicals in the upper modification which

increases with increasing temperature, as one sees from the results of the specific heat measurement.

In this connection, it is very interesting to observe a strong dependency of the transition temperature upon the concentration of thallium phosphate in Merz's experiment with the $\text{NH}_4\text{H}_2\text{PO}_4\text{-TlH}_2\text{PO}_4$ system.⁸⁾ The transition temperature decreases almost linearly with increasing Tl concentration up to 20 atomic %, shows a minimum at 25% Tl, and then increases with increasing concentration. The writer would like to point out that the linearly decreasing portion can almost perfectly be represented by the equation

$$T_c = T_{c0}(1-x)^2,$$

where x is the concentration of Tl. This can be interpreted as indicating the fact that the change in energy at the transition arises from the interaction among ammonium radicals; while the change in entropy at the transition arises from the order-disorder change of the acid proton arrangement. However, we have at present insufficient experimental data to infer to a concrete structural model.

§ 4. Generalized formulation of the electro-chemical properties of ferroelectric substances

Let the free energy of a ferroelectric crystal be of the form

$$F = F_0(\mu E + \beta x, T) + \frac{1}{2} N u x^2 - \frac{1}{2} \chi_1 E^2 - \zeta_1 x E, \quad (1)$$

where E is the strength of the electric field applied along a certain direction, x the corresponding piezoelectric strain, and the greek letters are constants. Then the polarization along the direction of the electric field, P , and the stress component corresponding with x , X , are given respectively by

$$P = -\frac{\partial F}{\partial E} = -\frac{\partial F_0}{\partial E} + \chi_1 E + \zeta_1 x, \quad (2)$$

$$X = -\frac{\partial F}{\partial x} = -\frac{\partial F_0}{\partial x} - N u x + \zeta_1 E_0. \quad (3)$$

Now

$$\frac{\partial F_0}{\partial x} = \frac{\beta}{\mu} \frac{\partial F_0}{\partial E}, \quad (4)$$

so that

$$X = \frac{\beta}{\mu} (P - \chi_1 E - \zeta_1 x) - N u x + \zeta_1 E. \quad (5)$$

For a free crystal ($X=0$), we have from (5)

$$P = \left(\frac{N u \mu}{\beta} + \zeta_1 \right) x + \left(\chi_1 - \frac{\zeta_1 \mu}{\beta} \right) E. \quad (6)$$

This is a generalization of the well-known Mueller's equation for Rochelle salt:¹³⁾

$$P_x = \text{const.} y_z \quad (7)$$

which can be obtained by neglecting χ_1 and ζ_1 , the constants corresponding to the normal electric susceptibility and the normal piezoelectric coupling constant.

In the temperature range, where there is no spontaneous polarization, we can expand $-\partial F_0/\partial E$ in powers of E and x . Retaining the linear terms, we have

$$P = \chi_0(T) \cdot \left(E + \frac{\beta}{\mu} x \right) + \chi_1 E + \zeta_1 x, \quad (8)$$

$$X = \frac{\beta}{\mu} \chi_0(T) \cdot \left(E + \frac{\beta}{\mu} x \right) - Nux + \zeta_1 E, \quad (9)$$

where χ_0 is a certain function of T . From these we have

$$-x = s \cdot X - d \cdot E, \quad P = -d \cdot X + \chi \cdot E, \quad (10)$$

where

$$s = \left[Na - \left(\frac{\beta}{\mu} \right)^2 \chi_0 \right]^{-1} = \frac{1}{Na} + \frac{\beta^2}{Na^2} \left(-\frac{N\mu^2}{\chi_0} - \frac{\beta^2}{a} \right)^{-1}, \quad (11)$$

$$d = \left(\frac{\beta}{\mu} \chi_0 + \zeta_1 \right) s = \frac{\beta}{a} \left(\mu + \frac{\zeta_1 \beta}{Na} \right) \left(\frac{N\mu^2}{\chi_0} - \frac{\beta^2}{a} \right)^{-1} + \frac{\zeta_1}{Na}, \quad (12)$$

$$\begin{aligned} \chi &= \chi_0 + \chi_1 + \left(\frac{\beta}{\mu} \chi_0 + \zeta_1 \right)^2 s \\ &= N \left(\mu + \frac{\zeta_1 \beta}{Na} \right)^2 \left(\frac{N\mu^2}{\chi_0} - \frac{\beta^2}{a} \right)^{-1} + \left(\chi_1 + \frac{\zeta_1^2}{Na} \right). \end{aligned} \quad (13)$$

(11), (12), (13) are the equations given in § 2 if we write s_{00} , d_{00} , χ_0^F for s , d , χ , and put $D = N\mu^2/\chi_0$. The most essential assumption used there is the equation

$$\chi_0 = \frac{N\mu^2}{kT + \text{const}} \quad (14)$$

which follows from Slater's model of KH_2PO_4 , but does not follow from Weiss' molecular field model for which a factor 1/3 appears on the right hand side.

We can apply our formulas for the case of Rochelle salt. Neglecting the terms containing χ_1 and ζ_1 , as mentioned above, and writing y_z , Y_z and P_z in place of x , X and P , we have from (8) and (9)

$$\begin{aligned} P_z &= \chi_0 \cdot \left(E + \frac{\beta}{\mu} y_z \right), \\ Y_z &= \frac{\beta}{\mu} \chi_0 \cdot \left(E + \frac{\beta}{\mu} y_z \right) - Nuy_z. \end{aligned}$$

Solving these equations with respect to Y_z and E , one has

$$\begin{aligned} -Y_z &= N\alpha y_z - (\beta/\mu)P_x & \Big| &= c_{44}y_z - f_{14}P_x, \\ E &= -(\beta/\mu)y_z + (1/\chi_0)P_z & \Big| &= -f_{14}y_z + \chi_1 P_x. \end{aligned}$$

The expression on the right of the vertical bar is that due to Mueller. The essential points of his "interaction theory" is the recognition of the constancy of the quantities c_{44} and f_{14} as functions of temperature, while all the 'anomalous' properties of Rochelle salt have their origin in the anomalous behavior of χ_1 (our $1/\chi_0$). As one sees, this is rather a trivial consequence from our free energy expression. In seeking for the molecular theory of Rochelle salt, one can thus confine himself to a clamped crystal for which the free energy is $F_0(\mu E, T)$.

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On the Structure of Heavy Nuclei*

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We make the following assumptions on the structure of heavy nuclei whose atomic weights A are greater than 110. The nuclear matter is nearly incompressible and their constituent neutrons have a tendency to cluster into closed shells, which are of a spherical shape and arranged concentrically in the spherical nucleus. Inner shells are bound more tightly than the outer ones. The proton density is determined only by the proper nuclear force and the Coulomb repulsion, and is given by the formula $\rho(r) = \text{const.} \{1 + 1/6 \cdot (r/7.3r_0)^2\}$. A shell mixed with the protons is called a core. A core is destroyed if at least one of its constituent neutrons or protons is knocked out. The part thus destroyed is assumed to make a liquid-like behavior in the fissioning of heaviest nuclei, and the thermodynamical treatment is applied to this part only. Under these assumptions, (i) we obtain $r_0 = 1.32 \cdot 10^{-13}$ cm when $A \gtrsim 200$, (ii) the shape of the fission yield curves of heaviest nuclei is explained, (iii) the released energy in the fission of U^{235} is about 195 Mev, (iv) the energy level spacing of heavy nuclei is greater than that obtained from the uniform liquid-drop model, and (v) the cross section of the neutron inelastic scattering by heavy nuclei is smaller, and the velocity of the scattered neutron is larger than those so far obtained. The introduction of the core explains the irregularity of the intensity of the fluorescent gamma-ray emitted from nuclei, and makes the specific heat of heavy nuclei smaller than that so far theoretically obtained.

Introduction

As the explanation of various properties of nuclei, by using the nuclear force acting between the constituent particles, is impossible, we explain them semi-classically, by using several models of nuclei. The liquid-drop model and the gas model are most useful. The former is a model devised on the assumption that the interaction between nucleons is large, and that of the latter, on the contrary, is small. In deducing various properties of heavy nuclei the liquid-drop model is more useful, in many respects, than the gas model. But it cannot be expected that all nuclear properties are deduced from a sole model, and some properties are, even in the region of heavy nuclei, explained by using the gas model. Especially, the idea of the magic number has thrown much light on the problems concerning the stability of nuclei and quadrupole moment etc.

In the present paper, we propose a new nuclear model which is obtained by bringing the idea of the closed shell structure into the liquid-drop model so far used, and explain, by using this model, the experimental results which have not been made clear so far. The experimental facts to be examined in this paper follow.

(1). *The shape of the fission yield curves of heaviest nuclei.* The fission yield curves produced by the neutron bombardment have two remarkable characteristics. One is the mass ratio of the fission products, i.e., the most probable mode of fission is when the

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mass ratio of the two fragments is 3:2. Thus the yield curve has two predominant peaks. The other is the breadth of each peak, i.e., each peak has the breadth almost constant from top to bottom whose mass range is about 20. The range of the charge number in each peak is only 4 to 8.

(2). *The value of r_0 .* Let the radius and the atomic weight of a nucleus be R and A , respectively. Then R and A are in the relation $R = A^{1/3} r_0$. The value of r_0 depends upon the way it is determined. By using our new model, we conclude that $1.32 \cdot 10^{-13}$ cm is the most probable value of r_0 when $A \gtrsim 200$.

(3). *The cross section of the inelastic neutron scattering.* The experiment¹⁾ in the measurement of the cross section of the inelastic neutron scattering, shows that the cross section increases with A linearly till A rises over about 100, after which it takes a rather downward trend with the increasing of A .

(4). *The intensity of the fluorescent gamma-ray emitted from a nucleus after the neutron bombardment.* When a nucleus is bombarded by a neutron, the fluorescent gamma-rays are emitted. Lea²⁾, Aoki³⁾, and Wakatsuki & Sugimoto⁴⁾ measured the intensity of these gamma-rays from nuclei ranging over almost all elements, excluding the rare earths. The energies of bombarding neutrons were 3 to 14 Mev. According to them, the intensity of the gamma-rays increases with the atomic weight A when $A \lesssim 100$, not only increasing with A but somewhat decreasing when $A \gtrsim 110$. What have been obtained from the experiments are far from showing a definite regular curve.

(5). *The photo-fission threshold energies of heavy nuclei.* Precise calculation of the fission threshold energies of heavy nuclei has been carried out by Frankel and Metropolis⁵⁾ by using the uniform liquid-drop model. On the other hand, Koch et al.⁶⁾ have made the measurement of the photo-fission threshold energies of U^{235} , U^{238} , Pu^{239} , U^{233} , and Th^{232} by using the betatron. Considerable discrepancies were found between these results. The threshold energies obtained by the experiment were all about 5 Mev, while the theoretical values of them are 7.0, 6.1, 4.9, 5.7, and 8.3 in Mev.

(6). *The specific heats of heavy nuclei* are not so large that the theory predicts^{7), 8)}, and almost constant for varying A ($\gtrsim 110$).

Among these six questions, (3), (4), and (6) suggest the necessity of giving a certain restriction to the freedom of the internal motion of heavy nuclei. And they are easily explained by using our new model. The calculation of the fission threshold energies is a very difficult mathematical task, so that we have not been able to estimate them by using the new model. But the possibility of obtaining the coincidence between the experimental and theoretical values has been semi-quantitatively proved in a separate paper⁹⁾ by the present writer and T. Yasaki, by using the new model.

In the present paper, we first put forward the new model in the subsequent paragraph, and then treat the questions stated above.

§ 1. Modified liquid-drop model

We assume that the nuclear matter is nearly incompressible and, when $A \gtrsim 110$, the

constituent neutrons have a tendency to cluster into closed shells, the number of whose constituent neutrons are 20, 50, 82, and 126. And these closed shells are assumed to be of a spherical shape and arranged concentrically in the spherical nucleus.

Let the closed shell consisting of 20 neutrons be called 20-shell. The 50-shell embraces the 20-shell within it and the 82-shell, the 50-shell, etc.. The reason why we consider the shell structure in heavy nuclei whose atomic weights are greater than 110, is: Firstly, the theoretical lower limit of atomic weights of nuclei which are alpha-radioactive, is about 120. Secondly, the anomaly of the cross sections of the inelastic neutron scattering and of the intensities of the fluorescent gamma-rays begin to appear when the atomic weight rises over 110.

The proton density is assumed to be determined only by the proper nuclear force and the Coulomb repulsion, and is given by the formula¹⁰⁾

$$\rho(r) = \text{const.} \{1 + 1/6 \cdot (r/7.4r_0)^2\}.$$

A shell mixed with the protons is called a core. For the sake of the simplicity, we assume that the binding energy of a proton belonging to a core is equal to that of the neutron belonging to the same core. The nuclei in the region $A \gtrsim 200$ contain 20-, 50-, 82-, and 126-core, the nuclei in the region $150 \lesssim A \lesssim 200$ contain 20-, 50-, and 82-core, and the nuclei in the region $110 \lesssim A \lesssim 150$ contain 20-, and 50-core. Table 1 gives the proton numbers and the magnitudes of the cores.

Atomic weight	20 core	50-core	82-core	126-core
$110 \lesssim A \lesssim 150$	14 (0.66)	34 (0.89)		
$150 \lesssim A \lesssim 200$	12 (0.59)	32 (0.80)	54 (0.95)	
$200 \lesssim A$	12 (0.51)	30 (0.70)	50 (0.82)	80 (0.95)

Table 1. Proton number in each of the cores. The number in the parentheses=radius of the core/nuclear radius.

It can be understood from the theories^{11),12)} and experiment¹³⁾ that the outermost core contained in any heavy nucleus can be destroyed when one or two of its members are knocked out, and the energy necessary for this ejection is 2 to 4 Mev. The inner core just beneath it contained in the outermost one is destroyed when one or two of its members are knocked out, the energy necessary for this ejection is 2 to 4 Mev. But in the case when the second core is destroyed by this amount of energy, the outermost and the other smaller ones remain unaffected. If one or two members of the second core are knocked outside the outmost core, by getting 4 to 8 Mev, both the outermost and the second are destroyed. In the same manner, when the energy is 2 to 4 Mev, the third core only can be destroyed. When 4 to 8 Mev, the second may be destroyed together with the third. When 6 to 12 Mev, the third including the second and the outermost cores may be destroyed; and so on.

Accordingly, a neutron may heat only the part outside of the outermost core by impinging into a nucleus. In some cases, it destroys the outermost core and heat the whole part outside of the second core. In another case, it destroys both the outermost and the second cores and heat the whole part outside of the third core. But it generally cannot destroy three cores simultaneously when the velocity of the neutron is small.

We regard the part thus degenerated as liquid, and apply the idea of the fissioning of heaviest nuclei and the thermodynamical treatment to this liquid part only.

In the region $110 \lesssim A \lesssim 150$, all the cores are destroyed simultaneously by a particle or a photon having the energy of about 8 Mev, including the binding energy. In the region $150 \lesssim A \lesssim 200$, all the cores can be destroyed simultaneously by a particle or a photon having the energy of more than about 12 Mev. In the region $200 \lesssim A$, an energy of more than about 16 Mev is necessary to destroy all of the cores simultaneously.

These properties of heavy nuclei clearly explain the experimental facts that when the bombarding neutron is not fast, the heaviest nuclei split themselves in asymmetric manner; when the incident energy amounts to more than ten Mev, the symmetric fission begins to occur; and when the incident energy reaches to several tens of Mev, the symmetric fission probability becomes predominant compared with the asymmetric one. Taking it the other way round, a precise measurement of the amount of energy at which the symmetric fission probability increases suddenly, will determine the rigidity of the cores.

§ 2. Alpha-disintegration; value of r_0

The nuclear radius R can be put as $R = A^{\frac{1}{3}} r_0$, where A is the atomic weight of the nucleus. T. Yasaki and the present writer¹¹⁾ determined the value of r_0 to be equal to $1.32 \cdot 10^{-13}$ cm, in the course of the calculation determining the fission threshold energy of uranium, by using our model. In this paragraph, the value of r_0 is determined, by applying the α -disintegration phenomena to the same model. We adopt the one-body model, i.e., assume that an α -particle already exists before it is emitted. The difference between our treatment and Gamow's is: In our model, the α -particle moves only in the part outside of the 50-core, which is considered to be loosely bound. Thus the motion of the particle has a narrower range than the whole nucleus. Accordingly, by the uncertainty principle, the kinetic energy of the particle will become larger than that in the case of Gamow. On the other hand, if we take the nuclear radius R to be equal to that of Gamow, the Coulomb barrier is unaltered, so that the penetrability of the particle becomes larger than that of Gamow, and the half-life of the nucleus will become shorter than that given by the experiment.

Thus, in order to make the half-life fit into the experimental data, the Coulomb barrier must be higher than that of Gamow. Thus the nuclear radius R , consequently also r_0 , will have to be made smaller than that of Gamow.

The quantitative result is obtained by using the method given in Bethe's paper¹⁵⁾ pp.161-166. Instead of the rectangular potential well, we use a potential, such as

$$V = \begin{cases} \infty & 0 \leq r < \beta R, \\ V_0 & \beta R \leq r < R, \\ 2Ze^2/r & R \leq r < \infty. \end{cases}$$

Here $V = \infty$ means that the α -particle does not penetrate into the 50-core, βR is the radius of the 50-core and $\beta = 0.7$ when $A \gtrsim 200$, and Z is the atomic number of the residual nucleus.

Let the wave function of the α -particle be ψ , and $\psi = u/r$, then the Schrödinger equation becomes as

$$u'' + 2M/\hbar^2 (E - V)u = 0,$$

where M and E are the mass and the kinetic energy at infinity of the α -particle, respectively. In solving this equation, the boundary condition

$$u = 0 \quad \text{for} \quad r = \beta R$$

is used. The number of the α -particles emitted from the nucleus in a unit time is $\lambda = 4\pi r^2 v |\psi|^2$, in which v is the velocity of the α -particle. The half-life of the disintegration is $\tau = (\log 2)/\lambda$. The calculation comes to

$$\tau = \tau_0 e^{2C} (1 - \beta)^3$$

with

$$\tau_0 = 3.3 \cdot 10^{-21} \text{ sec}, \quad C = \frac{4Ze^2}{\hbar v} \left[\arccos x^{\frac{1}{2}} - x^{\frac{1}{2}} (1 - x)^{\frac{1}{2}} \right],$$

$$x = ER/2Ze^2.$$

Let the quantities R , x , C , and τ in the case of Gamow be denoted as R' , x' , C' , and τ' , respectively. τ must be adjusted to fit into the experimental data, hence $\tau = \tau'$, which in turn is

$$e^{2C'} = e^{2C} (1 - \beta)^3.$$

Putting the value $\beta = 0.7$ in this relation, we obtain the relation between x and x' . We put as $x' - x = \delta$, expand the above relation into a power series of δ and adopt the term of the first degree of δ , then we obtain

$$\delta = 1.806 \frac{\hbar v}{2Ze^2} \{ 1/\sqrt{x'(1-x')} + \sqrt{(1-x')/x'} - \sqrt{x/(1-x)} \}^{-1}.$$

We use in this formula, the relation $E = Mv^2/2$, and adopt the experimental values of E and Gamow's values of λ' (Bethe, p. 166). Then the values of δ are determined for every element, so that the values of $R' - R$, consequently also the values of $r_0' - r_0$, are obtained. Here, r_0' given by Gamow is $1.46 \cdot 10^{-13}$ cm.

The values of $r_0' - r_0$ are approximately constant for all heavy nuclei and equal to $0.14 \cdot 10^{-13}$ cm on the average. Thus we obtain $r_0 = 1.32 \cdot 10^{-13}$ cm.

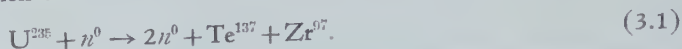
If the wave function of the α -particle is allowed to penetrate into the 50-core, the value of r_0 will become larger than the above one.

§ 3. Shape of the fission yield curve

The fission yield curve of U^{235} , which has been obtained by using the uranium pile, has two remarkable peaks. The height of the valley lying between these two peaks is only one-several hundredth of that of one of the peaks. This valley corresponds to the symmetric fission and will perhaps be brought about by the fast neutrons in the pile. So that the asymmetric fission only is considered to occur when a heaviest nucleus is bombarded by a neutron of $1 \sim 2$ Mev.

Each peak has a breadth almost constant from top to bottom, which is about 20 in nucleon number. This fluctuation of the nucleon number in each fragment is due mostly to that of the neutrons, while the fluctuation of the proton number is about 4 to

8. The mass ratio of the fission products lying on each top of two peaks is 3 : 2. The most probable mode of fission is



Here $2n^0$ on the right hand side are the spontaneous neutrons. Te^{137} and Zr^{97} emit β -particles three or five times and are settled to the stable isotopes Ba^{137} and Mo^{97} , respectively.

In order to explain the above experimental facts after our new nuclear model, we take U^{238} as an example.

When a U^{238} -nucleus is bombarded by a neutron of about 2 Mev, a compound nucleus U^{239} is formed and generally 126- and 82-core are destroyed. The part thus destroyed does the fissioning like liquid and the nucleus splits into two fragments, the larger containing the old 50-shell.

The compound nucleus contains 147 neutrons in all, two of which are emitted spontaneously, 82 of them remaining in the larger fragment, and 50 of them being transferred into the smaller one, thus $147 - (2 + 82 + 50) = 13$ neutrons remaining unsettled. These neutrons are suitably divided into two parts, the one remains in the larger fragment, and the other is transferred into the smaller one. Accordingly, each fragment has a fluctuation of 13 in neutron number. On the other hand, the proton density in the exterior of the 50-core is about $2/5$, so that the fluctuation of neutron number causes that of proton number 8 or 9. Thus the total fluctuation of nucleon number in each fragment is about 21 or 22. This number corresponds just to the breadth of each peak of the yield curve.

However, the 82-shell in the larger fragment and the 50-shell in the smaller one are not necessarily formed completely, i.e., if we do not take into consideration the binding energies, about 2 to 4 Mev, of the last one or two neutrons of the 82-shell, the larger fragment may contain 81 or 80 neutrons. By the same reason, the smaller fragment may contain 48 or 49 neutrons. In these cases, the fluctuation of the nucleon number becomes larger than that obtained above, 21 or 22. These cases occur when the incident energy is larger than the fission threshold energy, but the probabilities of their occurrence are relatively small. The fact that the bottom of the fission yield curve widen slightly, is thus explained.

We should not, however, determine the shape of the yield curve only by using the tendency of neutrons to cluster into the closed shells. The released energy in fission must be taken into account and it must be ascertained whether one of the partitions considered above actually releases the maximum energy.

The compound nucleus excited by a neutron splits into several fragments making maximum the energy difference $\Delta E = (M - \sum M_i)c^2$. Here M and M_i 's are the rest masses of the compound nucleus and the fission products, respectively. In this Einstein relation the energies released by the β -disintegrations which occur after the fission, are not included. As is known, the number of the fission products is two when the incident energy is not high.

Let the atomic weight of a nucleus whose neutron number, proton number and atomic weight are respectively N , Z , and $A (= N + Z)$, be $M(Z, A)$. Then the binding energy of this nucleus is given by

$$E = NM_n + ZM_p - M(Z, A). \quad (3.2)$$

Here M_n and M_p are the masses of the neutron and the proton, respectively.

Simplifying the fission mechanism of U^{238} , we dismiss the incident and the spontaneous neutrons as

$$\text{U}^{238} \rightarrow (Z)A + (92-Z)^{238-A}. \quad (3.3)$$

and estimate the released energy. Here $(Z)A$ and $(92-Z)^{238-A}$ are the larger and the smaller fragment, respectively. The atomic weight of the larger fragment falls within the region $110 \lesssim A \lesssim 150$, and that of the smaller fragment falls within the region $A \lesssim 100$. For the simplicity of the calculation, we make the following assumptions about the proton distribution in nuclei. In the region $A \lesssim 110$, the proton density is uniform. In the region $110 \lesssim A \lesssim 150$, 14 protons are distributed uniformly in the 20 core, and the remaining ones in the outer part of the 20-core. In the region $200 \lesssim A$, 30 protons are distributed uniformly in the 50 core, the remaining ones in the outer part of the 50-core.

We shall give here the empirical mass formula of nuclei having a core whose proton density differs from that of its exterior. Let the radii of the nucleus and the core be R and S , respectively. The core contains N_1 neutrons and Z_1 protons, and its exterior contains N_2 neutrons and Z_2 protons. And let $A_1 = N_1 + Z_1$, $A_2 = N_2 + Z_2$, $A = A_1 + A_2$, and $S/R = \beta$. Then the binding energy of the nucleus is given by the following formula:

$$E = aA - \left\{ \beta_1 \frac{(N_1 - Z_1)^2}{A_1} + \beta_2 \frac{(N_2 - Z_2)^2}{A_2} \right\} - 4\pi r_0^2 O A^{\frac{2}{3}} - \frac{1}{R} \left\{ \frac{3}{5} \frac{(Z_1^e)^2}{\beta} + \frac{3}{2} \frac{Z_2^2}{1 - \beta^3} \left(\frac{Z_1^e}{\beta^3} - \frac{Z_2^e}{1 - \beta^3} \right) \beta^3 (1 - \beta^2) + \frac{3}{5} \frac{(Z_2^e)^2}{(1 - \beta^3)^2} (1 - \beta^6) \right\}. \quad (3.4)$$

Here O is the surface tension, a , β_1 and β_2 are undetermined constants which are to be determined so that the formula (3.4) fits into the packing fraction curve as well as possible. And the last term on the right hand side is the Coulomb energy.

For a given value of A , there would be a value Z_A corresponding to the most stable isobar. Z_A is not necessarily an integer, but it would be near a stable isobar. Consider the odd nuclei, and assume that the formula (3.4) holds good in the region considered, then Z_A can be found by setting the derivative of equation (3.4) equal to zero:

$$\frac{d}{dZ} M(Z, A) = 0. \quad (3.5)$$

$M(Z, A)$ will vary parabolically with Z in the neighbourhood of Z_A :

$$M(Z, A) - M(Z_A, A) = \frac{1}{2} B_A (Z - Z_A)^2 \quad (3.6)$$

with

$$B_A = \left\{ \frac{d^2}{dZ^2} M(Z, A) \right\}_{Z_A}. \quad (3.7)$$

Eliminating β_2 from Eqs. (3.5), (3.7), we obtain the explicit form of B_A as:

$$B_A = \frac{2}{A - 2Z_A - (N_1 - Z_1)} \left\{ - (M_n - M_p) - \frac{3}{2} \frac{N_1 e^2 A^{\frac{2}{3}} - A_1^{\frac{2}{3}}}{r_0 A_2} + \frac{3}{5} \frac{e^2 A^{\frac{5}{3}} - A_1^{\frac{5}{3}}}{r_0 A_2} \right\}. \quad (3.8)$$

When $A \lesssim 110$, as there is no core,

$$B_A = \frac{2}{A - 2Z_A} \left\{ - (M_n - M_p) + \frac{3}{5} \frac{e^2 A^{\frac{5}{3}}}{r_0} \right\}. \quad (3.9)$$

Now Z_A lies between two values, Z and $Z+1$, one or both of which is the stable nucleus of number A , so that Z_A is at most $1/2$ unit from a stable isotope. We use this property of Z_A to estimate it directly. We use a smooth line through the table of stable, odd isotopes and use this line to define Z_A . Similarly, the most stable mass, $M(A, A)$ may be obtained directly from the average value, f_A , of the packing fraction over a small region at atomic weight A .

$$M(Z_A, A) = A(1 + f_A). \quad (3.10)$$

In averaging f_A , both even and odd isotopes are included, because then the positive term $1/2 \cdot B_A (Z - Z_A)^2$ is

largely cancelled by the negative $-\delta_A$ of the even isotopes, where δ_A is the odd-even term and equal to 0.5 Mev. Then we have

$$M(Z, A) = A(1 + f_A) + \frac{1}{2} B_A (Z - Z_A)^2 + 0 \begin{cases} A \text{ odd} \\ -\delta_A \begin{cases} A \text{ even } Z \text{ even} \\ +\delta_A \begin{cases} A \text{ even } Z \text{ odd} \end{cases} \end{cases} \end{cases} \quad (3.11)$$

By using these formulas, we estimate the value of the mass difference

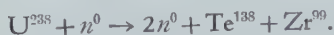
$$M(92, 238) - (M(Z, A) + M(92 - Z, 238 - A)) \quad (3.12)$$

for every partition of U^{238} . Here we neglect the rigidity of the core and use only the assumptions on the proton density stated above.

The maximum released energy is then obtained when the larger fragment is Te^{138} and the smaller is Zr^{100} . The energy released in this fission is equal to 200 Mev when $r_0 = 1.48 \cdot 10^{-13}$ cm. If $r_0 = 1.32 \cdot 10^{-13}$ cm, the maximum released energy is also obtained at the same partition as above and is equal to 195 Mev.

Generally, the smaller the value of r_0 is, the smaller is the released energy. We shall now compare the two values of the released energy obtained above with the experimental result. Measurement⁽⁶⁾ of the ionization produced by the fission products gives about 163 Mev. This is just the translational kinetic energy of the fragments.

We now consider, for example, the mode of fission:



And let the incident neutron energy and the kinetic energy of each of the spontaneous neutrons be 2 and 3 Mev respectively, and the binding and dissociation energy of a neutron be 6 Mev. Then the total kinetic energy of the two fragments is 185 or 190 Mev, according as $1.32 \cdot 10^{-13}$ cm or $1.48 \cdot 10^{-13}$ cm is adopted as the value of r_0 . The distortion of the nucleus immediately before splitting is so large that the product nuclei themselves have a large excitation energy aside from that due to their high neutron-proton ratios. This is probably of the order of 10 or 15 Mev and it will not be detected in the ionization chamber. Subtracting this energy from those obtained above, the total translational kinetic energy is 170 to 175 Mev or 175 to 180 Mev.

The mean released energy is lower than the maximum one by about 8 Mev. So that the mean total translational kinetic energy is 162 to 167 Mev or 167 to 172 Mev. These values are to be compared with the experimental value 163 Mev. Hence $r_0 = 1.32 \cdot 10^{-13}$ cm is more suitable than $r_0 = 1.48 \cdot 10^{-13}$ cm.

§ 4. Energy level spacing

In this paragraph, we adopt the thermodynamical method of Bethe (Bethe⁽⁷⁾, pp. 79-90) to our model, and deduce the energy level spacing formula of heavy nuclei. The comparison of the result with experiments will be made in the next paragraph.

In the first place, we introduce some notations, and relations between them necessary to the calculations. As to the details of the formulas (4.1) to (4.5) given below, refer to Bethe's paper.

The average spacing of the nuclear levels $D(U) = 1/\rho(U)$, where $\rho(U)dU$ is the number of levels with energy between U and $U+dU$. Let τ be an arbitrary parameter, and $F(\tau)$ a certain function of τ , satisfying the expression

$$\int \rho(E) e^{-E/\tau} dE = e^{-F/\tau}, \quad (4.1)$$

provided that the levels are very dense. When the system contains sufficiently many particles, the main contribution to the integral (4.1) will come from the energy levels E near U . Thus we may write instead of (4.1)

$$e^{-F/\tau} = \rho(U) e^{-U/\tau} / \lambda(U), \quad (4.2)$$

where $\lambda(U)$ is a quantity of the dimensions of an energy which is to be calculated by evaluating the integral in (4.1). It is a slowly varying function of the excitation energy U as compared with the rapidly varying functions $e^{-F/\tau}$, $e^{-U/\tau}$, and $\rho(U)$. We introduce the entropy $(U-F)/\tau = S$.

Then we obtain the following relations between the quantities defined above:

$$S = \int \frac{d\tau}{\tau} \frac{dU}{d\tau}, \quad (4.3)$$

$$\lambda(U) = (2\pi)^{1/2} \tau \left(\frac{dU}{d\tau} \right)^{1/2}, \quad (4.4)$$

$$D(U) = \lambda(U) e^{-S}. \quad (4.5)$$

In applying these formulas to our nuclear model, we use its following properties.

A particle impinging into a nucleus may be deflected elastically or amalgamized with the nucleus. The life time of the compound state is long enough to forget how it was formed.

When the energy is rapidly communicated to all particles in the part outside of the outermost core, before one of its member is knocked out, the thermal agitation occurs in that place only. In some cases, the outermost core is destroyed, and the thermal agitation occurs in the exterior part of the second core. In general, a particle or a photon having an energy of about the binding energy of neutron, 6 to 8 Mev, can destroy the two outermost cores, but not the third. High energy particles or photons can, however, destroy all cores, and, in this case, the nucleus behaves like a uniform liquid-droplet.

Under certain circumstances, even a high energy particle or photon can not destroy these cores. In some cases, an inner core is destroyed but the outer core might remain undestroyed. But the probability of such occurrence is very small. Thus, generally, the incident particles or photons having 6 to 8 Mev of energy, including the binding energy, destroy all cores of nuclei lying in the region $110 \lesssim A \lesssim 150$; all but the innermost core when $150 \lesssim A \lesssim 200$; 126-, and 82-core but not 50-, and 20-core when $200 \lesssim A$.

We apply the thermodynamical calculations to such a degenerated part in a nucleus, regarding it as liquid.

Let the nuclear radius be R and the radius of the greatest core remaining undestroyed be βR . The surface and volume waves are considered to occur in the degenerated part of the nucleus.

1. *The surface waves.* Bethe has assumed that the depth of the liquid is infinite. But this assumption is not suitable to our treatment. So we assume that the depth of the liquid is finite and put it as h . The increase in the potential and kinetic energy due to the occurrence of the surface waves are respectively

$$V = \frac{1}{4} G a^2 \cos^2 \omega t \cdot k^3 (e^{hk} + e^{-hk})^2$$

and

$$T = \frac{1}{4} \rho S a^2 \omega^2 \sin^2 \omega t \cdot k (e^{2hk} - e^{-2hk}),$$

which correspond to the formulas (310) and (310a) in Bethe's paper, respectively. Here G is the total surface energy of the liquid in equilibrium, S the total surface area, ρ the density of the liquid, ω the frequency of the wave, k the wave number, and a an arbitrary constant. If the sum of potential and kinetic energy is to be constant, we must put as

$$k^3 \coth hk = \frac{\rho S}{G} \omega^2. \quad (4.6)$$

When $h=\infty$, $\coth hk=1$, and the above relation is reduced to Bethe's. When the undestroyed core is large, or the wave length is large, we cannot put $\coth hk$ as 1. If, however, we leave the factor $\coth hk$ as it is, it is difficult to represent k as an explicit function of ω . So that, in place of $\coth x$, we use an approximate function of x which is equal to $1/x$ when $0 \leq x \leq 1$ and to 1 when $1 \leq x < \infty$. Then the relation (4.6) becomes as

$$k^3 = \begin{cases} \frac{\rho S}{G} \omega^2 h k, & 0 \leq k \leq k_0, \\ \frac{\rho S}{G} \omega^2, & k_0 \leq k < \infty, \end{cases} \quad (4.7)$$

where $h_0=1$. On the one hand, the number of normal vibrations with wave numbers between k and $k+dk$ is

$$\hat{p}(k)dk = \left(\frac{S}{4\pi^2}\right) 2\pi k dk. \quad (4.8)$$

Using the relation (4.7), the formula (4.8) is written as

$$\hat{p}(\omega)d\omega = \begin{cases} \frac{S}{2\pi} \cdot \frac{\rho S}{G} h \omega d\omega, & 0 \leq \omega \leq \omega_0 \\ \frac{S}{2\pi} \cdot \frac{2}{3} \left(\frac{\rho S}{G}\right)^{\frac{2}{3}} \omega^{\frac{1}{3}} d\omega, & \omega_0 \leq \omega < \infty, \end{cases} \quad (4.9)$$

where ω_0 is a fixed constant satisfying the equation

$$k_0^3 = \frac{\rho S}{G} \omega_0^2 \tanh 1. \quad (4.10)$$

For a spherical nucleus, the surface area S is equal to $4\pi R^2$, and the density ρ is $3AM/4\pi R^3$. The surface energy G can be put as $G=\Gamma A^{\frac{2}{3}}$ and we adopt the value

$$\Gamma = 13.3 \text{ Mev}, \quad (4.11)$$

which has been given by Feenberg.

If we introduce, instead of ω , the quantum energy $\epsilon = \hbar\omega$, (4.9) may be written as

$$\hat{p}_S(\epsilon)d\epsilon = \begin{cases} 6(h/r_0)(\Gamma\Gamma')^{-1}A^{\frac{2}{3}}\epsilon d\epsilon, & 0 \leq \epsilon \leq \epsilon_0 \\ 4 \cdot 3^{-\frac{1}{3}}(\Gamma P)^{-\frac{2}{3}}A^{\frac{2}{3}}\epsilon^{\frac{1}{3}}d\epsilon, & \epsilon_0 \leq \epsilon < \infty, \end{cases} \quad (4.12)$$

here

$$P = \hbar^2/Mr_0^2, \quad \epsilon_0 = \hbar\omega_0, \quad (4.13)$$

and when $r_0 = 1.4 \cdot 10^{-13} \text{ cm}$

$$P = 21 \text{ Mev}, \quad \epsilon_0 = 11.14(1-\beta)^{-1}A^{-\frac{1}{3}} \text{ Mev}. \quad (4.14)$$

By using these numerical values, we obtain Table 2 of values of ϵ_0 .

Atomic weight	20-core	50-core	82-core	126-core
120	6.643	20.52		
160	5.004	10.26	41.11	
232	3.776	6.041	10.66	45.28

Table 2. Values of ϵ_0 in Mev. $r_0 = 1.4 \cdot 10^{-13} \text{ cm}$, $\Gamma = 13.3 \text{ Mev}$. The cores in the first row are those remained undestroyed.

From (4.12), it is seen that, for a heavy nucleus ($A=200$), $P_S(\epsilon)$ gives about 1.3 normal modes of surface vibration with quantum energies below 1 Mev.

2. *The volume waves.* An ideal liquid will admit only longitudinal volume waves. The number of normal modes is

$$\Omega(2\pi)^{-3} 4\pi k^2 dk = \frac{2}{3\pi} R^3 (1-\beta^3) \omega^2 d\omega / u_0^3, \quad (4.16)$$

where u_0 is the velocity of sound and is represented as

$$u_0^2 = \frac{\sigma^2}{MA_0} \frac{d^2 E_0}{d\sigma^2} \quad (4.17)$$

where A_0 , E_0 and σ are the nucleon number, the energy and the density of the exterior of the core, respectively. Then $\sigma = 3MA_0/4\pi R^3(1-\beta^3)$. Put $R - \beta R$ as R_0 , then, from the condition $dE_0/dR_0 = 0$ and (4.17), we obtain

$$u_0^2 = \frac{R^2}{9MA_0} \frac{d^2 E_0}{dR_0^2} \quad (4.18)$$

$d^2 E_0/dR_0^2$ may be estimated from the statistical formula for the energy as a function of the radius. Here we must use the formulas (158) and (159a) in Bethe's paper¹⁵⁾, where N and Z are interpreted as the neutron and proton number in the part outside of the core, respectively. Then

$$E_0 = T + V \quad (4.19)$$

where T and V are respectively the total sums of kinetic and potential energies of the total nucleons in the part outside of the core, and

$$T = \frac{3}{10} T_0 \left(\frac{9\pi}{4} \right)^{2/3} (N+Z)^{1/3} \frac{N^{5/3} + Z^{5/3}}{(N+Z)^{5/3}} \frac{1}{a^2}, \quad (4.20)$$

$$V = -B(N+Z)a^{-1/2},$$

with

$$Q = x^{-3} \left\{ (2 - q_+^2) \exp\left(-\frac{1}{4} x^2 q_-^2\right) - (2 - q_-^2) \exp\left(-\frac{1}{4} x^2 q_+^2\right) \right. \\ \left. + \pi^{1/2} x^3 \theta\left(\frac{1}{2} q_+ x\right) - \pi^{1/2} (n-1) x^3 \theta\left(\frac{1}{2} q_- x\right) \right\},$$

where the constants B , a and the potential $J(r)$ between two nucleons are in the relations

$$J(r) = -B \exp\left(-\frac{r^2}{a^2}\right), \quad a = \frac{r_0}{a}, \quad r_0 = RA^{-1/3},$$

and

$$q_+ = n^{2/3} + n^{1/3} (2-n)^{1/3} + (2-n)^{2/3}, \quad q_- = n^{1/3} + (2-n)^{1/3},$$

$$p_+ = n^{2/3} - n^{1/3} (2-n)^{1/3} + (2-n)^{2/3}, \quad q_- = n^{1/3} - (2-n)^{1/3},$$

$$n = \frac{2N}{N+Z}, \quad x = \frac{3}{2} \left(\frac{\pi}{3} \right)^{1/3} \frac{1}{a}, \quad T_0 = \hbar^2 / Ma^2, \quad \theta(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx.$$

Using the formula (4.20), (4.19) is written as

$$\frac{E_0}{AT_0} = \frac{3^{2/3}}{5} x^2 \left\{ 1 + \left(\frac{n}{2-n} \right)^{5/3} \right\} \left(\frac{2}{2-n} \right)^{-5/3} - \frac{B}{T_0} \pi^{-1/2} Q. \quad (4.21)$$

R_0 is approximately proportional to a , therefore to x . So that we obtain

$$R^2 \frac{d^2 E_0}{dR_0^2} = (1-\beta)^{-2} R_0^2 \frac{d^2 E_0}{dR_0^2} = (1-\beta)^{-2} x^2 \frac{d^2 E_0}{dx^2}. \quad (4.22)$$

The value of $x^2 \cdot d^2 E_0 / dx^2$ is obtained from (4.21).

We take U^{238} as an example, and adopt the values $a = 2.82 \cdot 10^{-13}$ cm, $r_0 = 1.36 \cdot 10^{-13}$ cm and $B = 50$ Mev. Then

$$R^2 \frac{d^2 E_0}{dR_0^2} \approx (1-\beta)^{-2} \cdot 35 \cdot A_0 \text{ Mev}, \quad (4.23)$$

irrespective of whether we adopt the 50-core or the 20-core as the undestroyed one. Therefore, according to (4.18),

$$K = Mv_0^2 = \frac{K^2}{9A_0} \frac{d^2 E_0}{dK_0^2} = \frac{35}{9} (1-\beta)^{-2} \approx 4(1-\beta)^{-2} \text{ Mev.} \quad (4.24)$$

Substituting (4.24) in (4.16), we obtain the number of longitudinal waves of frequency between ω and $\omega + d\omega$:

$$p_l(\omega) d\omega = \frac{2}{3\pi} (M/K)^{3/2} K^3 (1-\beta^3) \omega^2 d\omega.$$

Or, using the quantum energy $\epsilon = \hbar\omega$,

$$p_l(\epsilon) d\epsilon = \frac{2}{3\pi} (K^2 P)^{-3/2} A_0 \epsilon^2 d\epsilon.$$

The number of volume waves of energy between ϵ and $\epsilon + d\epsilon$ is then given by $3p_l(\epsilon) d\epsilon$, i.e.,

$$p_v(\epsilon) d\epsilon = \frac{2}{\pi} (K^2 P)^{-3/2} A_0 \epsilon^2 d\epsilon. \quad (4.25)$$

3. *Thermal properties of the nucleus.* The total number of normal modes of quantum energy between ϵ and $\epsilon + d\epsilon$ is $p(\epsilon) d\epsilon = (p_s(\epsilon) + p_v(\epsilon)) d\epsilon$. The energy at the temperature τ is then, according to Planck's formula,

$$U = \int_0^\infty \frac{\epsilon p(\epsilon) d\epsilon}{e^{\epsilon/\tau} - 1} = \gamma_1 \tau^3 A^{3/2} f(\epsilon_0, \tau) + \gamma_2 \tau^{7/2} A^{3/2} g(\epsilon_0, \tau) + \gamma_3 \tau^4 A_0, \quad (4.26)$$

where

$$\gamma_1 = 6(h/r_0)(\Gamma P)^{-1}, \quad \gamma_2 = 4 \cdot 3^{-1/2} (\Gamma P)^{-3/2}, \quad \gamma_3 = \frac{2}{\pi} (K^2 P)^{-3/2} C_3,$$

$$f(\epsilon_0, \tau) = \int_0^{\epsilon_0/\tau} \frac{x^2 dx}{e^x - 1}, \quad g(\epsilon_0, \tau) = \int_{\epsilon_0/\tau}^\infty \frac{x^{1/2} dx}{e^x - 1}, \quad c_3 = \int_0^\infty \frac{x^3 dx}{e^x - 1} = 6.50,$$

and when $r_0 = 1.4 \cdot 10^{-13}$ cm and $\Gamma = 13.3$ Mev

$$\gamma_1 = 0.02148 (1-\beta) A^{1/2},$$

$$\gamma_2 = 0.06497, \quad (4.27)$$

$$\gamma_3 = 0.005375 (1-\beta)^3.$$

From (4.26), (4.27), (4.3), (4.4), and (4.5), the values of U , $S(U)$ and $D(U)$ for any value of τ are calculated. The curve of $D(U)$ vs. U is given in Fig. 1.

From (4.27), it is seen that the contribution of the volume wave to U is very small compared with that of the surface wave. In fact, when $\beta = 0.7$ or 0.5 , the critical energy at which the volume wave equally contributes to U as the surface wave, is several tens of Mev. The contribution of surface waves is thus predominant. On the other hand, the existence of the core restricts the occurrence of the surface waves, especially the long waves. So that the level spacing $D(U)$ obtained above will be somewhat larger than $D(U)$ deduced from the uniform liquid-drop model, by taking into account the surface waves only. We shall examine this difference in the following.

According to Bethe, when the depth of the liquid is infinite, we obtain, by taking into account the surface waves only,

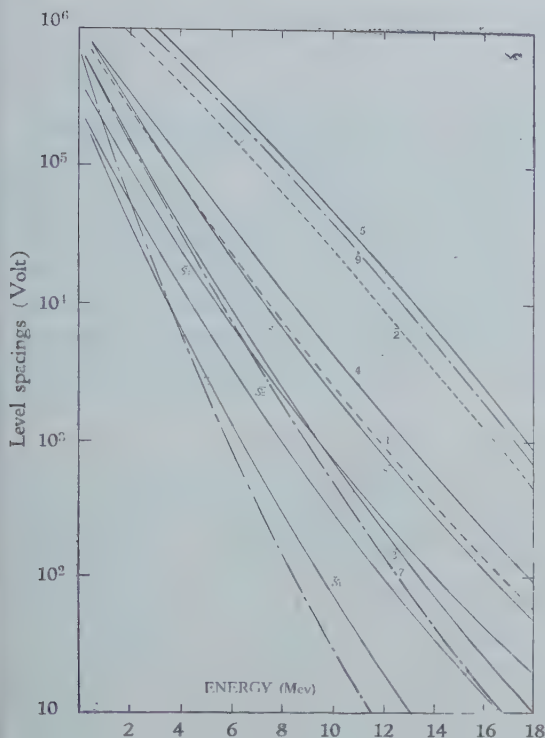


Fig. 1. The energy level spacings of Sn^{120} , Gd^{160} , and Th^{232} . $r_0 = 1.4 \cdot 10^{-13}$ cm. $I = 13.3$ Mev.

No Curve	Atomic Weight	Greatest undestroyed Core
1	120	20
2	120	50
3	160	20
4	160	50
5	160	82
6	232	20
7	232	50
8	232	82
9	232	126
S_3	120	Surface waves only are taken into account in the uniform liquid drop model.
S_2	160	
S_1	232	

$$U = kA^{2/3}I^{-7/6} \quad (4.28)$$

with

$$k = 4 \cdot 3^{-1/2} C_{4/3} (I')^{-2/3}, \quad C_{4/3} = 1.694, \quad \dots (4.29)$$

which corresponds to (4.26). The values $r_0 = 1.4 \cdot 10^{-13}$ cm and $I' = 13.3$ Mev give $k = 0.11$. k decreases when r_0 decreases or I' increases. From (4.28), we obtain

$$D(U) = aA^{-1/2}U^{5/2} \exp(-bA^{2/3}U^{4/3}), \quad (4.30)$$

where

$$a = \left(\frac{14\pi}{3}\right)^{1/2} k^{-3/4}, \quad b = \frac{7}{4} k^{3/4}.$$

The graphs of $D(U)$'s are given in Fig. 1. When U is fixed and r_0 decreases $D(U)$ increases.

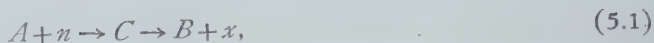
In the scope of our unsettled knowledge of the level spacing, it may be said that the two results deduced from (4.26) and (4.30) will coincide with each other when the undestroyed core is small.

§ 5. Inelastic scattering of neutrons by heavy nuclei

In this paragraph, we treat the questions (3), (4), and (6) stated in the introduction, by using our new model.

As a preliminary, we shall qualitatively examine what influences are exerted by the existence of the core on the cross sections of inelastic neutron scattering, the velocities of the inelastically scattered neutrons, the temperature of an excited nucleus, and the intensity of the fluorescent gamma-ray emitted after the nucleus is excited by a neutron.

We consider the process



where A , C , and B are the initial, the compound, and the residual nucleus, respectively. n is the incident neutron, and x an emitted neutron or a photon.

When the incident neutron energy is fixed, the fewer is the number of the nucleons which participate in the thermal agitation, the higher is the nuclear temperature. So that, when a core is remained undestroyed in the compound nucleus C , the temperature is higher than that in the case of the uniform model. Accordingly, when bombarded by neutrons of a constant energy, the nuclear temperature does not decrease monotonically with the increase of the atomic weight. This explains the experimental results of Gugelot and Stelson & Goodman.

When the nuclear temperature is higher, the probability that a part of the excited energy accumulates on a certain neutron to make it evaporate, turns out larger. So that, the larger the undestroyed core is, the larger is the probability that a neutron escapes from the nucleus, i.e., $I'_n(\text{large core}) > I'_n(\text{small core})$. Especially, $I'_n(\text{core}) > I'_n(\text{uniform})$. And at the same time, the higher the nuclear temperature is, the larger is the energy which accumulates on a neutron to make it evaporate. So that we obtain $v'_n(\text{large core}) > v'_n(\text{small core})$, where v'_n is the emitted neutron velocity. Especially, $v'_n(\text{core}) > v'_n(\text{uniform})$. This explains the experimental result of Dunlap and Little¹⁷, i.e., the energy of the neutrons emitted from Pb bombarded by 2.5 Mev neutrons, is in the intermediate of those of the elastic and inelastic scattering. The same result was obtained when Hg was bombarded by 2 Mev neutrons.

Thus, when a large core remains undestroyed, a large quantity of energy is carried out by the emitted neutron, and the residual nucleus B is remained at a low excited state, so that the intensity of the fluorescent gamma-ray is faint. On the contrary, when the undestroyed core is small or does not exist, the intensity is strong. This explains the experimental results of Aoki, Lea, and Wakatsuki & Sugimoto (Fig. 2).

The nucleus is considered to be a Fermi gas of nearly 0°. So that, the incident neutron can penetrate into the nucleus without being greatly interrupted by the nuclear matter. The amplitude of the wave function "dies out" exponentially inside the nucleus. When the undestroyed core is there, a part of the incident neutron wave is reflected on the surface of the core, and the elastic scattering probability of the incident neutron becomes larger, i.e., the sticking probability ξ becomes smaller. When the undestroyed core is large, ξ is small. Thus we obtain $\xi(\text{large core}) < \xi(\text{small core})$. When the undestroyed core is small, $\xi(\text{core}) \approx \xi(\text{uniform})$. On the other hand, the neutron capturing cross section is very small compared with the inelastic cross section, so that we obtain $\sigma(n, n')_{\text{large core}} < \sigma(n, n')_{\text{small core}} \lesssim \sigma(n, n')_{\text{uniform}}$. This may explain the experimental result of Grahame and Seaborg.

In order to treat some of the above results more quantitatively, in the next place, we modify Weisskopf's evaporation theory¹⁸, by using our model.

Let us examine the state of a heavy nucleus C when it is excited to an energy E_0

which we suppose to be greater than the binding energy of a neutron. This excitation may be produced by the process (5.1), in which n means a neutron or a hard gamma-ray. Before the collision the colliding particle must have a kinetic energy $E_C - E_0$, E_0 being its kinetic energy when bound to the nucleus C . When the wave length of the incident particle is smaller than the radius R of the nucleus, the collision can be described classically and the cross section for the inelastic scattering is equal to $\frac{2}{3}\pi R^2$. When $E_C - E_0 \gtrsim 3$ Mev, this condition is satisfied.

We now consider the emission of a neutron by the excited nucleus $C(E_C)$. According to Weisskopf, the probability per unit time of the nucleus C , excited to the energy E_C , emitting a neutron with kinetic energy between ϵ and $\epsilon + d\epsilon$, and thus transforming itself into a nucleus B with an excitation energy $E_B = E_C - E_0 - \epsilon$, is given by

$$W_n(\epsilon) d\epsilon = \sigma(E_C, \epsilon) \frac{2M}{\pi^2 \hbar^3} \epsilon \frac{\rho_B(E_B)}{\rho_C(E_C)}, \quad (5.2)$$

where $\sigma(E_C, \epsilon)$ is the cross section for the collision of a neutron with energy ϵ with a nucleus $B(E_B)$ producing a compound nucleus $C(E_C)$, $\rho_C(E)dE$ and $\rho_B(E)dE$ are the numbers of levels of the nuclei C and B , respectively, between E and $E + dE$, the energy being measured from the ground state of the particular nucleus under consideration. M is the mass of the neutron. Using the relation $\rho(E) = \exp(S(E))/\lambda(E)$ and the fact that $\lambda(E)$ is a slowly varying function of E , we express the above formula in terms of the entropy S . Then we obtain, by using the relations $\sigma(E_C, \epsilon) = \frac{2}{3}\pi R^2$, $S_B(E_C - E_0 - \epsilon) = S_B(E_C - E_0) - \epsilon/\tau_B(E_C - E_0) - f(\epsilon)$, and $f(\epsilon) = 1/2 \cdot d^2/d\epsilon^2 S_B(E_C - E_0 - \epsilon)|_{\epsilon=0}$,

$$W_n(\epsilon) d\epsilon = \text{const.} \cdot \frac{2}{3}\pi R^2 \exp(-\epsilon/\tau_B(E_C - E_0) - f(\epsilon)) d\epsilon. \quad (5.3)$$

If $f(\epsilon)$ is negligible, the formula (5.3) gives the Maxwellian distribution. In order to examine whether $f(\epsilon)$ is negligible or not, we adopt the formula (4.30) deduced by using the surface waves only, and estimate the order of the ratio:

$$f(\epsilon)/\epsilon/\tau_B(E_C - E_0) = \frac{3}{7} k^{-\frac{1}{2}} A^{-\frac{1}{2}} (E_C - E_0)^{-\frac{1}{2}}, \quad (5.4)$$

where k is the constant given in (4.28). This ratio is equal to 0.1 when $A=150$ and $E_C - E_0 = 5$ Mev. When the distribution (5.3) is Maxwellian, the mean value $\bar{\epsilon}$ of ϵ is equal to $2 \tau_B(E)$, E being the incident neutron energy. The values of U , which are obtained from the formula (4.26), are given in Table 3. The mean value $\bar{\epsilon}$ is to be calculated by using the formula (4.26) or Table 3.

The mean energy loss of the inelastically scattered neutron is $E - \bar{\epsilon}$. For example, when the incident neutron energy E is equal to 5 Mev, the energy loss is:

2.30 Mev when $(A, \text{core}) = (120, 20)$	2.98 Mev when $(A, \text{core}) = (232, 20)$
1.12 = (120, 50)	2.76 = (232, 50)
2.66 = (160, 20)	2.30 = (232, 82)
2.10 = (160, 50)	1.00 = (232, 126)
0.70 = (160, 82)	

τ	$A=120$		$A=160$			$A=232$			
	core		core			core			
	20	50	20	50	82	20	50	82	126
0.2	0.017	0.006	0.027	0.013	0.003	0.046	0.029	0.016	0.004
0.4	0.135	0.044	0.218	0.106	0.026	0.370	0.230	0.130	0.031
0.6	0.456	0.148	0.733	0.356	0.089	1.22	0.776	0.439	0.103
0.8	1.08	0.350	1.71	0.845	0.211	2.76	1.83	1.04	0.245
1.0	2.08	0.683	3.22	1.65	0.412	4.97	3.51	2.03	0.479
1.2	3.59	1.18	5.30	2.84	0.711	7.91	5.90	3.51	0.827
1.4	5.42	1.88	7.89	4.49	1.13	11.5	9.00	5.53	1.32
1.6	7.85	2.80	11.1	6.65	1.69	16.0	12.8	8.20	1.96
1.8	10.6	3.99	14.8	9.33	2.40	21.2	20.4	11.6	2.79
2.0	14.0	5.48	19.1	12.6	3.29	27.2	22.7	15.6	3.83

Table 3. The values of U , which are obtained numerically from the formula (4.26).

The intensity I of the fluorescent gamma-ray per atom is a product of the energy loss $E - \bar{\epsilon}$ and the neutron inelastic scattering cross section⁴⁾:

$$I = \text{const. } \xi A^{2/3} (E - \bar{\epsilon}). \quad (5.5)$$

The exact calculation of the sticking probability ξ is not easy. It is known only that when a nucleus is near by a closed shell nucleus, the ξ -value is small.

Thus the curve of the intensity I is not simple, but has several branches (Fig. 3). According to the ambiguity of the ξ -value, the positions of these branches can not be settled exactly.

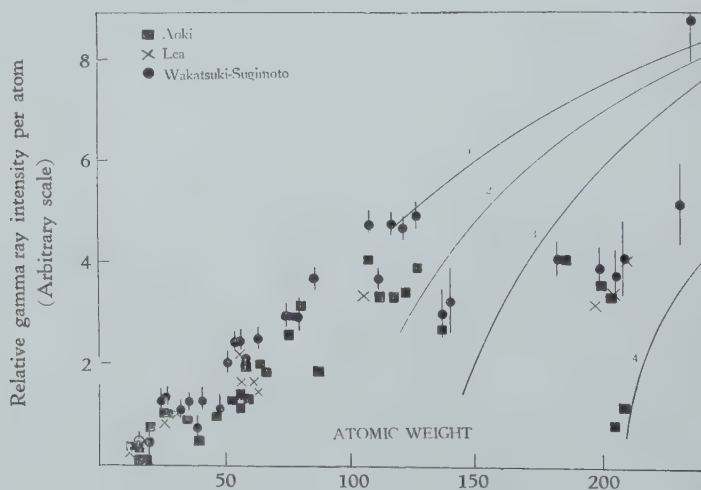


Fig. 2. The fluorescent gamma ray intensity per atom. Curves 1, 2, 3, and 4 are the intensities obtained when the 20-, 50, 82-, and 126 core are remained undestroyed, respectively. The sticking probability ξ is assumed to be constant, so that the positions of the branches are not exact. The true positions of the branches will perhaps be lower than those given in the figure. Curve 1 has been adjusted at $A=120$. The incident neutron energy is 5 Mev.

§ 7. Conclusion and acknowledgement

The necessities of the existence of cores in heavy nuclei have been discussed from two sides. One is the fission process, and the other is the inelastic scattering of neutrons. The remarkable feature of the asymmetric fission which has been explained in § 3 by using our model, will not adequately be explained by using the uniform liquid-drop model. And the uniform liquid-drop model is not strong enough to explain the experimental fact that the photo-fission threshold energies of heaviest nuclei are all almost equal to 5 Mev. The new model, having the core, has the possibility to explain this experimental result.

The experimental facts that when $A \gtrsim 110$, the inelastic neutron scattering cross section does not increase with A as the old theory predicts, the fluorescent gamma-ray per atom does not increase with A , and the nuclear specific heat does not increase with A , can all be explained by using the level densities of heavy nuclei ($A \gtrsim 110$) thinner than those given by the old theories. And, in fact, the level density can be made thinner when the freedom of internal motion of the nucleus is restricted. At one time, Grahame proposed the idea of local heating (hot spot) of the nucleus. The introduction of the core indeed restricts the freedom of the internal motion and leads one to Grahame's idea. And the conclusions of Grahame are also obtainable from our model. If, however, the heat conduction in the nucleus is very rapid, the idea of the local heating will not be so easily acceptable.

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The Mass Variation with Velocity in Bopp's Unitary Field Theory, I

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In the case of free motion of constant velocity, the total energy and total momentum of self-field produced by a point particle was calculated starting from the energy-momentum tensor derived by Bopp and Heisenberg for the Bopp-type non-local field theory. The correct velocity variation of energy and momentum required by relativistic mechanics is not guaranteed from the beginning for the above energy-momentum tensor. From the detailed calculations in this paper, however, the correct variation upon velocity is attainable if the characteristic function $\epsilon(x)$ satisfies a certain condition. As this condition is very flexible, there remain possibilities of the presence of some appropriate form of $\epsilon(x)$ which guarantees the finiteness of the self energy of a point particle and the correct velocity variation of energy and momentum of the self-field. The examination of the energy-momentum tensor itself causes some doubt about the correctness of the energy-momentum tensor derived by Bopp and Heisenberg.

§ 1. Introduction

Among various attempts which aim at solving many deep underlying difficulties in the present field theory of elementary particles is the unitary field theoretical contemplation of the properties of elementary particles. This unitary field theoretical standpoint is, as far as the following investigations are concerned, based on the understanding of the physical world, according to which matter and field in ordinary sense are not considered as independent entities in opposition to the usual dualistic standpoint. Particles have no mechanical mass at all. The inertia, which we observe as mass, should be entirely due to the inertia of the field which is produced by the particle around it and has a certain amount of energy and momentum and therefore also a certain inertia. Thus it is necessary for the above unitary standpoint to be acceptable that the total energy and momentum of the self-produced field of the particle show a correct variation upon velocity required by the relativistic mechanics. As is well known, current field theory does not satisfy the above requirement. F. Bopp¹⁾ investigated the modification of Maxwell's electromagnetic field theory by introducing higher derivatives into the Lagrangian and succeeded in obtaining the correct behavior of the total energy and momentum of the self-field. As this theory, however, has been proved to be equivalent to a mere mixed field theory and has a difficulty of negative energy, Bopp²⁾ considered later, as a further generalization, a modified electromagnetic field having non-local character. With respect to the above theory, the corresponding problem has not yet been investigated. It is the purpose of this paper to see whether it is possible to establish the correct velocity variation of the total energy and momentum of the self-field of a point particle and at the same time the finiteness of the total energy of the self-field of a point particle in the Bopp's new non-local unitary theory.

§ 2. Old Bopp's theory

In the old Bopp's unitary theory for electromagnetic field¹, originally developed by Mie and subsequently by Born, Infeld and Bopp, one replaces the usual Lagrangian

$$L = -\frac{1}{16\pi} \{ f_{\alpha\beta}^2 \} + \frac{1}{c} S_\alpha \varphi_\alpha \quad (1)$$

by

$$L = -\frac{1}{16\pi} \left\{ f_{\alpha\beta}^2 + \frac{1}{x^2} \left(\frac{\partial f_{\alpha\beta}}{\partial x_\gamma} \right)^2 \right\} + \frac{1}{c} S_\alpha \varphi_\alpha \quad (2)$$

where

$$f_{\alpha\beta} = \partial \varphi_\beta / \partial x_\alpha - \partial \varphi_\alpha / \partial x_\beta, \quad (2)'$$

$$\varphi_\alpha = (\mathbf{A}, i\phi), \quad S_\alpha = (\mathbf{i}, ic\rho)$$

and x is some constant having a dimension of reciprocal length and other notations have their usual meanings. The addition of higher derivatives of field quantities into Lagrangian has been proved to be equivalent to mixing neutral vector meson field in addition to Maxwell field. The energy-momentum tensor, which satisfies the conservation law

$$\frac{\partial T_{\alpha\beta}}{\partial x_\beta} = -\frac{1}{c} f_{\alpha\gamma} S_\gamma, \quad (3)$$

takes the form

$$T_{\alpha\beta}^* = T_{\alpha\beta}^M - T_{\alpha\beta}^Y, \quad (4)$$

where $T_{\alpha\beta}^M$ and $T_{\alpha\beta}^Y$ are the usual energy-momentum tensor for Maxwell field and neutral vector meson field respectively. The total energy E and the total momentum \mathbf{P} of the field can be calculated from the formula

$$E = - \int T_{44} dV, \quad \mathbf{P} = \frac{1}{ic} \int \vec{T}_{k4} dV. \quad (5)$$

For a uniform and rectilinear motion of a point source electron, the contributions of Maxwell and meson fields to total energy, E^M and E^Y , have the following values.

$$E^M = \frac{1}{\sqrt{1-\beta^2}} \left(E_0^M + \frac{1}{3} \beta^2 E_0^M \right), \quad (6)$$

$$E^Y = \frac{1}{\sqrt{1-\beta^2}} \left(E_0^Y + \frac{1}{3} \beta^2 E_0^M \right),$$

where the suffix 0 means the corresponding quantities for $\beta=0$. Accordingly total energy E is given by

$$E = E^M - E^Y = \frac{1}{\sqrt{1-\beta^2}} (E_0^M - E_0^Y) = \frac{1}{\sqrt{1-\beta^2}} E_0. \quad (7)$$

Quite similar results are obtained for total momentum \mathbf{P} :

$$\mathbf{P}^M = \frac{1}{c^2} \frac{\mathbf{V}}{\sqrt{1-\beta^2}} \left(E_0^M + \frac{1}{3} E_0^M \right),$$

$$\mathbf{P}^V = \frac{1}{c^2} \frac{\mathbf{V}}{\sqrt{1-\beta^2}} \left(E_0^V + \frac{1}{3} E_0^M \right), \quad (8)$$

and

$$\mathbf{P} = \mathbf{P}^M - \mathbf{P}^V = \frac{\mathbf{V}}{c^2 \sqrt{1-\beta^2}} E_0. \quad (9)$$

From equations (6)–(9), it is readily seen that the total energy and the total momentum show the correct variation with velocity and it is permitted to consider the energy and momentum of the electron to be stored in the surrounding field which is produced by the electron and the unitary standpoint can be admitted. It must be noted, however, that the separate field does not behave correctly in the above sense but their combination alone shows permissible dependence upon velocity. This form of Bopp's unitary field theory, though satisfactory in the above point, is nothing but a mixed field theory and has an unconquerable difficult point; the non-positive definite character of the Hamiltonian.

§ 3. New Bopp's theory

Later Bopp³⁾ proposed a non-local modification of electromagnetic field. If we introduce a Lorentz invariant "fernwirkungs-funktion" $\epsilon(x)$, which is a function of $\sqrt{-x_\alpha^2}$ alone, the modified Lagrange function is given

$$\bar{L} = -\frac{1}{16\pi} \int f_{\alpha\beta}(x) \epsilon(x-x') f_{\alpha\beta}(x') dx dx' + \frac{1}{c} \int \varphi_\alpha S_\alpha dx. \quad (10)$$

If we choose delta function as $\epsilon(x)$, the above Lagrangian reduces to the usual one. The Lagrangian defined in (10) means that a source function localized at some point determines not only the field quantities at that point but the ones at the surrounding points: $\epsilon(x)$ is a measure of nonlocality of the field. The above assignment, however, turns out to be equivalent to introducing infinitely many higher derivatives of field quantities, namely the generalization of the former expression (2). The problem of constructing the energy momentum tensor which satisfies the conservation law (3) in this case* was pursued by Bopp³⁾ and Heisenberg⁴⁾ and the following expression was obtained:

$$\begin{aligned} \tilde{T}_{\alpha\beta}(k) = & \frac{1}{256\pi^5} \int (k_\alpha - 2k'_\alpha)(k_\beta - 2k'_\beta) \frac{(k-k')^2 \tilde{\epsilon}(k-k') - k'^2 \tilde{\epsilon}(k')}{(k-k')^2 - k'^2} \\ & \times \tilde{\varphi}_\tau(k') \tilde{\varphi}_\tau(k-k') dk' - \frac{1}{32\pi^4 c} \partial_{\alpha\beta} \int \tilde{\varphi}_\tau(k') \tilde{S}_\tau(k-k') dk' \\ & + \frac{1}{16\pi^4 c} \int \tilde{\varphi}_\alpha(k') \tilde{S}_\beta(k-k') dk', \end{aligned} \quad (11)$$

* Recently Y. Ono (Prog. Theor. Phys. 6 (1951), 898) took up this problem and solved quite satisfactorily without having many difficulties concerning the expression obtained by Bopp and Heisenberg which will be discussed in the following.

where $\tilde{T}_{\alpha\beta}(k)$, $\tilde{\epsilon}(k)$, $\tilde{\varphi}_\alpha(k)$, $S_\alpha(k)$ are the Fourier coefficients of $T_{\alpha\beta}(x)$, $\epsilon(x)$, $\varphi_\alpha(x)$, $S_\alpha(x)$ and they are connected by the equation of the type

$$T_{\alpha\beta}(x) = \frac{1}{(2\pi)^4} \int \tilde{T}_{\alpha\beta}(k) e^{-ik_\tau x_\tau} dk,$$

$$\tilde{T}_{\alpha\beta}(k) = \int T_{\alpha\beta}(x) e^{+ik_\tau x_\tau} dx, \quad (12)$$

where dx and dk means $dx dy dz dt$ and $dk_x dk_y dk_z d\omega$ respectively.

The above form of energy-momentum tensor has many inadmissible defects: the most important one is that equation (11) does not tend to the usual form of energy momentum tensor for Maxwell field when one puts specially $\epsilon(x) = \delta(x)$ or in momentum space $\tilde{\epsilon}(k) = \text{const} \equiv 1$.** The corresponding limiting form is quite different from the usual one. Furthermore the expression (11) does not have the properties of gauge invariance and symmetric tensor. As regards the symmetric property, we can show it can be thought as symmetric when the equations of motion are used, as far as we consider the total energy and momentum of the field, as shown in the following section. Judging from these points, we can hardly believe that the above form of energy momentum tensor has been derived correctly. As the following calculations and discussions are entirely based upon the form (11), it might be more probable that the conclusions deduced from the following calculations must not be considered as the criticisms concerning the Bopp's nonlocal field theory itself. It might also be possible that we can only to say decisively that the energy momentum tensor (11) has not been derived correctly.

§ 4. Calculations and discussions

Now we consider a case of point electron which has a constant velocity \mathbf{V} and was situated at the origin at time $t=0$. The four current $S_\alpha(x)$ then takes the form

$$S_\alpha = (\mathbf{i}, i\rho)$$

and

$$\begin{aligned} \rho(x, y, z, t) &= e \delta(x - v_x t) \delta(y - v_y t) \delta(z - v_z t), \\ \mathbf{i}(x, y, z, t) &= e \mathbf{V} \delta(x - v_x t) \delta(y - v_y t) \delta(z - v_z t), \end{aligned} \quad (13)$$

which satisfy the continuity equation $\partial S_\alpha / \partial x_\alpha = 0$ or $\partial \rho / \partial t + \text{div } \mathbf{i} = 0$. If we integrate both sides of Equation (3) over the whole field, we obtain

$$\int \frac{\partial T_{\alpha\beta}}{\partial x_\beta} dV = -\frac{1}{c} \int f_{\alpha\beta} S_\beta dV \quad (14)$$

or

$$\begin{aligned} \partial E / \partial t &= -\mathbf{i} \cdot \mathbf{E}, \\ \partial \mathbf{P} / \partial t &= -(\rho \mathbf{E} + \frac{1}{c} [\mathbf{i} \times \mathbf{H}]), \end{aligned} \quad (14)'$$

** The most important purpose of the researchment of Y. Ono was to improve this point of defect.

where E and \mathbf{P} are total energy and momentum of the field and are given by equation (5) and \mathbf{E} and \mathbf{H} are derived from field quantities $\varphi_\alpha = (\mathbf{A}, i\phi)$ by the equation $\mathbf{H} = \text{rot} \mathbf{A}$ and $\mathbf{E} = -\text{grad} \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$. Thus so that total energy and momentum may be conserved, the total Lorentz force acting on the point electron or the right hand side of equation (14) or (14)' must vanish. In order to see this, we must first construct the solutions of the field equations

$$\frac{\partial}{\partial x_\beta} \int \varepsilon(x-x') f_{\alpha\beta}(x') dx' = \frac{4\pi}{c} S_\alpha(x) \quad (15)$$

or using the Lorentz condition $\partial \varphi_\alpha(x) / \partial x_\alpha = 0$,

$$\square \int \varepsilon(x-x') \varphi_\alpha(x') dx' = -\frac{4\pi}{c} S_\alpha(x). \quad (16)$$

Transforming into momentum representation, one obtains from (16)

$$k_\beta^2 \tilde{\varepsilon}(k) \tilde{\varphi}_\alpha(k) = \frac{4\pi}{c} \tilde{S}_\alpha(k), \quad (17)$$

from which it follows

$$\tilde{\varphi}_\alpha(k) = \frac{4\pi}{c} \frac{\tilde{S}_\alpha(k)}{k_\beta^2 \tilde{\varepsilon}(k)} \quad (18)$$

which satisfies Lorentz condition $\partial \varphi_\alpha(x) / \partial x_\alpha = 0$ or $k_\alpha \tilde{\varphi}_\alpha(k) = 0$ owing to the continuity equation $\partial S_\alpha(x) / \partial x_\alpha = 0$ or $k_\alpha \tilde{S}_\alpha(k) = 0$. The Fourier coefficients for the four current $S_\alpha(x)$ can be calculated as follows:

$$\begin{aligned} \tilde{S}_\alpha(k) &= \int S_\alpha(x) e^{ik_\tau x_\tau} dx = c(\mathbf{V}, ic) \int \delta(\mathbf{X} - \mathbf{V}t) e^{i(\mathbf{K}\mathbf{X} - \omega t)} d\mathbf{X} dt \\ &= \int e(\mathbf{V}, ic) \int e^{i(\mathbf{K} \cdot \mathbf{V} - \omega)t} dt = 2\pi e(\mathbf{V}, ic) \delta(\mathbf{K}\mathbf{V} - \omega) \\ &\equiv 2\pi e u_\alpha \delta(\mathbf{K} \cdot \mathbf{V} - \omega), \end{aligned} \quad (19)$$

where u_α is \mathbf{V} for $\alpha=1, 2, 3$ and ic for $\alpha=4$. It is here to be noted that every $\tilde{\varphi}_\alpha(k)$, therefore every $\tilde{S}_\alpha(k)$, contains delta function $\delta(\mathbf{K} \cdot \mathbf{V} - \omega)$. Because of the relation

$$\int T_{\alpha\beta}(x) dV = \frac{1}{(2\pi)^3} \int \tilde{T}_{\alpha\beta}(k) e^{ik_\tau x_\tau} dk dV = \frac{1}{(2\pi)^3} \int \tilde{T}_{\alpha\beta}(k) \delta(\mathbf{K}) e^{i\omega t} dk, \quad (20)$$

only the Fourier coefficients at $\mathbf{K}=0$ contribute to the volume integral. The Fourier coefficients of the integrand of right hand side of (14) are

$$\begin{aligned} (\widetilde{f_{\alpha\beta} S_\beta})(k) &= \int f_{\alpha\beta}(x) S_\beta(x) e^{ik_\tau x_\tau} dx \\ &= \frac{1}{(2\pi)^3} \int \tilde{f}_{\alpha\beta}(k') \tilde{S}_\beta(k'') e^{-i(k' + k'' - k)_\tau x_\tau} dk' dk'' dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^4} \int \tilde{f}_{\alpha\beta}(k') \tilde{S}_\beta(k-k') dk' \\
&= \frac{1}{(2\pi)^4} \int (-ik'_\alpha \tilde{\varphi}_\beta(k') + ik'_\beta \tilde{\varphi}_\alpha(k')) \tilde{S}_\beta(k-k') d\hat{k}'. \quad (21)
\end{aligned}$$

Owing to the solutions (18), the right hand sides of (21) and (11) have the form $\sum_{\alpha\beta} C_{\alpha\beta}(k, k') \tilde{S}_\alpha(k') \tilde{S}_\beta(k-k')$, where $C_{\alpha\beta}(k, k')$ are certain functions of k_α and k'_α , and therefore have the factor $\delta(\mathbf{K}\mathbf{I}-\omega)\delta(\mathbf{K}'\mathbf{I}-\omega')$, which can be said readily from (19). Combining these delta functions with the one in equation (20), we then obtain the factor $\delta(\mathbf{K})\delta(\omega)$ or $\delta(k)$. Accordingly, to the volume integral of Lorentz force density $1/c f_{\alpha\beta} S_\beta$ and energy momentum density $T_{\alpha\beta}$, which are the only interesting quantities in this paper, contributes only the Fourier coefficient at $k_\alpha=0$. It follows firstly that in order to calculate the total Lorentz force acting on the electron or the right hand side of (14) or (14') we may put $k_\alpha=0$ in equation (21). Because of the even character of $\epsilon(x)$ and the continuity equation $k_\alpha S_\alpha(k)=0$, the second term of right hand side of (21) vanishes. Concerning to the first term, we have only to remark that it is a odd function of k'_α . Thus it has been proved that the total Lorentz force acting on the electron vanishes and so the total energy and momentum of the field are conserved for the case of free motion of constant velocity. Secondly the statement with respect to the symmetry property of energy-momentum tensor mentioned in § 3 can be qualified. In order to investigate the symmetry property of energy-momentum tensor we have only to consider the last term of (11). Here again we may put $k_\alpha=0$, as far as we consider the volume integral of the energy-momentum of the field. Again remarking the even character of $\tilde{S}_\alpha(k)$ and replacing the $\tilde{\varphi}_\alpha(k')$ in the last term of (11) by the right hand side of (18), we can see that $T_{\alpha\beta}(x)$ has the following limited symmetry property :

$$\int T_{\alpha\beta}(x) dV = \int T_{\beta\alpha}(x) dV. \quad (22)$$

In order to see the meaning of this limited symmetry property we define as usual an angular momentum density :

$$M_{\alpha\beta\gamma} = x_\alpha T_{\beta\gamma} - x_\beta T_{\alpha\gamma} \quad (23)$$

whose divergence according to (3) becomes :

$$\sum_\gamma \frac{\partial}{\partial x_\gamma} T_{\alpha\beta\gamma} = T_{\beta\alpha} - T_{\alpha\beta} - \frac{1}{c} (x_\alpha f_{\beta\gamma} - x_\beta f_{\alpha\gamma}) S_\gamma.$$

If we integrate both sides over the total volume, the first two terms cancel owing to (22) and the third and the fourth term vanish respectively because the factor x_α and x_β can be replaced by numerical constants owing to the delta function $\delta(\mathbf{X}-\mathbf{I}t)$ contained in $S_\gamma(x)$ and the total Lorentz force acting on the electron vanishes as was proven before in this section. From this it follows that the tensor $\int dV M_{\alpha\beta\gamma}$ is constant in time. In particular, the components of the angular momentum are constant :

$$\frac{1}{ic} \int dV M_{\alpha\beta\gamma} = \int \left(x_\alpha \frac{1}{ic} T_{\beta\gamma} - x_\beta \frac{1}{ic} T_{\alpha\gamma} \right) dV, \quad (24)$$

where α, β run from 1 to 3. It is here to be recognized that the above energy momentum tensor does not satisfy the usual conservation law for energy and momentum :

$$\partial T_{\alpha\beta}/\partial x_\beta = 0, \quad (25)$$

and only the volume integral of Lorentz force $1/c \int_{\alpha 3} S_3$ vanishes, which alone was sufficient to the conservation of total energy and momentum of the field and the constancy of angular momentum of the field. But in order that the total energy and momentum of the field may form a 4-vector, the integral of Lorentz force $1/c \int_{\alpha 3} S_3$ over some 4-dimensional space must vanish⁶. The vanishment of integral over a flat surface $t=const.$ alone would not be sufficient. Therefore the total energy and momentum of the field do not in general show a correct behavior upon velocity of the singular point. It is the main purpose of this paper to see whether the total energy and momentum of the field do have correct dependence upon velocity and if it is the case to find the condition for $\epsilon(x)$ which makes it sure to have correct behavior. This will be done below.

First of all we calculate the third term of $\tilde{T}_{\alpha 1}(k)$ or the right hand side of equation (11) for $\beta=4$. If we substitute the solutions (18) and (19) into field potentials $\tilde{\varphi}(k)$, we obtain

$$\begin{aligned} & \frac{1}{16\pi^4 c} \int \tilde{\varphi}_\alpha(k') \tilde{S}_4(k-k') dk' \\ &= \frac{2}{c^2 (2\pi)^3} \int \frac{\tilde{S}_\alpha(k') \tilde{S}_4(k-k')}{k'^2_\gamma \tilde{\epsilon}(k')} dk' \\ &= \frac{2c^2}{2\pi c^2} \int \frac{u_\alpha u_4 \delta(\mathbf{K}'\mathbf{V}-\omega') \delta((\mathbf{K}-\mathbf{K}')\mathbf{V}-(\omega-\omega'))}{k'^2_\gamma \tilde{\epsilon}(k')} d\mathbf{K}' d\omega' \\ &= \frac{c^2}{\pi c^2} u_4 u_\alpha \delta(\mathbf{K}\mathbf{V}-\omega) \int \frac{\delta(\mathbf{K}'\mathbf{V}-\omega')}{(k'^2-\omega'^2/c^2) \tilde{\epsilon}(k', \omega'/c)} d\mathbf{K}' d\omega' \\ &= \frac{c^2}{\pi c^2} u_4 u_\alpha \delta(\mathbf{K}\mathbf{V}-\omega) \int \frac{d\mathbf{K}'}{k'^2 (1-\beta^2 \cos^2 \theta) \tilde{\epsilon}(k', k'\beta \cos \theta)}, \end{aligned} \quad (26)$$

where θ is the angle between \mathbf{K}' and \mathbf{V} and $k'=|\mathbf{K}'|$ and $\beta=v/c$. If we remember that $\tilde{\epsilon}(k)$ is a function of only $k^2_\alpha = \mathbf{K}^2 - \omega^2/c^2$ and introduce new integration variable $K=k'\sqrt{1-\beta^2 \cos^2 \theta}$ instead of k' , we obtain further

$$\begin{aligned} (26) &= \frac{c^2}{\pi c^2} u_4 u_\alpha \delta(\mathbf{K}\mathbf{V}-\omega) \int \frac{K^2 dK' \sin \theta d\theta d\varphi}{K^2 \tilde{\epsilon}(K) (\sqrt{1-\beta^2 \cos^2 \theta})^3} \\ &= \frac{2c^2}{c^2} u_4 u_\alpha \delta(\mathbf{K}\mathbf{V}-\omega) \int_0^\infty \frac{dK}{\tilde{\epsilon}(K)} \int_{-1}^{+1} \frac{dz}{(\sqrt{1-\beta^2 z^2})^3}. \end{aligned} \quad (27)$$

The last factor can be integrated as follows :

$$\int_{-1}^{+1} \frac{dz}{(\sqrt{1-\beta^2 z^2})^3} = 2 \int_0^1 \frac{dz}{(\sqrt{1-\beta^2 z^2})^3} = 2 \int_0^{\sin^{-1} \beta} \frac{d\theta}{\beta \cos^2 \theta} = \frac{2}{\sqrt{1-\beta^2}}. \quad (28)$$

Thus we obtain

$$\begin{aligned} & \frac{1}{16\pi^4 c} \int \tilde{\varphi}_\alpha(k') \tilde{S}_1(k-k') dk' \\ &= \frac{4e^2}{c^2} u_4 u_\alpha \delta(\mathbf{KV} - \omega) \frac{1}{\sqrt{1-\beta^2}} \int_0^\infty \frac{dK}{\tilde{\varepsilon}(K)}. \end{aligned} \quad (29)$$

In order to see the meaning of remaining integral, we calculate the static potential for a point charge at the origin :

$$\begin{aligned} \varphi_4(x) &= i\varphi_0(x) = \frac{1}{(2\pi)^4} \int \tilde{\varphi}_4(k) e^{-ik_\alpha x^\alpha} dk \\ &= \frac{2e}{c(2\pi)^3} u_4 \int \frac{\delta(\omega)}{(\mathbf{K}^2 - \omega^2/c^2) \tilde{\varepsilon}(\mathbf{K}, \omega/c)} e^{-i(\mathbf{KX} - \omega t)} d\mathbf{K} d\omega \\ &= \frac{eu_4}{c\pi} \int \frac{e^{-iKx \cos \theta}}{K^2 \tilde{\varepsilon}(K)} K^2 dK \sin \theta d\theta \\ &= \frac{2eu_4}{c\pi} \int_0^\infty \frac{\sin Kr}{Kr} \frac{dK}{\tilde{\varepsilon}(K)}, \end{aligned} \quad (30)$$

where we used (18) and (19) putting $\mathbf{V}=0$ and $r=|\mathbf{X}|$. Therefore, remembering $u_4=ic$, we obtain as the static potential at the origin

$$\varphi(0) = \frac{2e}{\pi} \int_0^\infty \frac{dK}{\tilde{\varepsilon}(K)}, \quad (31)$$

which is just the integral appeared in the equation (29) and diverges in the usual Maxwell theory where $\tilde{\varepsilon}(k)=1$. Substituting $\varphi(0)$ for the integral appearing in (29), we get finally

$$\begin{aligned} & \frac{1}{16\pi^4 c} \int \tilde{\varphi}_\alpha(k') \tilde{S}_4(k-k') dk' \\ &= \frac{2\pi e}{c^2} u_4 u_\alpha \delta(\mathbf{KV} - \omega) \varphi(0) / \sqrt{1-\beta^2}. \end{aligned} \quad (32)$$

Thus we obtain as the total energy and momentum of the field arising from the third term of $\tilde{T}_{\alpha\beta}(k)$, using $u_4=ic$ and $\mathbf{U}=\mathbf{V}$,

$$\begin{aligned} E^{(3)} &= - \int T_{44} dV = - \frac{1}{(2\pi)} \int \tilde{T}_{44}(k) e^{i\omega t} \delta(\mathbf{K}) d\mathbf{K} d\omega \\ &= c\varphi(0) / \sqrt{1-\beta^2} \end{aligned} \quad (33)$$

and

$$\mathbf{P}^{(3)} = \frac{1}{ic} \int \tilde{T}_{\alpha 4} dV = e\mathbf{V}\varphi(0) / c^2 \sqrt{1-\beta^2}, \quad (34)$$

which show correct dependence upon velocity of the particle.

Next we calculate the second term only :

$$\begin{aligned}
 & -\frac{1}{32\pi^4 c} \partial_{4\alpha} \int \tilde{\varphi}_\tau(k') \tilde{S}_\tau(k-k') dk' \\
 & = -\frac{1}{(2\pi)^3 c^2} \partial_{4\tau} \int \frac{\tilde{S}_\tau(k') \tilde{S}_\tau(k-k')}{k'^2_\beta \tilde{\epsilon}(k')} dk' \\
 & = -\frac{c^2}{2\pi c^2} \partial_{\alpha 1} u_\tau u_\tau \delta(\mathbf{KV}-\omega) \int \frac{\delta(\mathbf{K}'\mathbf{V}-\omega')}{(\mathbf{K}'^2-\omega'^2/c^2) \tilde{\epsilon}(\mathbf{K}', \omega'/c)} d\mathbf{K}' d\omega' \\
 & = \frac{c^2}{2\pi} \partial_{\alpha 1} (1-\beta^2) \delta(\mathbf{KV}-\omega) \int \frac{dK \sin \theta d\theta d\varphi}{\tilde{\epsilon}(K) (\sqrt{1-\beta^2 \cos^2 \theta})^3} \\
 & = 2c^2 \partial_{\alpha 1} (1-\beta^2) \delta(\mathbf{KV}-\omega) \frac{1}{\sqrt{1-\beta^2}} \int_0^\infty \frac{dK}{\tilde{\epsilon}(K)} \\
 & = \pi c \partial_{\alpha 1} \sqrt{1-\beta^2} \delta(\mathbf{KV}-\omega) \varphi(0), \tag{35}
 \end{aligned}$$

where use was made of (28) and (31). Thus we get as the total energy and momentum of the field arising from the second term of $\tilde{T}_{\alpha\beta}(k)$

$$E^{(2)} = - \int T_{44}(x) dV = - \frac{1}{2} c \varphi(0) \sqrt{1-\beta^2} \tag{36}$$

and

$$\mathbf{P}^{(2)} = 0. \tag{37}$$

The contributions due to this term, therefore, do not give correct velocity dependence.

Finally we calculate the first term of $\tilde{T}_{\alpha 1}(k)$. This time we must remember that the contributions to the total energy and momentum of the field arises only from the Fourier coefficient $\tilde{T}_{\alpha 1}(k)$ at $k_\alpha=0$ and we must take an appropriate limiting process before performing calculations because this term contains a factor whose denominator and numerator both approach zero when k_α approach zero. If we expand $\tilde{\epsilon}(k-k')$ in Taylor series

$$\begin{aligned}
 \tilde{\epsilon}(k-k') &= \tilde{\epsilon}(k'-k) \equiv f(k'^2_\alpha - 2k'_\alpha k_\alpha + k^2_\alpha) \\
 &= f(k'^2_\alpha) - 2k_\alpha k'_\alpha f'(k'^2_\alpha) + \dots \\
 &= \tilde{\epsilon}(k') - 2k_\alpha k'_\alpha f'(k'^2_\alpha) + \dots, \tag{38}
 \end{aligned}$$

where $f(k'^2_\alpha)$ is a function of k'^2_α and is equal to $\tilde{\epsilon}(k)$ which is a function of $\sqrt{|k'_\alpha|^2}$ and $f'(k'^2_\alpha)$ is the derivative of $f(k'^2_\alpha)$ with respect to its argument k'^2_α , we obtain

$$\begin{aligned}
 & \lim_{k_\alpha \rightarrow 0} \frac{(k-k')^\nu \tilde{\epsilon}(k-k') - k'^\nu \tilde{\epsilon}(k')}{(k-k')^2 - k'^2} \\
 &= \lim_{k_\alpha \rightarrow 0} \frac{k'^2_\alpha \tilde{\epsilon}(k') - 2k'^2_\beta k'_\alpha k'_\alpha f'(k'^2_\beta) - 2k_\alpha k'_\alpha \tilde{\epsilon}(k') + \dots - k'^2_\alpha \tilde{\epsilon}(k')}{k'^2_\alpha - 2k'_\alpha k'_\alpha + \dots - k'^2_\alpha} \\
 &= \tilde{\epsilon}(k') + k'^2_\alpha f'(k'^2_\alpha). \tag{39}
 \end{aligned}$$

Putting the above limiting value into the respective term, we can calculate the first term as follows:

$$\begin{aligned}
& \frac{1}{(256\pi^5)} \int (k_\alpha - 2k'_\alpha) (k_4 - 2k'_4) \{ \tilde{\epsilon}(k') + k'^2_\alpha f'(k'^2_\alpha) \} \tilde{\varphi}_T(k') \tilde{\varphi}_T(k - k') dk' \\
&= \frac{\pi e^2}{(2\pi)^2} \epsilon^{\mu_1 \mu_2 \mu_3} \delta(\mathbf{KV} - \omega) \int \frac{\tilde{\epsilon}(k') + k'^2_\alpha f'(k'^2_\alpha)}{(k'^2_\alpha)^2 \tilde{\epsilon}(k')^2} \delta(\mathbf{K}'\mathbf{V} - \omega') 4k'_\alpha k'_4 dk', \quad (40)
\end{aligned}$$

where we have put $k_n = 0$ and used similar manipulations as before. We shall consider the case $u=4$ and $u=1, 2, 3$ separately. In the former case, one gets

$$\begin{aligned}
(40) &= \frac{e^2}{\pi} (1 - \beta^2) \delta(\mathbf{KV} - \omega) \int \frac{\tilde{\epsilon}(k') + k'^2_\alpha f'(k'^2_\alpha)}{(k'^2_\alpha)^2 (\tilde{\epsilon}(k'))^2} \frac{\omega'^2}{e^2} \delta(\mathbf{K}'\mathbf{V} - \omega') d\mathbf{K}' d\omega' \\
&= \frac{e^2}{\pi} (1 - \beta^2) \delta(\mathbf{KV} - \omega) \int \frac{\tilde{\epsilon}(\mathbf{K}', k' \beta \cos \theta) + k'^2 (1 - \beta^2 \cos^2 \theta) f'(k'^2 (1 - \beta^2 \cos^2 \theta))}{(k'^2 (1 - \beta^2 \cos^2 \theta))^2 (\tilde{\epsilon}(\mathbf{K}', k' \beta \cos \theta))^2} \\
&\quad \times k'^2 \beta^2 \cos^2 \theta k'^2 dk' \sin \theta d\theta d\varphi. \quad (41)
\end{aligned}$$

Again introducing the new integration variable $K = k' \sqrt{1 - \beta^2 \cos^2 \theta}$ instead of k' , above formula becomes

$$\begin{aligned}
& \frac{e^2}{\pi} (1 - \beta^2) \delta(\mathbf{KV} - \omega) \int \frac{\tilde{\epsilon}(K) + K^2 f'(K^2)}{K^4 (\tilde{\epsilon}(K))^2} \frac{K^4 \beta^2 \cos^2 \theta \sin \theta d\theta d\varphi dK}{(\sqrt{1 - \beta^2 \cos^2 \theta})^5} \\
&= 2e^2 \beta^2 (1 - \beta^2) \delta(\mathbf{KV} - \omega) \int_0^\infty \frac{\tilde{\epsilon}(K) + K^2 f'(K^2)}{(\tilde{\epsilon}(K))^2} dK \int_{-1}^{+1} \frac{z^2 dz}{(\sqrt{1 - \beta^2 z^2})^5}. \quad (42)
\end{aligned}$$

The last integral will be calculated separately:

$$\begin{aligned}
\int_{-1}^{+1} \frac{z^2 dz}{(\sqrt{1 - \beta^2 z^2})^5} &= \frac{2}{\beta^2} \int_0^{\sin^{-1} \beta} \frac{\sin \theta \cos \theta d\theta}{\cos^5 \theta} = \frac{2}{\beta^3} \int_0^{\sin^{-1} \beta} \frac{\sin^2 \theta d\theta}{\cos^4 \theta} \\
&= \frac{2}{\beta^3} \frac{1}{3} \left[\frac{\sin \theta}{\cos^3 \theta} - \tan \theta \right]_0^{\sin^{-1} \beta} = \frac{2}{3} \frac{1}{(\sqrt{1 - \beta^2})^3}. \quad (43)
\end{aligned}$$

Introducing a new abbreviation

$$\tilde{\xi} \equiv \frac{2e}{\pi} \int_0^\infty \frac{\tilde{\epsilon}(K) + K^2 f'(K^2)}{(\tilde{\epsilon}(K))^2} dK, \quad (44)$$

one obtains

$$(42) = \frac{2\pi e}{3} \frac{\beta^2}{\sqrt{1 - \beta^2}} \delta(\mathbf{KV} - \omega) \tilde{\xi}. \quad (45)$$

Thus the contribution to the total energy of the field arising from the first term of $\tilde{T}_{a1}(k)$ becomes

$$\begin{aligned}
E^{(1)} &= - \int T_{41}(x) dV = - \frac{1}{(2\pi)} \int \tilde{T}_{41}(k) e^{i\omega t} \delta(\mathbf{K}) d\mathbf{K} d\omega \\
&= - \frac{e}{3} \frac{\beta^2}{\sqrt{1 - \beta^2}} \tilde{\xi}. \quad (46)
\end{aligned}$$

For the case $u=1, 2, 3$ we can readily see that only the component parallel to \mathbf{V} does

remain. Choosing specially the direction of \mathbf{V} as the third axis $a=3$, one gets for $a=3$

$$\begin{aligned}
 (40) &= \frac{c^2}{i\pi} (1-\beta^2) \delta(\mathbf{KV}-\omega) \int \frac{\tilde{\varepsilon}(k') + k'^2_{\alpha} f'(k'^2_{\alpha})}{(k'^2_{\alpha})^2 (\tilde{\varepsilon}(k'))^2} \delta(\mathbf{K}'\mathbf{V}-\omega') k' \cos \theta \frac{\omega'}{c} d\mathbf{K}' d\omega' \\
 &= -2ic^2\beta(1-\beta^2) \delta(\mathbf{KV}-\omega) \int_0^{\infty} \frac{\tilde{\varepsilon}(K) + K^2 f'(K^2)}{(\tilde{\varepsilon}(K))^2} \int_{-1}^{+1} \frac{z^2 dz}{(\sqrt{1-\beta^2 z^2})^5} \\
 &= -\frac{2i\pi e}{3} \frac{\beta}{\sqrt{1-\beta^2}} \delta(\mathbf{KV}-\omega) \hat{\xi},
 \end{aligned} \tag{47}$$

where we have introduced the same variable change and used quite the same manipulations and abbreviation as before. Thus the contribution to the total momentum of the field arising from the first term of $\tilde{T}_{a4}(k)$ becomes

$$\begin{aligned}
 \mathbf{P}^{(1)} &= \frac{1}{ic} \int \dot{T}_{a4}(x) dV = \frac{1}{ic(2\pi)} \int \dot{T}_{a4}(k) e^{i\omega t} \delta(\mathbf{K}) d\mathbf{K} d\omega \\
 &= -\frac{e}{3} \frac{\mathbf{V}}{c^2 \sqrt{1-\beta^2}} \hat{\xi}.
 \end{aligned} \tag{48}$$

Therefore we get as the total energy and momentum, adding (33), (36), (46) and (34), (37), (48),

$$\begin{aligned}
 E &= -\frac{e\beta^2}{3\sqrt{1-\beta^2}} \hat{\xi} - \frac{1}{2} e \sqrt{1-\beta^2} \varphi(0) + \frac{e}{\sqrt{1-\beta^2}} \varphi(0) \\
 &= -\frac{e}{3\sqrt{1-\beta^2}} \hat{\xi} + \frac{e}{\sqrt{1-\beta^2}} \varphi(0) + \frac{e(1-\beta^2)}{3\sqrt{1-\beta^2}} \hat{\xi} - \frac{e(1-\beta^2)}{2\sqrt{1-\beta^2}} \varphi(0)
 \end{aligned} \tag{49}$$

and

$$\mathbf{P} = -\frac{e\mathbf{V}}{3c^2\sqrt{1-\beta^2}} \hat{\xi} + \frac{e\mathbf{V}}{c^2\sqrt{1-\beta^2}} \varphi(0). \tag{50}$$

We can see readily from (49) and (50) that if the following relation holds between $\varphi(0)$ and $\hat{\xi}$, the energy E and momentum \mathbf{P} show correct behavior upon velocity:

$$2\hat{\xi} = 3\varphi(0)$$

or

$$2 \int_0^{\infty} \frac{\tilde{\varepsilon}(K) + K^2 f'(K^2)}{(\tilde{\varepsilon}(K))^2} dK = 3 \int_0^{\infty} \frac{dK}{\tilde{\varepsilon}(K)}, \tag{51}$$

in which case the energy and momentum are given

$$E = \frac{1}{2} e \varphi(0) / \sqrt{1-\beta^2} \tag{52}$$

and

$$\mathbf{P} = \frac{1}{2} e \mathbf{V} \varphi(0) / c^2 \sqrt{1-\beta^2}. \tag{53}$$

If we identify the total energy of the field for a point particle at rest with its rest energy,

we must put

$$E_0 = \frac{1}{2} e \varphi(0) = mc^2. \quad (54)$$

Substituting the rest mass m into (52) and (53), we obtain the correct expression for energy and momentum of a moving particle required by relativistic mechanics :

$$E = mc^2 / \sqrt{1 - \beta^2}$$

and

$$\mathbf{P} = m\mathbf{V} / \sqrt{1 - \beta^2}. \quad (55)$$

The condition (51) for the fulfilment of the requirement of the unitary field theory is only a very weak condition upon the functional form of $\tilde{\varepsilon}(k)$ or $\varepsilon(x)$, so that an unique conclusion cannot be drawn from (51).

In the usual Maxwell theory, where $\varepsilon(x) = \delta(x)$ or $\tilde{\varepsilon}(k) = 1$, the condition (51) cannot be satisfied because $\xi = \varphi(0) = \infty$. In order to satisfy the condition (51), we may, for example, equal the integrands of both sides of (51):

$$\frac{\tilde{\varepsilon}(K) + K'^2 f'(K'^2)}{(\tilde{\varepsilon}(K))^2} = \frac{3}{2} \frac{1}{\varepsilon(K)}, \quad (56)$$

or

$$2K'^2 f'(K'^2) = \tilde{\varepsilon}(K'). \quad (57)$$

Remembering $\tilde{\varepsilon}(K) \equiv f(K'^2)$ and $f'(K'^2) = df(K'^2)/dK'^2$ and writing $K'^2 \equiv x$ for simplicity, we obtain the following differential equation for $f(x)$:

$$2x f'(x) = f(x),$$

which can be satisfied by

$$f(x) = \text{Const. } x^{\frac{1}{2}}$$

or

$$f(K'^2) = \text{Const. } \sqrt{K'^2} \quad (58)$$

or

$$\tilde{\varepsilon}(K) = \text{Const. } K. \quad (59)$$

This solution gives an infinite value to $\varphi(0)$:

$$\varphi(0) = \frac{2e}{\pi} \int_0^\infty \frac{dK}{\tilde{\varepsilon}(K)} = \frac{2e}{\pi} \int_0^\infty \frac{dK'}{K} \rightarrow \infty \quad (60)$$

according to (31), thus tending rest mass m to infinity. This assignment for $\tilde{\varepsilon}(k)$, which cannot give a finite energy of self field or a finite rest mass and a correct velocity variation of energy and momentum of the field at the same time, is easily be shown to have many other unsatisfactory points. We have now examined, however, only a special example which satisfies the condition (51) and there is not yet excluded the possibility of the presence of some more general assignment for $\tilde{\varepsilon}(k)$ which gives a finite rest mass and a correct mass

variation with velocity. In order to examine further this point we now consider another familiar assignment for $\tilde{\varepsilon}(k)$ or $\varepsilon(x)$ which reduces the general Bopp's Lagrangian (10) to old one (2). This has the form

$$\varepsilon(x) = \left(1 - \frac{1}{x^2} \square\right) \delta(x) \quad (61)$$

or in momentum space

$$\tilde{\varepsilon}(k) = \int \varepsilon(x) e^{ik_\tau x_\tau} dx = 1 + (k_a^2/x^2). \quad (62)$$

For this form of $\tilde{\varepsilon}(k)$, the condition (51) becomes

$$2 \int_0^\infty \frac{1 + (K^2/x^2) + (K'^2/x^2)}{(1 + (K^2/x^2))^2} dK = 3 \int_0^\infty \frac{dK}{1 + (K^2/x^2)}, \quad (63)$$

which can be verified by elementary calculations. The static potential, then, becomes according to (30)

$$\begin{aligned} \varphi_0(x) &= \frac{2c}{\pi} \int_0^\infty \frac{\sin Kr}{Kr} \frac{dK}{\tilde{\varepsilon}(K)} \\ &= \frac{2c}{\pi} \int_0^\infty \frac{\sin Kr}{Kr} \frac{dK}{1 + (K^2/x^2)} \\ &= ex^2(1 - e^{-r/x})/r, \end{aligned} \quad (64)$$

which is finite at the origin and approaches Coulomb potential as r becomes large. Therefore the assignment (62) is a very satisfactory one from the standpoints interested in this paper. If we abandon the old assignment (62) from other reasons, there occurs naturally the problem if there is any other selection which is free from those defects. The answer to this problem has not yet been given, but in view of the flexibility of the condition (51) we are sure that there are some other appropriate forms of $\tilde{\varepsilon}(k)$ or $\varepsilon(x)$ which satisfy (51) and give a finite energy of self field.

§ 5. Conclusions

Two important points which must be satisfied in unitary field theory, the finiteness of the total energy of self field produced by a point particle and the correct velocity variation of total energy and total momentum of self field produced by a free point particle moving with constant velocity, was examined in detail with respect to Bopp-type non-local unitary field theory. The calculations are entirely based on the energy-momentum tensor given by Bopp. As the Lorentz force acting on the particle by the self field does not vanish in general, the correct behavior upon velocity of total energy and momentum of the surrounding field is not always guaranteed from the beginning. As a result of the calculations performed in this paper, the correct variation is assured if the characteristic "fernwirkungsfunktion" $\varepsilon(x)$ does satisfy a certain relation (51). According to this condition, it is easily shown that usual Maxwell theory does not satisfy the above mentioned two

requirements and old Bopp's modification of electromagnetic field which introduces derivatives into the Lagrangian does satisfy both requirements. Because that condition is a very general one including integrals, it is not easy to find its general solutions. However, in view of a great flexibility of the condition, there is some hope of the presence of some other assignments of $\varepsilon(x)$ which satisfy the above mentioned two requirements. As remarked at the end of section 3, the above calculations and discussions starts from the expression (11) for the energy-momentum tensor derived by Bopp and Heisenberg and there are many unsatisfactory points concerning the form (11) for the energy-momentum tensor. Therefore it becomes also a very important problem to construct a energy-momentum tensor free from those difficult points for the Bopp-type non-local field theory. At any rate it is concluded that it might be possible to choose an appropriate functional form for the characteristic function $\varepsilon(x)$ which guarantees the correct variation upon velocity of the total energy and total momentum of the field produced by a point particle moving with constant velocity and the finiteness of the energy of the self field of a point particle.

The author is indebted to Mr. Y. Ono for his many instructive discussions.

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Beta-Spectrum of the Third Forbidden Transition of Rb^{87}

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The Rb^{87} beta-spectrum of the third forbidden transition is well explained by the combination of two matrix elements $Q_3(\beta[\sigma \times \mathbf{r}], \mathbf{r})$ and $Q_3(\beta\alpha, \mathbf{r})$. Complex gamma-functions are calculated according to Martin. Screening correction is taken into account.

§ I. Introduction

Recently it was pointed out that radiative¹⁾ and mesonic²⁾ corrections did not affect Konopinski's forbidden theory of beta-decay that is, the shape of spectrum as well as the selection rules expected from Konopinski's forbidden theory. Since this conclusion about mesonic correction is obtained through the consideration of rotational property only, this can be applied to the case of any interaction between nucleons. These facts make us believe firmly the success of Konopinski's forbidden theory in highly forbidden cases. Recently several investigators have been succeeded to explain some beta-spectra in the first and second forbidden transitions by the combinations of two matrix elements $Q_n(\beta[\sigma \times \mathbf{r}], \mathbf{r})$ and $Q_n(\beta\alpha, \mathbf{r})$.³⁾ Then it is very interesting to confirm the success in explaining the beta-spectra in the third forbidden case by the combinations of these two matrix elements.

Recently Rb^{87} beta-spectrum was measured by Curran, Dixon and Wilson⁴⁾ and is found to be of quite different shape from allowed ones. Since the known spin and magnetic moment suggest that the configurations of Rb^{87} and Sr^{87} seems to be $(p_{3/2})^{-1}$ and $(g_{9/2})^{-1}$ respectively, Rb^{87} beta-decay is expected to be the third forbidden one with three spin change. In the following we will try to explain Rb^{87} beta-spectrum by the combination of two matrix elements $Q_3(\beta[\sigma \times \mathbf{r}], \mathbf{r})$ and $Q_3(\beta\alpha, \mathbf{r})$.

§ 2. Procedure, results and discussion

The NT spectrum involves contributions from two matrix elements $Q_n(\beta[\sigma \times \mathbf{r}], \mathbf{r})$ and $Q_n(\beta\alpha, \mathbf{r})$ and also from the matrix element $Q_{n+1}(\beta\sigma, \mathbf{r})$, but a rough estimate shows that the contribution from matrix element $Q_{n+1}(\beta\sigma, \mathbf{r})$ is comparatively small. Therefore we take into account the contributions from the matrix elements $Q_n(\beta[\sigma \times \mathbf{r}], \mathbf{r})$ and $Q_n(\beta\alpha, \mathbf{r})$ only. They are, in the terminology of Greuling⁵⁾,

$$C_{2T} = \left| \sum_{i,j,k} Q_{i,j,k}(\beta[\sigma \times \mathbf{r}], \mathbf{r})^2 / n! \right|^2 f(w) - \left[\sum_{i,j,k} Q_{i,j,k}(\beta[\sigma \times \mathbf{r}], \mathbf{r}) / n! \right. \\ \left. \times Q_{i,j,k}(\beta\alpha, \mathbf{r}) / n! + \text{c.c.} \right] g(w) + \left| \sum_{i,j,k} Q_{i,j,k}(\beta\alpha, \mathbf{r}) / n! \right|^2 h(w)$$

$$\begin{aligned}
= & \left| \sum_{i,j,k} Q_{i,j,k}(\beta[\sigma \times \mathbf{r}], \mathbf{r})/n! \right|^2 \sum_{\nu=0}^n (A_{n\nu} g^{2(n-\nu)-2} M_\nu - 2C_{n\nu} g^{2(n-\nu)-1} N_\nu \\
& + [D_{n\nu} - B_{n\nu}/(n+1)] g^{2(n-\nu)} L_\nu) + \left| \sum_{i,j,k} Q_{i,j,k}(\beta\alpha, \mathbf{r})/n! \right|^2 \sum_{\nu=0}^n (A_{n\nu} g^{2(n-\nu)-2} L_\nu) \\
& - \left[\sum_{i,j,k} Q_{i,j,k}(\beta[\sigma \times \mathbf{r}], \mathbf{r})/n! \cdot Q_{i,j,k}^*(\beta\alpha, \mathbf{r})/n! + \text{c.c.} \right] \sum_{\nu=0}^n (-A_{n\nu} g^{2(n-\nu)-2} N_\nu \\
& + C_{n\nu} g^{2(n-\nu)-2} L_\nu)
\end{aligned}$$

where $n=3$ and

$$Q_{ijk}(\mathbf{a}, \mathbf{r}) = \int d\tau V^* \left[a \begin{smallmatrix} x & x & x \\ (i & j & k) \end{smallmatrix} - 1/5 \frac{\partial}{\partial \tau} a \begin{smallmatrix} r^2 \\ (ij & k) \end{smallmatrix} - 2/5 \frac{\partial}{\partial \tau} x \begin{smallmatrix} (\mathbf{a} \cdot \mathbf{r}) \\ (ij & k) \end{smallmatrix} \right] U,$$

$$A_{n\nu} = \frac{(n-\nu) 2^{n-2\nu} (2\nu+1)!}{(2n-2\nu)! (\nu!)^2}, \quad B_{n\nu} = \frac{2^{n-2\nu} (2\nu+1)!}{(2n-2\nu+1)! (\nu!)^2},$$

$$C_{n\nu} = \frac{(n-\nu) 2^{n-2\nu} (2\nu+1)!}{(2n-2\nu+1)! (\nu!)^2}, \quad D_{n\nu} = \frac{2^{n-2\nu} (\nu+1) (2\nu)!}{(2n-2\nu+1)! (\nu!)^2},$$

$$L_\nu = (F_\nu/F_0) \left(\frac{2^\nu \nu!}{(2\nu+1)!} \hat{p}^\nu \right)^2 \frac{\nu+1+S_\nu}{2\nu+2},$$

$$\begin{aligned}
M_\nu = & (F_\nu/F_0) \left(\frac{2^{\nu+1} \nu!}{(2\nu+2)!} \hat{p}_\nu \right)^2 \left[\frac{2\nu+2}{\nu+1+S_\nu} \left(\frac{\alpha Z}{2\rho} \right) + \left(\frac{S_\nu}{2S_\nu+1} \hat{p}^2/W \right. \right. \\
& \left. \left. - \frac{(2\nu+1)(\alpha Z)^2 W}{(2S_\nu+1)(\nu+1+S_\nu)} \right) \left(\frac{\alpha Z}{\rho} \right) + \frac{(\nu+1)(S_\nu-\nu)S_\nu}{(2S_\nu+1)^2} \left(1 - \frac{4S_\nu+3}{S_\nu(S_\nu+1)} (\alpha Z)^2 \right) \hat{p}^2, \right. \\
& \left. + \left(1 + \frac{\nu(4S_\nu+3)}{(S_\nu+1)(\nu+1+S_\nu)} (\alpha Z)^2 \right) \left(\frac{\alpha Z}{2S_\nu+1} \right)^2 \right],
\end{aligned}$$

$$N_\nu = - (F_\nu/F_0) \left(\frac{2^\nu \nu!}{(2\nu+1)!} \hat{p}^\nu \right)^2 \frac{1}{\nu+1} \left[\left(\frac{\alpha Z}{2\rho} \right) + \frac{S_\nu}{2S_\nu+1} \hat{p}^2/W - \frac{2(\alpha Z)^2}{2S_\nu+1} W \right],$$

$$F_\nu(W, Z) = \left(\frac{(2\nu+2)!}{\nu!} \right)^2 (2\rho)^{(2S_\nu-\nu-1)} \exp(\pi y) |\Gamma(S_\nu + iy)|^2 / \Gamma^2(1+2S_\nu),$$

$$S_\nu = [(\nu+1)^2 - (\alpha Z)^2]^{\frac{1}{2}}, \quad y = \alpha Z W / \rho.$$

It was shown by Longmire and Messiah⁶⁾ that the ratio

$$\rho^{MM'} \equiv Q_{jk}(\beta[\sigma \times \mathbf{r}], \mathbf{r})^{MM'} / Q_{jk}(\beta\alpha, \mathbf{r})^{MM'}$$

is independent of M, M', i, j and k , and is real. And consequently

$$C_{3T} = \sum_{i,j,k} |Q_{ijk}(\beta[\sigma \times \mathbf{r}], \mathbf{r})/n!|^2 \{ f(\mathbf{w}) - 2\rho g(\mathbf{w}) + \rho^2 h(\mathbf{w}) \}.$$

To obtain F_0, F_1, F_2 , and F_3 we must calculate the complex Γ -function, and we calculate the latter according to Martin⁷⁾ (appendix). The results are shown in Fig. 1.

With the following ratio of the matrix elements

$$\rho = 4.2,$$

we obtain a good fit for C_{3T} to the experimental spectrum from about 100 kev to the upper limit of 275 kev (Fig. 2). In the region of energy below 100 kev, we took into account the screening correction according to Longmire and Brown⁸⁾. The values of

Fig. 1. a.

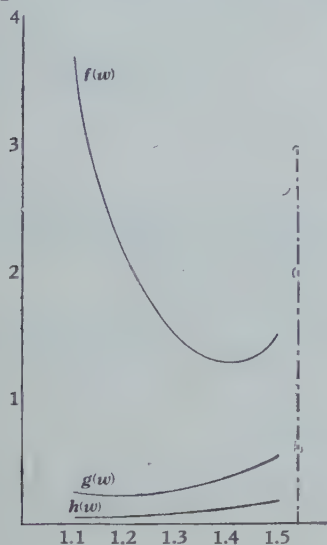


Fig. 1. b.

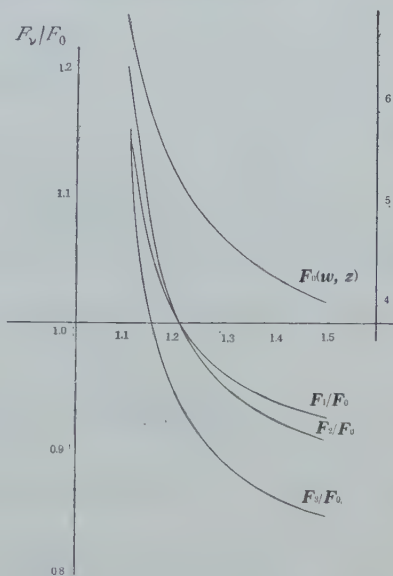


Fig. 2.

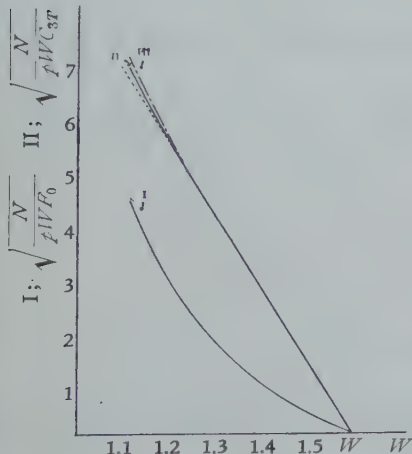


Fig. 2. Curve I is Fermi prot of Rb^{87} beta-spectrum.

Curve II is the one corrected by the third forbidden correction function C_{3T} for the tensor interaction, in which the screening correction is taken into account.

Curve III is the corrected one in which the screening correction is not taken into account.

Table I. The screening correction.

W mc ²	1.1	1.2
$F_0(W-D_0)/F_0(W)$	1.04	1.01
$F_1(W-D_0)/F_1(W)$	1.05	1.02
$F_2(W-D_0)/F_2(W)$	1.06	1.02
$F_3(W-D_0)/F_3(W)$	1.06	1.02

$F(W-D_0)/F(W)$ are shown in Table I. The result is shown in Fig. 2. The slight shift in 50 kev energy seems to be within the uncertainty caused by the experimental difficulty. However, it is to be noted that the shift may be due to the I. B. electron.⁹⁾ The calculation of the I. B. electron correction is now in progress.

The C_{3V} spectrum is different from the C_{3T} one by only one small term B_n in the terminology of Greuling. As the contribution from B_n is so small, we obtain a good fit for C_{3T} to the experimental spectrum too. On the other hand, all other matrix elements in first, second

and third forbidden cases fail to explain the Rb^{87} spectrum.

We calculate the ft -value and obtain $ft = 4.2 \times 10^{16}$ where $t = 6 \times 10^{10} \text{ y}$ and f -value is obtained by the graphical integration,

$$ft = (2\pi^3/G^2) \log^2 / \left[\sum_{i,j,k} Q_{ijk} (\beta[\sigma \times \mathbf{r}], \mathbf{r}) / n! \right]^2.$$

Appendix

Since

$$\ln \Gamma(x+iy) = \ln R e^{i\phi} = \ln R + i\phi$$

the modulus of the complex gamma-function can be obtained by the real part of $\ln \Gamma(x+iy)$. We obtain the following series by making use of the Taylor expansion,

$$\begin{aligned} \ln \Gamma(x+iy) &= \ln \Gamma(x) + iy \left\{ \frac{\partial}{\partial(iy)} \ln \Gamma(x+iy) \right\}_0 - \frac{y^2}{2!} \left\{ \frac{\partial^2}{\partial(iy)^2} \ln \Gamma(x+iy) \right\}_0 + \dots \\ &= \ln \Gamma(x) + iy\psi(x) - \frac{y^2}{2} \psi'(x) - \frac{iy^3}{3!} \psi''(x) \dots, \end{aligned}$$

where $\psi(x), \psi'(x), \dots$ are polygamma-functions which are real and have been tabulated. Then

$$\ln R = \ln \Gamma(x) - \frac{y^2}{2!} \psi'(x) + \frac{y^4}{4!} \psi'''(x) \dots$$

This series may be made to converge more rapidly by increasing the argument x .

Thus

$$\ln R = \ln \Gamma(x+1) - \frac{y^2}{2!} \psi'(x+1) + \frac{y^4}{4!} \psi'''(x+1) \dots - \frac{1}{2} \ln(x^2+y^2),$$

since by the property of the gamma-function

$$\ln \Gamma(z) = \ln(z+1) - \ln z$$

and

$$\Re[\ln(x+iy)] = \frac{1}{2} \ln(x^2+y^2).$$

This process may be repeated until the argument become so large that series converge sufficiently to give the accuracy desired. Thus

$$\begin{aligned} \ln R &= \ln \Gamma(x+n) - \frac{y^2}{2!} \psi'(x+n) + \frac{y^4}{4!} \psi'''(x+n) - \dots \\ &\quad - \frac{1}{2} \ln(x^2+y^2) [(x+1)^2+y^2] \dots [(x+n-1)^2+y^2]. \end{aligned}$$

Thus, for $Z=38$ and $x=s_0=0.9608$,

$$\log_{10} |F(S_0 + iy)| = 1.35463 - 0.04848y^2 + 0.00040y^4$$

$$- \frac{1}{2} \log (0.92314 + y^2) (3.84474 + y^2) (8.76634 + y^2) (15.6879 + y^2),$$

where $y^2 = (aZW/p)^2 = (aZ)^2 (+\epsilon)^2 / \{ (1 + \epsilon)^2 - 1 \} \sim 0.077/2\epsilon$ ($\epsilon < 1$)

and for 50 kev, $y^2 = 0.44$,

for 20 kev, $y^2 \sim 1$,

for 10 kev $y^2 \sim 2$.

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Note on the Forbidden Transitions in Beta-Decay

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Recently several investigators have succeeded in explaining beta-spectra in the highly forbidden transitions by the matrix element $Q_{n+1}(\beta\sigma, \mathbf{r})^{(1)}$ or by the combinations of two matrix elements $Q_n(\beta[\sigma \times \mathbf{r}], \mathbf{r})$ and $Q_n(\beta\alpha, \mathbf{r})$ or $Q_n(\mathbf{r}, \mathbf{r})$ and $Q_n(\alpha, \mathbf{r})$.^{(2),(3)} These results are shown in Table I.

First Konopinski and Uhlenbeck^{(4),(5)} roughly estimated these matrix elements as follows

$$\begin{aligned} |Q_2(\mathbf{r}, \mathbf{r})|^2 &\approx (4/3)R^4, & |Q_2(\beta[\sigma \times \mathbf{r}], \mathbf{r})^2 &\approx (4/3)R^4, \\ |Q_2(\alpha, \mathbf{r})|^2 &\approx |Q_2(\beta^2\alpha, \mathbf{r})|^2 \approx (20/9)(v/c)^2R^2, & |Q_3(\beta\sigma, \mathbf{r})|^2 &\approx (56/5)R^4, \\ |Q_4(\beta\sigma, \mathbf{r})|^2 &\approx (3456/35)R^6. \end{aligned}$$

However, in Table I, $Q_{n+1}(\beta\sigma, \mathbf{r})$ is not so large compared with $Q_n(\beta[\sigma \times \mathbf{r}], \mathbf{r})$ and $Q_n(\beta\alpha, \mathbf{r})$, so this estimate seems to be wrong. On the other hand, it was attempted to make a sharper estimate of these matrix elements by making use of the symmetrical nuclear hamiltonian. It was pointed out by Konopinski and Uhlenbeck that if one neglects the difference in Coulomb energy in the initial and final nuclei, one can derive for example relations like

$$Q_2(\alpha, \mathbf{r}) \approx Q_2(\beta\alpha, \mathbf{r}) = i(W_0/2)Q_2(\mathbf{r}, \mathbf{r}).$$

But it is soon understood that in Table I such relation did not hold good in experimental results.

Therefore, it is very difficult to estimate these matrix elements accurately; it seems, however, to us that the fact that $Q_n(\beta\alpha, \mathbf{r})$ is several times greater than $Q_n(\beta[\sigma \times \mathbf{r}], \mathbf{r})$ and $Q_{n+1}(\beta\sigma, \mathbf{r})$ may show that v/c is several times greater than \mathbf{r} .

Then, in the case of the first forbidden transitions similar relations is expected, that is, the matrix element $\int \alpha$ is several times greater than the other ones $\int \sigma \cdot \mathbf{r}$, $\int \sigma \times \mathbf{r}$ and B_{ij} .⁽⁶⁾ Recently, Takebe, Nakamura and one of the writers have been succeeded to explain Tm^{170} beta-spectrum of the first forbidden transition by the combination of two matrix elements $\int \alpha$ and $\int \sigma \times \mathbf{r}$ or $\int \alpha$ and $\int \sigma \cdot \mathbf{r}$. The result are shown in Table II and the ratio of these two matrix elements suggests this fact mentioned above. Therefore it is soon expected that f -value of beta-decay of spin change 1 and parity change yes must be several powers of ten smaller than that of beta-decay of spin change 2 and parity change yes and lie on the region about 10^6 .

It was first pointed out by Davidson, Nakamura and one of the writers⁷⁾ that ft -values of beta-decay due to matrix element B_{ij} fall into distinct range of 10^4 to 10^9 . On the other hand ft -values of allowed transitions fall into the range of 10^3 to 10^6 , excepting a few special cases. And it has been noticed by several investigators⁷⁾ that beside above groups there are several elements where parity change are expected from Mayer's shell model and ft -values are 10^5 to 10^7 and these have not been explained satisfactorily. We call another's attention to the fact that these are well explained by above mentioned fact, that is to say, beta-decay of the elements of Table III are possibly the first forbidden transitions of $\Delta J=1$.

In the case of the transitions due to matrix elements $\{\sigma \times r$ or $\{r$ the shapes of energy spectra are allowed ones when Z is large, but is considerably different from allowed ones when Z is small; when main contribution to spectrum is, however, that from $\{a$, above mentioned beta-decay are probably much the same shape as allowed one.

Table I. The highly forbidden transitions.

Element	W_0 mc ²	t	$ft = (2\pi^3/G^2) \log 2 / \sum Q_{n+1}(\beta \sigma, r)/n! ^2$	n
Be ¹⁰	2.1	2.7×10^6 y	2.1×10^{11}	2
K ⁴⁰	3.6	2.7×10^9 y	2.1×10^{15}	3

f are calculated by the formula given by R. E. Marshak, Phys. Rev. **61** (1942), 435.

Element	W_0 mc ²	t	$Q_n(\beta a, r)/Q_n(\beta [\sigma \times r], r)$	$ft = (2\pi^3/G^2) \log 2 / \sum Q_n(\beta [\sigma \times r], r)/n! ^2$	n
Cl ³⁶	2.4	4.4×10^5 y	$\sqrt{25.73}$ or $\sqrt{18}$	0.7 to 2.3×10^{12}	2
Rb ⁸⁷	1.54	6×10^{10} y	4.2	$4.2 \times 10^{16*}$	3
Tc ⁹⁹	1.57	2.1×10^5 y	7.7	2.7 to 8.0×10^{11}	2
Sb ¹²⁴	5.49	60 d	13	0.8 to 1.2×10^{11}	2

F are calculated by multipling allowed one by the maximam or minimum value of C_{nT} .

*F of Rb is exactly calculated by graphical integration.

$Q_n(a, r)/Q_n(r, r)$ is much the same as $Q_n(\beta a, r)/Q_n(\beta [\sigma \times r], r)$.

Table III. Elements which decay across the closed shell 40 and into the ground state of the final nuclei directly and have ft -values of order of about 10^5 to 10^7 .

Element	Ga ⁷³	Ge ⁷⁵	Se ⁸¹	Br ⁷⁵	Br ⁸⁰	Br ⁸³	Br ⁸⁵	Zr ⁸⁹
Log ft^*	5.96	5.00	4.88	5.69	5.49	5.13	5.06	6.26

Similar elements will be easily seen about closed shell 82 and 126.

* A. M. Feingold, Rev. Mod. Phys. **23** (1951), 10.

Table II. The first forbidden transition.

Element	W_0 mc ²	t	$Q_n(\beta a, r)/Q_n(\beta [\sigma \times r], r)$	$ft = 2\pi^3/G^2 \log 2 / \sum Q_n(\beta [\sigma \times r], r)/n! ^2$	n
Tm ¹⁷⁰	2.94	1.09×10^7 s	10.3	1.7×10^9	1

f is calculated by the graphycal integration.

$Q_n(\beta a, r)/Q_n(r, r)$ and $ft = 2\pi^3/G^2 \log 2 / |\sum Q_n(r, r)/n!|^2$ is much the same as $Q_n(\beta a, r)/Q_n(\beta [\sigma \times r], r)$ and $ft = 2\pi^3/G^2 \log 2 / |\sum Q_n(\beta [\sigma \times r], r)/n!|^2$.

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Letters to the Editor

“Weisskopf-Wigner Method” in S-Matrix Formalism

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As is well known, Weisskopf-Wigner's method¹⁾ has been successfully applied to analysis of such problems as the natural line breadth of spectral lines, resonance scattering phenomena and so on. In these problems, the time-decrease of initial-state amplitude plays an essential role, and have to be properly taken into account. According to Weisskopf and Wigner, the Schrödinger equations describing transitions of a system from its state A to states B 's

$$i\hbar\dot{b}_A = \sum_B \langle A|\bar{H}|B\rangle b_B, \\ i\hbar\dot{b}_B = \langle B|\bar{H}|A\rangle b_A \quad (1)$$

are solved with the initial condition; $t \rightarrow 0$, $b_A \rightarrow e^{-\gamma t/2}$, $b_B \rightarrow 0$, which properly describes the exponential decrease of initial state A in its start of time-developments, in contrast to ordinarily chosen one: $t \sim 0$, $b_A = 1$, $b_B = 0$. As the solution involves the damping factor γ in its energy denominator in the form of half value breadth, the state amplitude b_B remains finite even in its resonance-points.

In the Dyson-Feynman's S-matrix formalism of quantum field theory, the effect of decrease of state amplitudes can not manifestly be expressed on account of its power series expansion with respect to the coupling constant. And this gives rise to the terms which diverge at the resonance-points, leading to the appearance of the displaced poles, infra-red catastrophe and so on.

In this note, we demonstrate that the effects of the time-decrease in state amplitudes are also taken into account explicitly by rearranging the terms of power series expansion of S-matrix, especially, Weisskopf-Wigner's results are obtained by approximate calculation of the higher order terms of S-matrix.

Let us begin with the transformation function

$$U(t, 0) = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar} \right)^n \int_0^t dt_1 \int_0^{t_1} dt_2' \cdots \int_0^{t_{n-1}'} dt_n \bar{H}(t_1) \bar{H}(t_2') \cdots \bar{H}(t_n). \quad (2)$$

The diagonal element of which is

$$\begin{aligned} \langle A|U(t, 0)|A\rangle &= -\frac{1}{\hbar^2} \int_0^t dt_1 \int_0^{t_1} dt_2' \\ &\times \langle A|\bar{H}(t_1)\bar{H}(t_2')|A\rangle \\ &+ \frac{1}{\hbar^4} \int_0^t dt_1 \int_0^{t_1} dt_2' \int_0^{t_2'} dt_3' \int_0^{t_3'} dt_4' \\ &\times \langle A|\bar{H}(t_1)\bar{H}(t_2')\bar{H}(t_3')\bar{H}(t_4')|A\rangle + \cdots \end{aligned} \quad (3)$$

The second order term can be written as

$$-\frac{t}{\hbar^2} \langle A|\bar{H}(t) \int_0^t \bar{H}(t') dt' |A\rangle. \quad (4)$$

This involves the fact that the diagonal element is independent of time. The amplitude increases linearly with time and tends to ∞ as $t \rightarrow \infty$, in the second approximation. This is a characteristic feature of dampingless resonance (“resonance catastrophe”), and is intimately connected with the divergence due to vanishing energy denominators in its momentum space representation.

From the third term of (3), decomposing the $\langle A|\bar{H}(t_1)\bar{H}(t_2')\bar{H}(t_3')\bar{H}(t_4')|A\rangle$ into $\langle A|\bar{H}(t_1)\bar{H}(t_2')\bar{H}|A\rangle \langle A|\bar{H}(t_3')\bar{H}(t_4')|A\rangle$

$\bar{H}(t_4)|A\rangle + \sum \langle A|\bar{H}(t_1)\bar{H}(t_2)|B\rangle\langle B|\bar{H}(t_3)\bar{H}(t_4)|A\rangle$ and picking up the first term, we can extract the term

$$\frac{1}{\hbar^4} \int_0^t dt_1 \langle A|\bar{H}(t_1) \int_0^{t_1} dt_2 \bar{H}(t_2) |A\rangle \times \int_0^{t_2} dt_3 \langle A|\bar{H}(t_3) \int_0^{t_3} dt_4 \bar{H}(t_4) |A\rangle, \quad (5)$$

and transform it into

$$\frac{1}{\hbar^4} \langle A|\bar{H}(t) \int_0^t \bar{H}(t') dt' |A\rangle^2 \int_0^t dt_1 \int_0^{t_1} dt_3' = \frac{1}{2!} \left(\frac{-i}{\hbar^2} \langle A|\bar{H}(t) \int_0^t \bar{H}(t') dt' |A\rangle \right), \quad (6)$$

where we extended the upper limit t_2 of t_3 -integration to t_1 without any justification. This approximation constitutes an essential manipulation for the derivation of Weisskopf-Wigner's results. The physical implication of this approximation will be discussed in the next paper in connection with the S-matrix form of Heitler's damping theory.

In the same way, the $2n$ -th term is also transformed into

$$\left(\frac{-1}{\hbar^2} \langle A|\bar{H}(t) \int_0^t \bar{H}(t') dt' |A\rangle \right)^n \times \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n = \frac{1}{n!} \left(\frac{-i}{\hbar^2} \langle A|\bar{H}(t) \int_0^t \bar{H}(t') dt' |A\rangle \right)^n. \quad (7)$$

Introducing these results into (2), we can extract the following approximate result

$$\langle A|U(t, 0)|A\rangle = e^{-\frac{i}{\hbar^2} \langle A|\bar{H}(t) \int_0^t \bar{H}(t') dt' |A\rangle} = e^{-\frac{\gamma t}{2}} \quad (8)$$

This corresponds to the Weisskopf-Wigner's description of time-decrease of initial state amplitude.

Next, we consider the off-diagonal elements of the matrix U .

$$\langle B|U(t, 0)|A\rangle = \left(\frac{-i}{\hbar} \right) \int_0^t dt' \langle B|\bar{H}(t')|$$

$$A\rangle + \left(\frac{-i}{\hbar} \right)^3 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \times \langle B|\bar{H}(t_1)\bar{H}(t_2)\bar{H}(t_3)|A\rangle + \dots \quad (9)$$

From the second term we can pick up the term of form $\langle B|\bar{H}(t_1)|A\rangle\langle A|\bar{H}(t_2)\bar{H}(t_3)|A\rangle$ and from the third term, a term of form $\langle B|\bar{H}(t_1)|A\rangle\langle A|\bar{H}(t_2)\bar{H}(t_3)|A\rangle\langle A|\bar{H}(t_4)\bar{H}(t_5)|A\rangle$, etc. Estimating those integrals by same approximate calculations, we can obtain

$$\langle B|U(t, 0)|A\rangle = \left(\frac{-i}{\hbar} \right) \int_0^t dt' \langle B|\bar{H}(t')|A\rangle \left(1 - \frac{1}{\hbar^2} \langle A|\bar{H}(t) \int_0^t dt'' \bar{H}(t'') |A\rangle \frac{t'}{1!} + \frac{1}{\hbar^4} \langle A|\bar{H}(t) \int_0^t dt'' \bar{H}(t'') |A\rangle^2 \frac{t'^2}{2!} + \dots \right) = \left(\frac{-i}{\hbar} \right) \langle B| \int_0^t dt' \bar{H}(t') e^{-\frac{\gamma t'}{2}} |A\rangle. \quad (10)$$

It is readily seen that this result corresponds to Weisskopf-Wigner's one, if we observe its Fourier transform.

From above consideration, we may conclude:

- 1) Weisskopf-Wigner's results are contained implicitly in S-matrix formalism, mixing up each other in its higher order approximations.
- 2) By extracting them and summing up, we can get the same results as Weisskopf and Wigner's one entirely within the framework of Feynman-Dyson's S-matrix formalism.

Detailed accounts will be published in near future.

Author expresses his cordial thanks to Prof. S. Tomonaga and Dr. T. Miyazima, for their suggestion on this problem and continual guidance.

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On the Green-Functions of the Quantum Electrodynamics

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Recently Schwinger¹⁾ has proposed a new theory of Green's function by using his own formalism of the quantized fields. The physical meanings of some quantities introduced in his theory, however, seem somewhat ambiguous, though his new theory is quite attractive. Accordingly it is worthwhile to re-formulate his formalism by making use of the well-known theory of Tomonaga and Schwinger and to clarify the ambiguous points mentioned above. The results thus obtained are the following:

Schwinger's G and \mathfrak{G} correspond to Dyson's S'_F and D'_F in the theory of S-matrix II.²⁾ The operators Γ^μ , M - m and $P_{\mu\nu}$ correspond to the proper vertex part, the proper self-energy part of an electron and the proper self-energy part of a photon respectively. The relation between the Green's function G and the S-matrix is given by

$$G = i(\text{vac}|T(\psi, \bar{\psi}, S)|\text{vac})'/(\text{vac}|S|\text{vac}).$$

Similarly the relation between \mathfrak{G} and S is given by

$$\mathfrak{G}_{\mu\nu} = i(\text{vac}|T(A, A, S)|\text{vac})'/(\text{vac}|S|\text{vac}),$$

where T denotes the chronological operator given by Wick,³⁾ and $(\quad)'$ means that A_μ and A_ν are connected in Feynman graph.

In order to remove the divergences arising in G , \mathfrak{G} etc., it is necessary to modify the fundamental equations for G and \mathfrak{G} . The modified equations are given as follows,

$$(-i\gamma^\mu\partial_\mu + m + K)G = 1,$$

$$K = (Z_2 - 1)(-i\gamma^\mu\partial_\mu + m) - Z_2 e_1 \gamma^\mu \langle A \rangle + \Sigma^* - Z_2 \delta m,$$

$$[\square\delta_{\mu\rho} - \{P_{\mu\rho} + f(\delta_{\mu\rho}\square - \partial_\mu\partial_\rho)\}]\mathfrak{G}_{\rho\nu}$$

$$= -\delta_{\mu\nu},$$

$$f = 1 - Z_3,$$

$$\Sigma^* = i e_1^2 \gamma^\mu G \Gamma^\nu \mathfrak{G}_{\nu\mu},$$

$$\delta\Sigma^* = -e_1 A^\mu,$$

$$\delta\langle A_\mu \rangle$$

$$P_{\mu\nu} = -i e_1^2 \text{tr}(\gamma^\mu G \Gamma^\nu G),$$

$$(1-f)\square\langle A_\mu \rangle = -J - i e_1 \text{tr}(\gamma^\mu G),$$

$$\Gamma^\mu = \gamma^\mu + \{A^\mu + (Z_2 - 1)\gamma^\mu\}.$$

The proof of the assertion that the quantities G , \mathfrak{G} etc., thus modified are free from divergences can be easily given by using the mathematical induction.

Ward's identity⁴⁾ is essential, as Dyson⁵⁾ has pointed out, for the success of the removal of the divergences. This identity can be derived quite easily from the fact that the whole system of equations written above covariant under the gauge transformation. The detailed explanation will be published in this journal in the near future.

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Note on the Physical Connection between Electron's Electro-Magnetic Mass and its Charge

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March 30, 1952

It is well-known that the so-called "ultra-violet" divergences appearing in quantum electrodynamics can be isolated in such unobservable renormalization factors as electron's electro-magnetic mass its additional charge except ambiguous photon self-energy

by Tomonaga-Schwinger theory and the remaining finite effects of the field reactions show very good accordance with experiments.

Now in this letter we shall indicate a little interesting point of view relating to the connection between electron's electro-magnetic mass and its charge without consideration about the additional charge due to field reactions.

First we take notice of the following two points about electron's electro-magnetic mass (or self-energy) though both of them seem to be more or less trivial. (1) Unless the experimental mass of the electron consists of its electro-magnetic mass only, it is essentially impossible to observe electron's electro-magnetic mass and its intrinsic mass separately at least in the frame of the current field theory. Or in other words electron's electro-magnetic mass is essentially "unobservable" quantity. (2) Coulomb self-energy which forms a part of electron's electro-magnetic self-energy appears on the score of summing up Coulomb interaction energy between infinitely small charges in the electron. Unless electron's volume is able to vary or electron decays into small fragments which have smaller charge than electron's one, such energy is one which we can not use practically even if it is finite. Since we have been not yet encountered such phenomena, it seems to be quite doubtful whether the Coulomb self-energy of the electron has a physical meaning as "energy" or not. Of course, if we suppose that electron's electro-magnetic self-energy and positron's one are transmuted into created photon's energy together with the energy due to its intrinsic mass in the case of electron-positron annihilation, we can give enough physical meaning to electron's electro-magnetic self-energy which contains Coulomb self-energy. But

the remarkable fact "one" real electron (or one real positron) does not decay "spontaneously" in vacuum seems to show us again the possibility that we may consider the electron's electro-magnetic self-energy to be meaningless physically.

Next we pay our attention to the electron's charge corresponding to these two points about its electro-magnetic mass (or self-energy). (1) Of course electron's charge is "observable" quantity and this is in striking contrast to the unobservability of electron's electro-magnetic mass. (2) The experimental value of electron's charge is always constant and this fact is connected closely with the doubt for the physical meaning of the Coulomb self-energy in the sense mentioned already.

From these contrast between electron's electro-magnetic mass and its charge — mainly from the point (1) — we may say that electron's electro-magnetic mass and its charge are connected each other in a sort of relation which is slightly analogous to the so-called complementarity-relation; namely our new relation is not same as the ordinary quantum mechanical one which is called "uncertainty-relation", but so to speak "well-known and knownothing relation" because electron's charge is always observable precisely with a constant value, while its electro-magnetic mass is always unobservable and furthermore between electron's charge and a part of its electro-magnetic self-energy there is such physical connection as mentioned above.

The author hopes that such point of view concerning the connection between electron's charge and its self-energy give a new spotlight on the problem of electron's electro-magnetic self-energy.

The General Discussion of the Self-Stress

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The relativistic covariance of the present quantum theory of field is guaranteed only on the basis of the point model of the elementary particle. In connection with the problem of the self-stress, Pais-Epstein¹⁾ and Yukawa-Umezawa²⁾ clarified these circumstances in the second order approximation of the perturbation calculation for the systems of the electron-photon and the electron-scalar meson.

In such a system (with the interaction of the second kind) that the infinite number of the counter terms are required by the renormalization³⁾, one may expect that the elementary particle has, as it were, the finite extension, and so it becomes an important problem to examine such system the stand point of the relativistic covariance.

It is proved in this paper that, as long as the present field theory is concerned, the vanishing self-stress of an arbitrary elementary particle is also obtained by the point model (in any order of perturbation approximation).

We start, for example, from the following general Lagrangian :

$$L = \mathcal{L} + \mathcal{L}^0 + \mathcal{L}'$$

$$\mathcal{L} = -\frac{1}{2} (X_{\alpha\beta}^{\mu\nu} Q_{\beta\nu} Q_{\alpha\mu} + x^2 Y_{\alpha\beta} Q_{\beta} Q_{\alpha}),$$

$$\mathcal{L}^0 = -\bar{\psi}_i (X_{ij}^{\mu} \phi_{j\mu} + \mu \Gamma_{ij} \psi_j),$$

$$\mathcal{L}' = f \bar{\psi}_i O_{ij}^{\alpha} \psi_j Q_{\alpha} + g \bar{\psi}_i O_{ij}^{\alpha\mu} \psi_j Q_{\alpha\mu},^{*)}$$

where ψ_i and Q_{α} are the field quantities, the subscript μ, ν stands for the derivatives with respect to x_{μ} and X , Γ, X, Y and O are the dimensionless constants independent of the field quantities and the masses.

Thus we obtain

$$T_{\mu\mu} = \mu \bar{\psi}_i \Gamma_{ij} \psi_j + x^2 Y_{\alpha\beta} Q_{\beta} Q_{\alpha} - f \bar{\psi}_i O_{ij}^{\alpha} \psi_j Q_{\alpha} + Q$$

with

$$Q = X_{\alpha\beta}^{\mu\nu} Q_{\beta\nu} Q_{\alpha\mu} + x^2 Y_{\alpha\beta} Q_{\beta} Q_{\alpha}.$$

Let us denote the propagation functions of the fields Q_{α} and Ψ_i by $\Delta_{F\alpha\beta}$ and S_{Fij} respectively, we have, just then,

$$\begin{aligned} \mu \frac{\partial}{\partial \mu} S_{Fij}(x-x') &= -i\mu \int S_{Fik}(x-x'') \\ &\times dx'' \Gamma_{kl} S_{Flj}(x''-x'), \end{aligned}$$

$$\begin{aligned} \mu \frac{\partial}{\partial \mu} \phi_i(x) &= -i\mu \int S_{Fij}(x-x'') \\ &\times dx'' \Gamma_{jk} \Psi_k(x''), \end{aligned}$$

$$\begin{aligned} x \frac{\partial}{\partial x} \Delta_{F\alpha\beta}(x-x') &= 2ix^2 \int \Delta_{F\alpha\gamma}(x-x'') \\ &\times dx'' Y_{\gamma\delta} \Delta_{F\delta\beta}(x''-x'). \end{aligned}$$

From above relations the expectation values for one fermion at rest become, in any order of approximation,

$$\langle \mu \int \bar{\Psi}_i \Gamma_{ij} \Psi_j dv \rangle = \mu \frac{\partial \delta \mu}{\partial \mu} + \mu,$$

$$\langle x^2 \int Y_{\alpha\beta} Q_{\beta} Q_{\alpha} dv \rangle = x^2 \frac{\partial \delta \mu}{\partial x},$$

$$\langle -f \int \bar{\Psi}_i O_{ij}^{\alpha} \Psi_j Q_{\alpha} dv \rangle + \langle \int Q dv \rangle = -g \frac{\partial \delta \mu}{\partial g},$$

where $\delta \mu$ means the self-energy of fermion due to the boson field. Thus we can generalize the Pais' formula¹⁾ as follows :

$$\begin{aligned} \langle \int T_{\mu\mu} dv \rangle &= \mu \frac{\partial \delta \mu}{\partial \mu} + x \frac{\partial \delta \mu}{\partial x} - g \frac{\partial \delta \mu}{\partial g} + \mu \\ &= 3S(0) + E(0) \end{aligned}$$

where $S(0)$ and $E(0)$ mean the self-stress and the self-energy of the fermion in the rest system. As we have the quantities $[\mu] = [x] = L^{-1}$, $[g] = L$ with the dimension of mass in this system, the self-energy must have the form

$$\delta\mu = \sum a_{ij}^n g^n \mu^i x^{j**})$$

with $i+j-n=1$ and dimensionless constants a_{ij}^n . We have, therefore,

$$S(0) = \frac{1}{3} \left(\mu \frac{\partial}{\partial \mu} + x \frac{\partial}{\partial x} - g \frac{\partial}{\partial g} - 1 \right) \delta\mu = 0.$$

The same is true for the system with more complex interaction. For example, if $\hbar\bar{\psi}O\psi Q^m$ is added in the interaction, we have

$$\begin{aligned} \langle \int T_{\mu\mu} dv \rangle = & \left(\mu \frac{\partial}{\partial \mu} + x \frac{\partial}{\partial x} - g \frac{\partial}{\partial g} \right. \\ & \left. - (m-1) \hbar \frac{\partial}{\partial \hbar} \right) S\mu + \mu \end{aligned}$$

and

$$\delta\mu = \sum a_{ij}^n g^n \hbar^n \mu^i x^j$$

with

$$-l - (m-1)n + i + j = 1.$$

Thus we have always the vanishing self-stress as long as we stand on the point model of the elementary particle. Detailed discussion will appear elsewhere.

We thank Prof. S. Sakata and Mr. S. Kamefuchi for their valuable discussions and pertinent criticisms.

- 1) A. Pais and S. T. Epstein, Rev. Mod. Phys. **21** (1949), 445.
- 2) J. Yukawa and H. Umezawa, Prog. Theor. Phys. **6** (1951), 112.
- 3) S. Sakata, H. Umezawa and S. Kamefuchi, Prog. Theor. Phys. **7** (1952), No. 4.

*) We use the natural unit i. e., $\hbar = c = 1$.

**) If we use the self-energy including any cut off momentum Λ_0 i. e.

$$\delta\mu = \sum a_{ijk}^n g^n \mu^i x^j f_0^k$$

with $i+j-n+k=1$, and $k \neq 0$, the non-vanishing self-stress is obtained.

Orthogonal Functions in the Complex Domain

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Eigenfunctions associated to an operator in the complex domain constitute a system of orthogonal functions along the path of integration in the complex plane. There are orthogonal functions free from operators, examples of which are given. At the end of this paper are collected some transformation functions constructed with orthogonal functions.

§ 1. Orthogonal functions in the complex plane

The relations¹⁾

$$(y_m, y_n) = 0, \quad (z_m, z_n) = 0, \quad (y_m, z_n) = \delta_{mn} \quad (1)$$

that are satisfied by the eigenfunctions associated to an operator in the complex domain remind us of the relations that are satisfied by a system of basis vectors in the euclidian space of even dimensions. Instead of the usual form of the innerproduct

$$(q, p) = \sum q^\rho p^\rho, \quad \rho = 1, 2, \dots, n, 1', \dots, n'$$

of the two vectors (q^ρ) , (p^ρ) in the $2n$ -dimensional euclidian space, here will be used the innerproduct of the form

$$(x, y) = \sum (x^r y^{r'} + x^{r'} y^r) = g_{rs} x^r y^s, \quad (2)$$

$$r = 1, 2, \dots, n$$

where the vectors (x^ρ) , (y^ρ) may be regarded as those obtained by the change of variables

$$x^r = (q^r + i q^{r'}) / \sqrt{2}, \quad x^{r'} = (q^r - i q^{r'}) / \sqrt{2},$$

$$y^r = (p^r + i p^{r'}) / \sqrt{2}, \quad y^{r'} = (p^r - i p^{r'}) / \sqrt{2}$$

from the vectors (q^ρ) , (p^ρ) .

There exists then a system of basis vectors e_ρ that satisfies the orthogonality relations

$$\left. \begin{aligned} (e_r, e_s) &= 0, & (e_{r'}, e_{s'}) &= 0, & (e_r, e_{s'}) &= \delta_{rs} \\ (e_\rho, e_\alpha) &= g_{\rho\alpha} \end{aligned} \right\} \quad (3)$$

or

The relations (1) and (3) are identical except that while the innerproduct of two vectors is defined by (2) the innerproduct of two functions is defined to be the complex integral of the product of the two functions divided by $2\pi i$ and taken along a certain closed curve C in the complex plane, viz.

$$(f, g) = \frac{1}{2\pi i} \int_C f(z)g(z)dz.$$

Hence the eigenfunctions associated to the operators may be called orthogonal functions along the curve C .

There are infinitely many systems of orthogonal functions on the same curve. For simplicity, let C be the unit circle with its center at the origin, then

$$c_r = z^r, \quad c_{r'} = z^{-r-1}, \quad r = 0, 1, 2, \dots$$

constitute a system of orthogonal functions.

Let f_p be another system of orthogonal functions on the same curve C , there must be then the orthogonality relations

$$(f_p, f_\sigma) = g_{p\sigma}.$$

Since f_p is a linear combination of c_p , there is a linear transformation A such that

$$f_p = c_\lambda A^\lambda_p,$$

the sign of summation with respect to λ being omitted. The orthogonality relations require the linear transformation to be orthogonal, viz.

$$g_{p\sigma} = g_{\lambda\mu} A^\lambda_p A^\mu_\sigma. \quad (4)$$

With regard to the orthogonal transformations the suffices may be raised or lowered with the aid of the fundamental tensors $g_{p\sigma}$ and $g^{p\sigma}$ which is inverse to it.

§ 2. Decomposable systems of orthogonal functions

If f_r are regular inside the curve C , $f_{r'}$ regular outside C , the linear transformation A decomposes into the direct sum of a non-singular matrix and its transposed inverse matrix, since A must be of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

where the elements of a and b are labeled with the indices r, r' respectively, and that the orthogonality relation (4), requires that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a^* & 0 \\ 0 & b^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

(* : transposition) or $a^*b = 1$

i.e. $b = (a^{-1})^* = a^{-*}.$

For simplicity, $(e_r), (f_r)$ shall be denoted by $e, f, (e_{r'}), (f_{r'})$ by e', f' . The relations between e, e' and f, f' will be then represented by

$$f = ea, \quad f' = e'a^{-*}.$$

Hence

$$f(z)f'^*(z') = e(z)aa^{-1}e'^*(z') = e(z)e'^*(z') = (z|z') = 1/(z' - z).$$

In case f are regular inside C while f' are not regular inside as well as outside C , the linear transformation A takes the form

$$A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$$

so that the orthogonality relations require

$$b = a^{-*}, \quad ac^* + ca^* = 0$$

$$\text{or } (ac^*)^* = -ac^*.$$

One gets then

$$\begin{aligned} f(z)f'^*(z') &= c(z)a(e(z')c + e'(z')b)^* \\ &= c(z)a(c^*e^*(z') + b^*e'^*(z')) \\ &= c(z)ac^*e^*(z') + c(z)e'^*(z'). \end{aligned}$$

Since ac^* is skew symmetric, the first term in the right member will vanish when $z' = z$.

Here is an example. The Bessel's equation

$$\frac{d}{dz} z^2 \frac{dy}{dz} + z^2 y = ly$$

has

$$\text{eigenvalues } l_n = n(n+1), \quad n = 0, 1, 2, \dots,$$

$$\text{eigenfunctions } f_n(z) = (z|n) = z^{-\frac{1}{2}} J_{n+\frac{1}{2}}(z),$$

$$f_{n'}(z) = (n'|z) = (-)^n \left(n + \frac{1}{2} \right) \pi z^{-\frac{1}{2}} J_{-n-\frac{1}{2}}(z).$$

The orthogonality conditions are satisfied, but one gets

$$(z|n)(n|z') = \frac{\cos(z' - z)}{z' - z}$$

which is one of the addition theorems of Bessel functions. One sees

$$\left[\frac{\cos(z' - z)}{z' - z} - \frac{1}{z' - z} \right]_{z'=z} = 0.$$

In the worst case when there exist no basis vectors regular inside or outside the curve C , the sum $f(z)f'^*(z')$ takes no definite form other than the δ -function.

When $f(z)f'^*(z')$ has a definite form, Laurent's expansion of any function will be readily given.

§ 3. The change of variable

When the correspondence between the variables z and $w = f(z)$ is required to be one-to-one, $f(z)$ must be of the form $(\alpha z + \beta)(\gamma z + \delta)^{-1}$.

The transformation rule for eigenfunctions $(w|n)$, $(n|w)$ shall be given here for the transformation $w \rightarrow z$.

Let

$$(z|n) = A(z) (w(z)|n),$$

$$(n|z) = B(z) (n|w(z)),$$

then the orthogonality conditions for $(z|n)$, $(n|z)$ require that

$$\begin{aligned} 1) \quad (z|n)(n|z') &= A(z)B(z') (w(z)|n)(n|w(z')) \\ &= A(z)B(z') (w(z') - w(z))^{-1} \\ &= A(z)B(z') (\gamma z + \delta)(\gamma z' + \delta)(a\delta - \beta\gamma)^{-1}(z' - z)^{-1}, \\ \therefore A(z)B(z) &= (a\delta - \beta\gamma)^{1/2}(\gamma z + \delta)^{-1} = \sqrt{\frac{dw}{dz}}. \quad (5) \\ 2) \quad (n|z)(z|n') &= B(z)A(z)(n|w(z))(w(z)|n') = B(z)A(z)(n|w)(w|n') \frac{dz}{dw}, \\ \therefore A(z)B(z) &= \frac{dw}{dz}. \end{aligned}$$

This condition is already satisfied by (5).

Hence one gets

$$\begin{aligned} (z|n) &= (w(z)|n) \sqrt{\frac{dw}{dz}}, \\ (n|z) &= (n|w(z)) \sqrt{\frac{dw}{dz}}. \end{aligned}$$

§ 4. The equation $f[(az + \beta)(\gamma z + \delta)^{-1}] = l f(z)$

As an application of the change of variable, we take the equation

$$f\left(\frac{az + \beta}{\gamma z + \delta}\right) = l f(z).$$

The transformation $z \rightarrow (az + \beta)(\gamma z + \delta)^{-1}$ leaves fixed the two points z_1 , z_2 ,

$$z_1 = \frac{a - \delta + \sqrt{D}}{2\gamma}, \quad z_2 = \frac{a - \delta - \sqrt{D}}{2\gamma}, \quad D = (a - \delta)^2 + 4\beta\gamma,$$

which are the two roots of the equation $z = (az + \beta)(\gamma z + \delta)^{-1}$.

The two cases are distinguished.

1) $D \neq 0$. If $(z - z_1)(z - z_2)^{-1}$ is denoted by w , the transformation will be represented in terms of w as

$$w \rightarrow kw$$

where k is the ratio of the two characteristic roots of the transformation, viz.

$$k = \frac{a + \delta - \sqrt{D}}{a + \delta + \sqrt{D}}.$$

For simplicity it is assumed that $k^m \neq 1$ ($m=1, 2, \dots$). If we put

$$f(z) = g(w)$$

the equation will be changed into the equation

$$g(kw) = ly(w)$$

which has

$$\text{eigenvalues} \quad l_n = k^n, \quad n = 0, 1, 2, \dots,$$

$$\text{eigenfunctions} \quad (w|n) = w^n, \quad (n|w) = w^{-n-1},$$

or

$$(z|n) = \frac{(z_1 - z_2)^{1/2} \left(\frac{z - z_1}{z - z_2} \right)^n}{z - z_2}, \quad (n|z) = \frac{(z_1 - z_2)^{1/2} \left(\frac{z - z_1}{z - z_2} \right)^{-n-1}}{z - z_2}.$$

2) $D=0$. If c denotes $(a + \delta)/2\gamma$, w does $c(z - z_1)^{-1}$, the transformation will be changed into the transformation

$$w \rightarrow w + 1.$$

So the equation for $g(w) = f(z)$ will be

$$g(w+1) = l_g(w).$$

A continuous spectrum appears in this case in which are made three assumptions ;

- 1) the path of integration is the imaginary axis or a straight line parallel to the imaginary axis,
- 2) $(w|a) \rightarrow 0$ as $w \rightarrow -\infty$,
- 3) $(a|w) \rightarrow 0$ as $w \rightarrow \infty$.

One gets then

$$\text{eigenvalues} \quad l = e^a, \quad a > 0,$$

$$\text{eigenfunctions} \quad (w|a) = e^{aw}, \quad (a|w) = e^{-aw},$$

the orthogonality relations :

$$(a|w)(w|a') = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-aw + a'w} dw = \delta(a - a') = (a|a'),$$

$$(w|a)(a|w') = \int_0^{\infty} e^{aw - a'w'} da = 1/(w' - w) = (w|w'),$$

$$\text{Real part of } w' - w > 0.$$

Returning to the old variable we have

$$(z|a) = \exp\left(\frac{ac}{z - z_1}\right) (-c)^{1/2} (z - z_1)^{-1},$$

$$(a|z) = \exp\left(\frac{-ac}{z-z_1}\right)(-c)^{1/2}(z-z_1)^{-1}.$$

An easy generalization of the preceding functional equation may be the equation

$$f(a_1 z + \beta_1) + f(a_2 z + \beta_2) + \cdots + f(a_m z + \beta_m) = l f(z) \quad (6)$$

to which corresponds the operator L with its matrix element

$$(z|L|z') = \frac{1}{z' - a_1 z - \beta_1} + \cdots + \frac{1}{z' - a_m z - \beta_m}.$$

So the equation adjoint to (6) will be

$$\frac{1}{a} g\left(\frac{z - \beta_1}{a_1}\right) + \cdots + \frac{1}{a_m} g\left(\frac{z - \beta_m}{a_m}\right) = l g(z).$$

Eigenvalues and eigenfunctions are as follows ;

$$\text{eigenvalues} \quad l = a_1^n + a_2^n + \cdots + a_m^n, \quad n = 0, 1, 2, \dots,$$

$$\text{eigenfunctions} \quad (z|n) : \text{polynomial of degree } n \text{ in } z,$$

$$(n|z) : \text{ascending power series in } z^{-1} \text{ starting at } z^{-n-1}.$$

A special case is of a little interest. The equation

$$f\left(\frac{z}{m}\right) + f\left(\frac{z+1}{m}\right) + \cdots + f\left(\frac{z+m-1}{m}\right) = l f(z)$$

has

$$\text{eigenvalues} \quad l_n = m^{1-n}, \quad n = 0, 1, 2, \dots,$$

$$\text{eigenfunctions} \quad (z|n) = B_n(z), \quad \text{Bernoulli's polynomial,}$$

$$(n|z) = \frac{1}{n!} \left(-\frac{d}{dz}\right)^n \log \frac{z}{z-1},$$

the path of integration being taken to be a circle of radius > 1 with its center at the origin. The result

$$B_n\left(\frac{z}{m}\right) + B_n\left(\frac{z+1}{m}\right) + \cdots + B_n\left(\frac{z+m-1}{m}\right) = m^{1-n} B_n(z)$$

is known as the multiplication theorem for Bernoulli's polynomials.²⁾

§ 5. Free orthogonal functions

So far the orthogonal functions have been considered as those associated to their operators. However, as was seen in § 2, a system of orthogonal functions can be generated with the aid of a non-singular matrix and transposed matrix inverse to it. If we can expand $(z|z')$ in the series

$$\frac{1}{z' - z} = f_0(z) g_0(z') + f_1(z) g_1(z') + \cdots + f_n(z) g_n(z') + \cdots$$

where $f_n(z)$ are regular in a domain of the complex plane confined by a closed curve while $g_n(z)$ are regular in the other domain of the complex plane, $f_n(z)$, $g_n(z)$ constitute a system of orthogonal functions, viz.

$$f_n(z) = (z|n), \quad g_n(z) = (n|z).$$

Here are a few examples.

1) Appell's series.³⁾ P. Appell expanded $(z'-z)^{-1}$ in the series

$$\frac{1}{z'-z} = \frac{Q_0(z)}{P_1(z')} + \frac{Q_1(z)}{P_2(z')} + \dots + \frac{Q_n(z)}{P_{n+1}(z')} + \dots$$

where $P_n(z)$ is given polynomial of degree n having all its zeros inside the unit circle. $Q_n(z)$ is a polynomial of degree n uniquely determined by the orthogonality conditions.

2) Neumann's series.⁴⁾

$$(z|n) = \{J_n(\sqrt{z})\}^2, \quad J_n(z) : \text{Bessel function,}$$

$$(n|z) = \text{Neumann's polynomial } Q_n(\sqrt{z}) \text{ multiplied by } \epsilon_n,$$

$$(0|z) = \frac{1}{z}, \quad (1|z) = \frac{2}{z} + \frac{4}{z^2}, \quad (2|z) = \frac{2}{z} + \frac{16}{z^2} + \frac{64}{z^3}, \dots$$

3) Sonine's series.⁵⁾ Let $\psi(w)$ be any function of w regular in the neighbourhood of $w=0$; and, if $\psi(w)=x$, let $w=\phi(x)$ so that ϕ is the function inverse to ψ . We have then

$$(z|n) = \frac{n!}{2\pi i} \int_{0+}^{0+} e^{zx} \frac{dw}{w^{n+1}}, \quad (n|z) = \frac{1}{n!} \int_0^\infty e^{-zx} w^n dx,$$

$$\begin{aligned} \text{e.g.} \quad w = \phi(x) = x + x^2, \quad x = \psi(w) &= \frac{1}{2} (\sqrt{1+4w} - 1) \\ &= w - w^2 + 2w^3 - 5w^4 + \dots, \end{aligned}$$

$$(z|0) = 1, \quad (0|z) = \frac{1}{z},$$

$$(z|1) = z, \quad (1|z) = \frac{1}{z^2} + \frac{2}{z^3},$$

$$(z|2) = z^2 - 2z, \quad (2|z) = \frac{1}{z^3} + \frac{6}{z^4} + \frac{60}{z^5},$$

$$(z|3) = z^3 - 6z^2 + 12z, \quad (3|z) = \frac{1}{z^4} + \frac{12}{z^5} + \frac{60}{z^6} + \frac{120}{z^7}, \dots$$

The orthogonality conditions are satisfied.

4) General interpolation series.⁶⁾

$$\frac{1}{x-z} = \frac{1}{x-a_1} + \frac{z-a_1}{(x-a_1)(x-a_2)} + \frac{(z-a_1)(z-a_2)}{(x-a_1)(x-a_2)(x-a_3)} + \dots,$$

$$(z|n) = (z - a_1) \cdots (z - a_n),$$

$$(n|z) = 1/(z - a_1) \cdots (z - a_{n+1}).$$

The path of integration shall encircle all the points a_1, a_2, a_3, \dots in the positive sense. There are three special cases.

4') Newton's interpolation series.

$$\frac{1}{x-z} = \frac{1}{x} + \frac{z}{x(x-1)} + \frac{z(z-1)}{x(x-1)(x-2)} + \dots$$

4'') Everett's interpolation series.

$$\frac{1}{x-z} = \frac{\xi}{x} - \frac{z}{\xi} + \frac{1}{3} \frac{(\xi+1)\xi(\xi-1)}{(x+1)x(x-1)} - \frac{1}{3} \frac{(z+1)z(z-1)}{(\xi+1)\xi(\xi-1)} + \dots,$$

$$(\xi = 1-z, \quad \tilde{\xi} = 1-x).$$

4''') Stirling's interpolation series.

$$\frac{1}{x-z} = \frac{1}{x} + \frac{z}{x^2-1} + \frac{z^2}{x(x^2-1)} + \frac{z(z^2-1)}{(x^2-1)(x^2-2^2)} + \frac{z^2(z^2-1)}{x(x^2-1)(x^2-2^2)} + \dots$$

The orthogonality relations are easily verified.

§ 6. Some transformation functions

When there are two decomposable systems of orthogonal functions e_ρ, f_ρ ; e, f being regular inside the curve C , e', f' regular outside the curve C , we can denote $e(z)f'^*(l)$ by $(z|l)$, $f(l)e'^*(z)$ by $(l|z)$. In place of the conditions for $(z|l)$, $(l|z)$ in the preceding paper⁷⁾, $(x|y)$ is conditioned to be regular qua function of x inside the path of integration, regular qua function of y outside the path of integration. One sees then

$$e(z)f'^*(l) = e_r(z)f^r(l) = (z|l),$$

$$f(l)e'^*(z) = f_r(l)e^r(z) = (l|z),$$

by virtue of the relations

$$(z|l)(l|z') = e(z)f'^*(l)f(l)e'^*(z') = e(z)e'^*(z') = (z|z'),$$

$$(l|z)(z|l') = f(l)e'^*(z)e(z)f'^*(l') = f(l)f'^*(l') = (l|l'),$$

$$(e'^*(z)e(z) = f'^*(l)f(l) = 1).$$

The fundamental system of orthogonal functions is the system

$$e_r(z) = z^r, \quad e_{r'}(z) = e^r(z) = z^{-r-1}$$

which is modified to the system

$$f_r(z) = z^r/e_r, \quad f_{r'}(z) = f^r(z) = e_r z^{-r-1}.$$

We have then

$$(z|l) = \sum_{r=0}^{\infty} \frac{z^r}{l^r}, \quad r=0, 1, 2, \dots,$$

$$(l|z) = \sum_{r=0}^{\infty} \frac{l^r}{z^r}.$$

Here are some examples.

$$1) \quad (z|l) = \frac{1}{l} + \frac{z}{l^2} + \frac{z^2}{l^3} + \dots = \frac{1}{l-z},$$

$$(l|z) = \frac{1}{z} + \frac{l}{z^2} + \frac{l^2}{z^3} + \dots = \frac{1}{z-l}.$$

$$2) \quad (z|l) = \frac{1}{l} + \frac{2z}{l^2} + \frac{3z^2}{l^3} + \dots = \frac{l}{(l-z)^2},$$

$$(l|z) = \frac{1}{z} + \frac{l}{2z^2} + \frac{l^2}{3z^3} + \dots = \frac{1}{l} \log \frac{z}{z-l}.$$

$$3) \quad (z|l) = \frac{1}{l} + \frac{3}{2} \frac{z}{l^2} + \frac{3 \cdot 5}{2 \cdot 4} \frac{z^2}{l^3} + \dots = \frac{l^{1/2}}{(l-z)^{3/2}},$$

$$(l|z) = \frac{1}{z} + \frac{2}{3} \frac{l}{z^2} + \frac{2 \cdot 4}{3 \cdot 5} \frac{l^2}{z^3} + \dots = \sqrt{\frac{1}{l(z-l)}} \arcsin \sqrt{\frac{l}{z}}.$$

$$4) \quad (z|l) = \frac{1}{l} + \frac{3z}{l^2} + \frac{5z^2}{l^3} + \dots = \frac{l+z}{(l-z)^2},$$

$$(l|z) = \frac{1}{z} + \frac{l}{3z^2} + \frac{l^2}{5z^3} + \dots = \frac{1}{(lz)^{1/2}} \operatorname{arctanh} \sqrt{\frac{l}{z}}.$$

$$5) \quad (z|l) = \frac{1}{l} + \frac{a}{1} \frac{z}{l^2} + \frac{a(a+1)}{1 \cdot 2} \frac{z^2}{l^3} + \dots = \frac{l^{a-1}}{(l-z)^a},$$

$$(l|z) = \frac{1}{z} + \frac{1}{a} \frac{l}{z^2} + \frac{1 \cdot 2}{a(a+1)} \frac{l^2}{z^3} + \dots = \frac{1}{z} F\left(1, 1; a, \frac{l}{z}\right).$$

$$6) \quad (z|l) = \frac{1}{l} + \frac{ab}{1 \cdot c} \frac{z}{l^2} + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} \frac{z^2}{l^3} + \dots = \frac{1}{l} F\left(a, b; c, \frac{z}{l}\right),$$

$$(l|z) = \frac{1}{z} + \frac{1 \cdot c}{ab} \frac{l}{z^2} + \frac{1 \cdot 2 \cdot c(c+1)}{a(a+1)b(b+1)} \frac{l^2}{z^3} + \dots = \frac{1}{z} F\left(1, 1, c; a, b, \frac{l}{z}\right).$$

$$7) \quad (z|l) = \frac{1}{l} - \frac{1}{1!} \frac{z}{l^2} + \frac{1}{2!} \frac{z^2}{l^3} - \dots = \frac{1}{l} e^{-z/l},$$

$$(l|z) = \frac{1}{z} - 1! \frac{l}{z^2} + 2! \frac{l^2}{z^3} - \dots = \frac{1}{l} \hat{\zeta}\left(\frac{l}{z}\right),$$

$$\hat{\zeta}(x) = 1 - \frac{1!}{x} + \frac{2!}{x^2} - \dots.$$

$$8) \quad (z|l) = \frac{1}{l} - \frac{1}{1!^2} \frac{z}{l^2} + \frac{1}{2!^2} \frac{z^2}{l^3} - \dots = \frac{1}{l} J_0 \left(2\sqrt{\frac{z}{l}} \right),$$

$$(l|z) = \frac{1}{z} - \frac{1!^2 l}{z^2} + \frac{2!^2 l^2}{z^3} - \dots = \frac{1}{l} \beta \left(\frac{l}{z} \right),$$

$$\beta(x) = \frac{1}{x} - \frac{1!^2}{x^2} + \frac{2!^2}{x^3} - \dots$$

The cases pertaining to well-known polynomials:

$$9) \quad (z|l) = \frac{P_0(z)}{l} + \frac{3P_1(z)}{l^2} + \frac{5P_2(z)}{l^3} + \dots = \frac{l^2 - 1}{(l^2 - 2lz + 1)^{3/2}},$$

$$(l|z) = Q_0(z) + lQ_1(z) + l^2Q_2(z) + \dots = \frac{1}{(l^2 - 2lz + 1)^{1/2}} \operatorname{arc} \coth \frac{z - l^{-1}}{(l^2 - 2lz + 1)^{1/2}},$$

$P_n(z)$, $Q_n(z)$: Legendre's functions.

$$10) \quad (z|l) = \frac{L_0(z)}{l} + \frac{L_1(z)}{l^2} + \frac{L_2(z)}{l^3} + \dots = \frac{e^{-z/(l-1)}}{l-1},$$

$$(l|z) = M_0(z) + lM_1(z) + l^2M_2(z) + \dots = \frac{1}{l-1} \xi \left(\frac{l-1}{z} \right),$$

$L_n(z)$: Laguerre's polynomial,

$$M_m(z) = (-)^m \sum n! \binom{n}{m} z^{-n-1}, \quad n = m, m+1, \dots$$

$$11) \quad (z|l) = \frac{H_0(z)}{l} + \frac{H_1(z)}{l^2} + \frac{H_2(z)}{l^3} + \dots = \varepsilon(l-z),$$

$$(l|z) = K_0(z) + lK_1(z) + l^2K_2(z) + \dots = i\varepsilon(iz-il),$$

$H_n(z)$: Hermite's polynomial,

$$K_n(z) = \frac{1}{z^{n+1}} + \frac{(n+1)(n+2)}{2z^{n+3}} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 \cdot z^{n+5}} + \dots,$$

$$\varepsilon(x) = \int_x^\infty \exp \frac{x^2 - t^2}{2} dt = \frac{1}{x} - \frac{1}{x^3} + \frac{1 \cdot 3}{x^5} - \frac{1 \cdot 3 \cdot 5}{x^7} + \dots$$

$$12) \quad (z|l) = \frac{B_0(z)}{l} + \frac{B_1(z)}{l^2} + \frac{B_2(z)}{l^3} + \dots = \int_0^\infty \frac{te^{(z-l)t}}{e^t - 1} dt,$$

$$(l|z) = C_0(z) + lC_1(z) + l^2C_2(z) + \dots = \log \frac{z-l}{z-1-l},$$

$B_n(z)$, Bernoulli's polynomial,

$$C_n(z) = \frac{1}{n!} \left(-\frac{d}{dz} \right)^n \log \frac{z}{z-1}.$$

The relation between (7) and (10) is obviously observed. The change of variables will engender various systems of orthogonal functions.

Postscriptum To fill up the vacant space two remarks may be inserted here.

1) The path of integration considered so far is a closed curve without any node or a straight line extending to infinity at most. If the path of integration encircles each of the singular points of a differential equation a number of times in any sense, there appears the need of Riemann surfaces with topological considerations. The treatment of the case shall be reserved for a later occasion elsewhere.

2) Many examples of the system of orthogonal functions are known in the real domain as well as in the complex domain. We don't know any system of symplectic functions in the proper sense. A system of functions $(f_1, f_2, \dots, f_1', f_2', \dots)$ may be called symplectic when it satisfies the conditions

$$[f_r, f_s] = 0, \quad [f_{r'}, f_{s'}] = 0, \quad [f_r, f_{s'}] = -[f_{r'}, f_s] = \delta_{rs},$$

where $[f, g]$ is defined as

$$[f, g] = \int_C F(f, g) dx,$$

$F(f, g)$ being a function bilinear and alternate in f, g , C an appropriate curve.

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Electronic States of Ethylene Molecule

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Electronic states of ethylene molecules are discussed according to the LCAO method of molecular orbitals. The molecule is regarded as a system which consists of two $2p\pi$ -electrons in a field of a skeleton. It is assumed that effective charges for $2p\pi$ orbitals are not the same as those for orbitals in the skeleton. Values of the effective charge are determined so as to minimize the energy of each electronic state. The result for excited electronic energy is compared with experiment, and their agreement is found though the observed values are very ambiguous. A relation between LCAO and valence bond methods is discussed.

Introduction

The nature of double bonds was discussed by many authors on the basis of quantum mechanics¹⁾⁻⁸⁾. It was found that the double bonds between carbons, oxygens, and nitrogens consist of a σ -bond and a π -bond, and further that peculiar properties of organic substances with double bonds are mostly due to π -electrons associated with the double bonds. Excited states of these electrons have usually been discussed according to the LCAO method of molecular orbitals. However, the hitherto calculation can not be regarded as being sufficient, though their results were more or less in agreement with experiment. It will be tried, in what follows, to improve the method for an ethylene molecule by introducing some modification into the conventional LCAO method. This is the purpose of the present paper.

The electronic energy levels of the ethylene molecule were calculated by Parr and Crawford⁴⁾ and by Craig⁵⁾. Their method contained certain unsatisfactory points. One of them was to adopt an inadequate value for an effective charge for $2p\sigma$, $2p\pi$ and $2s$ orbitals. In some cases they assumed for effective charge a value 3.18 which had been determined by Zener⁶⁾ for the neutral carbon atom. The orbitals do not belong to a neutral carbon atom, but belong to a molecule. Therefore we should take into account an effect of a modified field for these orbitals, though the above mentioned assumption has often been accepted³⁾. Effective charges were assumed to be the same for both σ and π orbitals. However, it may be reasonable to assume different values of the effective charges for σ and π orbitals, because effective fields for them are different. Moreover, in the calculations by the above mentioned authors there was no principle of determining the effective charges.

As the first step of approaching to a more reasonable approximation we shall assume that effective charges for $2p\sigma$ orbitals are the same for all electronic states but those for the $2p\pi$ orbitals are not always the same. Further, it will be assumed that the former

is equal to Zener's value, though this point should be improved as is mentioned above. The latter will be determined so as to minimize the energy of each molecular orbital level. This is an essential improvement in the present work. We shall find in our result that the above mentioned effective charges are different in general in accordance with our first expectation. The calculated result for excited electronic energy will be found to agree with experiment though hitherto observed values yet contain much ambiguity.

§ 1. Energy formulas

We shall derive energy formulas in this section according to the LCAO method of molecular orbitals. The derivation can be carried out in the similar way as the calculation by Parr and Crawford.⁴⁾ We shall therefore describe only an outline of the derivation.

We first assume a plane ethylene molecule. The symmetry of this molecule is represented by a group V_h . Its electronic states are therefore A_g , A_u , B_{1g} , B_{1u} , B_{2g} , B_{2u} , B_{3g} and B_{3u} .

We regard the molecule as a system which consists of two electrons in a field of a skeleton. The skeleton is assumed to consist of two nuclei of atomic number 4 and six hybrid sp^2 -electrons in the molecular plane. The influence of hydrogen atoms is wholly neglected. Let $I(k)$ and $II(k)$ be $2p\pi$ atomic orbitals of the k -th electron belonging to the first and second nuclei respectively ($k=1, 2$). Orthonormal molecular orbitals are given by linear combinations of these atomic orbitals as follows:

$$\begin{aligned} B_{1u}: \phi_0(k) &= (1/2\sigma_0)^{1/2}(I(k) + II(k)), \\ B_{2g}: \phi_1(k) &= (1/2\sigma_1)^{1/2}(I(k) - II(k)), \end{aligned} \quad (1)$$

where $\sigma_0 = 1 + S$, $\sigma_1 = 1 - S$, and S is an overlap integral. Symmetry properties are shown on the left of the functions.

From these molecular orbitals we have four linearly independent anti-symmetric wave functions of the molecule in the singlet and triplet states as follows:

$$\begin{aligned} {}^1A_g: \phi_0(1)\phi_0(2) \cdot \chi_s, \\ {}^3B_{3u}: (1/2)^{1/2}[\phi_0(1)\phi_1(2) - \phi_1(1)\phi_0(2)] \cdot \chi_t, \\ {}^1B_{3u}: (1/2)^{1/2}[\phi_0(1)\phi_1(2) + \phi_1(1)\phi_0(2)] \cdot \chi_s, \\ {}^1A_g: \phi_1(1)\phi_1(2) \cdot \chi_s, \end{aligned} \quad (2)$$

where χ_s and χ_t are the singlet and triplet spin functions of the two electrons respectively.

The Hamiltonian for this system in atomic units can be written in the following form:

$$H = H(1) + H(2) + (1/r_{12}), \quad (3)$$

where $H(k)$ is a sum of kinetic energy and potential energy, due to the skeleton, of the k -th electron ($k=1, 2$), and $(1/r_{12})$ is the repulsive potential between two electrons. A secular equation for energy, W , of two 1A_g states is given by

$$\begin{vmatrix} (2/\sigma_0)(u+\beta) + (1/2\sigma_0^2)(M+J+2K+4L) - W & (1/2\sigma_0\sigma_1)(M-J) \\ (1/\sigma_0\sigma_1)(M-J) & (2/\sigma_1)(u-\beta) + (1/2\sigma_1^2)(M+J+2K-4L) - W \end{vmatrix} = 0 \quad (4)(a)$$

and energy values of 1B_u and 3B_u states are given by

$$\begin{aligned} {}^1B_u & \quad (2/\sigma_0\sigma_1)(u-S\beta) + (1/\sigma_0\sigma_1)(J-K), \\ {}^3B_u & \quad (2/\sigma_0\sigma_1)(u-S\beta) + (1/\sigma_0\sigma_1)(M-K), \end{aligned} \quad (4)(b)$$

where

$$\begin{aligned} u &= \int I(I)H(1)I(1)dv_1, & \beta &= \int I(1)H(1)II(1)dv_1, \\ J &= \int (1/r_{12})I(1)^2II(2)^2dv_1dv_2, & & \text{(Coulomb integral),} \\ K &= \int (1/r_{12})I(1)II(1)I(2)II(2)dv_1dv_2, & & \text{(exchange integral),} \\ L &= \int (1/r_{12})I(1)^2I(2)II(2)dv_1dv_2, & & \text{(hybrid Coulomb exchange integral),} \\ M &= \int (1/r_{12})I(1)^2I(2)^2dv_1dv_2, & & \text{(mononuclear integral).} \end{aligned} \quad (4)(c)$$

When we discuss excited states we should consider, as was pointed out by Mulliken⁸, a "perpendicular ethylene molecule" in which planes of two CH_2 groups are perpendicular to each other. In this case $II(k)$ should be replaced by $II_{\perp}(k)$ in which $\cos \phi$ in $II(k)$ given by (8) in the second section is replaced by $\sin \phi$. Consequently S , L and β all vanish. Energy values of the "perpendicular ethylene" are thus given by

$$\begin{aligned} {}^1A_g &: 2u_{\perp} + M, \quad 2u_{\perp} + J_{\perp}, \\ {}^1B_u &: 2u_{\perp} + (M - K_{\perp}), \\ {}^3B_u &: 2u_{\perp} + (J_{\perp} - K_{\perp}), \end{aligned} \quad (5)$$

where \perp 's indicate that $II(k)$ is replaced by $II_{\perp}(k)$ in the respective formula.

Ionization energy is calculated as follows. An ionized ethylene molecule has only one π -electron in the ground state molecular orbital represented by ψ_0 . Therefore the energy, ϵ_0 , of this electron, is equal to orbital energy of ψ_0 . It is given by

$$\epsilon_0 = \int \psi_0(1)H(1)\psi_0(1)dv_1 = (1/\sigma_0)(u+\beta). \quad (6)$$

The difference between ϵ_0 and ground state energy corresponds to the ionization energy.

Now, we shall consider the problem according to the valence bond method. In this method wave functions of homopolar states of the molecule are given by

$$\begin{aligned} {}^1A_g &: (1/2)^{1/2}[I(1)II(2) + II(1)I(2)] \cdot \chi_s, \\ {}^3B_u &: (1/2)^{1/2}[I(1)II(2) - II(1)I(2)] \cdot \chi_s, \end{aligned} \quad (7)(a)$$

and those of heteropolar states are given by

$$\begin{aligned}
 {}^1\text{B}_u &= (1/2)^{1/2} [I(1)I(2) - II(1)II(2)] \cdot \chi_s, \\
 {}^1\text{A}_g &= (1/2)^{1/2} [I(1)I(2) + II(1)II(2)] \cdot \chi_s.
 \end{aligned}
 \tag{7} (b)$$

A set of functions given by linear combinations of (2) and that given by (7) are both linear assemblages whose bases consist of the same functions. Therefore their irreducible invariant subsets are the same. Thus we see that the valence bond and LCAO molecular orbital methods give the same result. In fact we can find easily that a secular equation for ${}^1\text{A}_g$ and energy formulas for ${}^1\text{B}_u$ and ${}^3\text{B}_u$ are the same in both method if we take into account all the overlap integrals, though these integrals have customarily neglected in the conventional valence bond method⁽⁵⁾.

§ 2. Results and comparison with experiment

In this section we shall explain an outline of our principle of calculation and shall compare the result with experiment. The method for calculating various integrals will be accounted for in the last section. We assume for the $2p\pi$ orbital an usual form in polar coordinates as follows:

$$(\kappa^5/32\pi)^{1/2} r \cos \theta \cos \phi \exp(-\kappa r/2) \tag{8}$$

where κ denotes an effective charge of the orbital, and $\phi=0$ coincides with the molecular plane. The one-electron energy $H(k)$ in (3) is the same as that previously assumed by Mayer and Sklar⁽¹¹⁾. We neglect an influence of hydrogen atoms altogether. The potential energy in $H(k)$ is therefore equal to a potential energy due to two neutral carbon atoms minus a potential energy due to two $2p\pi$ -electrons:

$$H(1) = -\frac{1}{2}A_1 + U_{I_0}(1) + U_{II_0}(1) - \int (1/r_{12}) I_0(2)^2 dv_2 - \int (1/r_{12}) II_0(2)^2 dv_2 \tag{9}$$

Table 1.

κ	3.2	2.4	1.6
κ_0	3.2	3.2	3.2
α	-0.6688	-0.643	-0.525
α_\perp	-0.6522	-0.633	-0.40
β	-0.3225	-0.395	-0.411

All quantities are given in atomic units.
 \perp indicates that the value is calculated for the perpendicular ethylene.

where all quantities are measured in atomic units, $I_0(2)$ and $II_0(2)$ are atomic $2p\pi$ -orbitals with a fixed effective charge κ_0 , of the second electron, belonging to the first and second nuclei respectively, and $U_{I_0}(1)$ and $U_{II_0}(2)$ are potential energies respectively due to the first and second neutral carbon atoms in which effective charges of all atomic orbitals is equal to κ_0 .

For the sake of simplicity we assume $\kappa_0 = 3.2$ (Zener's value was equal to 3.18).

Calculated values of α and β for three values of κ are shown in Table 1. Numerical values of S , J , K , L , and M were calculated by Araki and Watari⁽⁹⁾. Values of the parameters given by them are shown in Table 2.* Using these values of parameters we calculate electronic energy of an ethylene molecule with its C-C distance of 2.5 atomic

* I wish to express here my deepest gratitude to Professor G. Araki and Mr. W. Watari for their kindness of showing me their result before publication.

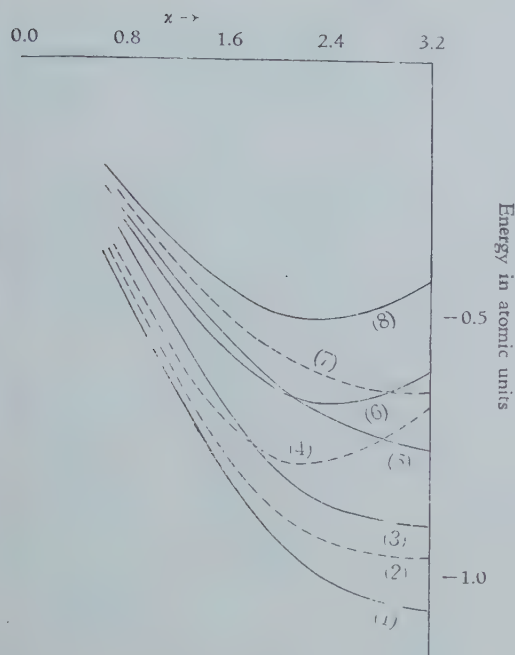


Fig. 1. Dependence of electronic energy of ethylene on effective charge.

Full lines indicate plane ethylene

(1) 1A_g , (3) $^3B_{3u}$, (5) $C_2H_4^+$, (6) $^1B_{3u}$, (8) 1A_g .

Dotted lines indicate perpendicular ethylene

(2) 1A_g , $^3B_{3u}$, (4) 1A_g , $^1B_{3u}$, (7) $C_2H_4^+$.

Table 3.

Dependence of electronic energy of ethylene on effective charge.

x	Electronic Energy in atomic units					
	Plane Ethylene			Perpendicular Ethylene		
	3.2	2.4	1.6	3.2	2.4	1.6
x_0	3.2	3.2	3.2	3.2	3.2	3.2
1A_g	-0.44	-0.52*	-0.45	-0.69	-0.80*	-0.71
$^1B_{3u}$	-0.62	-0.69*	-0.58	-0.70	-0.80*	-0.72
$^3B_{3u}$	-0.93*	-0.89	-0.69	-0.99*	-0.97	-0.80
1A_g	-1.10*	-1.04	-0.83	-0.98*	-0.98	-0.79
ϵ_0	-0.77*	-0.71	-0.55	-0.66*	-0.64	-0.51

C.C distance = 1.323A

* shows minimum.

units (1.323A) for three values of x . The result is shown in Table 3 and Fig. 1. According to the experimental result on infrared spectra¹⁴⁾ the C-C distance of ethylene molecules in the ground state is equal to 1.353A. The above mentioned distance is adopted merely for the sake of simplicity in the numerical calculation.

Table 2.
Araki and Watari's values of parameters in atomic units.

x	3.2	2.4	1.6	0.8
S	0.2869	0.4680	0.6947	0.9074
M	0.6263	0.4697	0.3131	0.1566
J	0.3455	0.3139	0.2533	0.1474
J_{\perp}	0.3350	0.2982	0.2343	0.1330
L	0.1377	0.1846	0.1988	0.1385
K	0.0426	0.0901	0.1409	0.1263
K_{\perp}	0.00244	0.00478	0.00755	0.00680

\perp indicates that the value is calculated for the perpendicular ethylene.

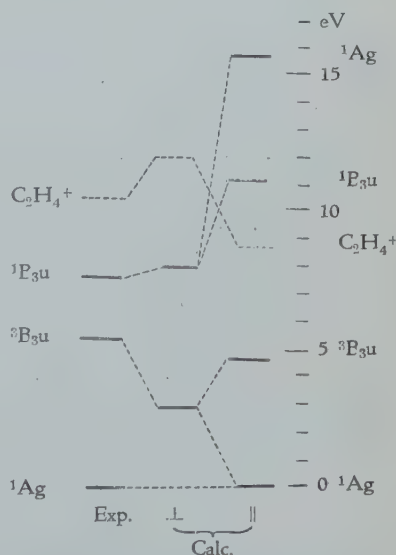


Fig. 2. Electronic energy of ethylene referred to its ground state.

We see that minimum values of electronic energy correspond to $x=3.2$ for the lower ${}^1\text{Ag}$, ${}^3\text{B}_{3u}$ and ionic states and to $x=2.4$ for the higher ${}^1\text{Ag}$ and ${}^1\text{B}_{3u}$ states. The former value is equal to x_0 and the latter is less than x_0 . This shows the adequacy of our expectation that x and x_0 are different in general.

The calculated energy values (minima for varying x) for excited electronic states are shown in Table 4 and they are compared with observed values⁽⁷⁾⁽⁸⁾⁽⁹⁾. The agreement may be regarded as being satisfactory, except for ionization energies, if we remember the observed values to contain some ambiguities. The calculated values by Craig⁽⁵⁾ were not in agreement with experiment. Parr and Crawford⁽⁴⁾ calculated these energies for several values of x assuming $x_0=x$ and some of their results were in agreement with experiment, but there was no principle determining which values were final.

Table 4.
Electronic energy of ethylene molecule referred to its ground state.

States	Calc. values in eV.		Exp. values in eV.
	Plane	Perp	
${}^1\text{Ag}$	15.8	8.2	
${}^1\text{B}_{3u}$	11.2	8.0	7.6 ⁷⁾
${}^3\text{B}_{3u}$	4.6	3.0	5.4 ⁸⁾
${}^1\text{Ag}$	0	3.0	0
ion	8.8	12.0	10.45 ⁷⁾ 10.60 ⁹⁾

C-C distance=1.323Å.

As for the values of ionization energy the present calculation is not in agreement with experiment. If we calculate this value according to Mulliken's method³⁾ we have 10.24 eV as the ionization energy of ethylene. This is in better agreement with experiment though the former method may be considered as being more reasonable. The disagreement suggests a need for improving our method of approximation in a more reasonable way.

§ 3. Integrals

In this section we shall give an account for the method of evaluating integrals α and β . Substituting Eq. (9) for $H(1)$ and $H(2)$ in the definition of α and β given by (4) (c) we have

$$\alpha = -\frac{1}{2} \int I \Delta I dv + \int U_{I_0} I(1)^2 dv_1 + \int U_{II_0} I(1)^2 dv_1 \\ - \int (1/r_{12}) I_0(2)^2 I(1)^2 dv_1 dv_2 - \int (1/r_{12}) II_0(2)^2 I(1)^2 dv_1 dv_2, \quad (10) (a)$$

$$\beta = -\frac{1}{2} \int I \Delta II dv + 2 \int U_{I_0} I(1) II(1) dv_1 - 2 \int (1/r_{12}) I_0(2)^2 I(1) II(1) dv_1 dv_2, \quad (10) (b)$$

where atomic orbitals $I(1)$ and $II(1)$ are defined by (8). Kinetic energy integrals are easily evaluated. The result is given by

$$\alpha_{\text{kin}} = -\frac{1}{2} \int I \Delta I dv = x^2/8, \\ \beta_{\text{kin}} = -\frac{1}{2} \int I \Delta II dv = x^2 [(1/12) \exp(-\rho/2) \{3 + (3/2)\rho + (1/4)\rho^2\} - (1/8)S], \quad (11)$$

where $\rho = xR$ and R is a nuclear distance. The mononuclear integral is evaluated in polar coordinates. The result is given by

$$M_0 = \int (1/r_{12}) I_0(1)^2 I(2)^2 dv_1 dv_2 \\ = (x/20) [5\lambda + 4\lambda^3 - \lambda^5 \{ 45(1+\lambda)^{-7} + 45(1+\lambda)^{-6} + 31(1+\lambda)^{-5} \\ + 17(1+\lambda)^{-4} + 8(1+\lambda)^{-3} + 4(1+\lambda)^{-2} \}] \quad (12)$$

where $\lambda = x/x_0$. Coulomb and hybrid Coulomb exchange integrals which are defined by

$$J_0 = \int (1/r_{12}) I(1)^2 I_0(2)^2 dv_1 dv_2 \text{ and } I_0 = \int (1/r_{12}) I(1) II(1) I_0(2)^2 dv_1 dv_2 \quad (13)$$

have never hitherto been evaluated. Their evaluation is most tedious. We calculate these integrals in the following way. Mulligan¹²⁾ obtained general expressions, in polar coordinates, for potential functions due to all possible combinations of atomic orbitals in an atom. We transform Mulligan's potentials into those which are represented in elliptical coordinates. By making use of this result the above mentioned integrals can be represented in terms of $A_n(a)$, $B_n(b)$ and $F_n(a, b)$, where $A_n(a)$ and $B_n(b)$ are functions which were defined by Kotani, Amemiya and Simose¹³⁾ and $F_n(a, b)$ is defined by

$$F_n(a, b) = \int_{-1}^1 \eta^n e^{-b\eta} d\eta \int_1^\infty \frac{e^{-a\xi}}{\xi + \eta} d\xi. \quad (14)$$

Numerical values of $A_n(a)$ and $B_n(b)$ were given by Kotani, Amemiya and Simose.¹³⁾ $F_n(a, b)$ can be evaluated by the aid of the recurrence formula as follows:*

$$F_n(a, b) = (-1)^n \frac{e^{-\delta}}{\delta} \log \frac{a}{b} - (-1)^n \frac{e^{-\delta}}{\delta} [-Ei(-2b)] + \frac{e^{\delta}}{\delta} [-Ei(-2a)] \\ - (-1)^n \frac{e^{-a} a^{n-1}}{\delta} \sum_{k=1}^n (-1)^k B_k(b) - \frac{n}{\delta} F_{n-1}(a, b), \quad (15)$$

$$F_0(a, b) = \frac{e^{-\delta}}{\delta} \log \frac{a}{b} - \frac{e^{-\delta}}{\delta} [-Ei(-2b)] + \frac{e^{\delta}}{\delta} [-Ei(-2a)], \quad (16)$$

$$\delta = a - b > 0, \quad b > 0.$$

Penetration integrals which are defined by

$$P_0 = \int U_{I_0} I(1)^2 dv_1, \\ Q_0 = \int U_{I_0} II(1)^2 dv_1 \quad (17)$$

and

$$R_0 = \int U_{I_0} I(1) II(1) dv_1$$

* Mulligan's equations seem to contain errors. There was no logarithmic term in his equation. (cf. Kotani, Amemiya and Simose's paper⁽¹³⁾)

are evaluated using the expression for U_{10} given by Parr and Crawford⁽¹¹⁾. This is a potential due to a neutral carbon atom consisting of a 2s-electron, three 2p-electrons and a nucleus of atomic number 4 (perfectly shielding by 1s). The first of these integrals is given by

$$P_0 = x_0(1/2)(\lambda/1 + \lambda)^5[2(1 + \lambda) + 6 \\ + 10(1 + \lambda)^{-1} + 15(1 + \lambda)^{-2}]. \quad (17)$$

The other integrals can be represented in terms of A_n and B_n . Thus we can evaluate all integrals contained in Eq (10). Numerical values of these integrals are shown in Table 5.*

The present calculation was carried out under the guidance of Professor G. Araki. I wish to express my sincere thanks to Professor Araki for the valuable suggestions and continued encouragements he has given in the course of the present work.

Table 5.

x	3.2	2.4	1.6	0.8
$\alpha_{kin.}$	1.28	0.72	0.32	0.08
$\beta_{kin.}$	0.1172	0.1649	0.1530	0.0648
P_0	0.9375	0.4666	0.1472	0.0129
R_0	0.0822	0.0809	0.0486	0.0079
Q_0	0.0396	0.0365	0.0243	0.0052
M_0	0.6263	0.5296	0.3893	0.1955
J_0	0.3455	0.3277	0.2840	0.1820
$J_{\pm 0}$	0.3350	0.3100	0.2715	0.1753
L_0	0.1376	0.1990	0.2334	0.1600

All quantities are given in atomic units.
 $x_0 = 3.2$, $R = 2.5$ atomic units.

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* A full account of recurrence formulas of the integrals will be published elsewhere in connection with other calculations.

Disintegration of Light Nuclei by σ -Meson Capture

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σ -stars in photographic plates are classified into those originated from heavy and light nuclei. The branching ratio of disintegration modes, the α - p ratio, and the energy distribution of emitted particles are measured for light stars. Following characteristic features of light stars are ascertained; 1) the large prong number, 2) the large α - p ratio, 3) the low kinetic energy of emitted particles, and 4) the frequent emission of heavy fragments. The disintegration of C^{12} into two α -particles and a proton are studied in detail, and the possible emission of Be^8 nuclei is discussed. It turns out to be the main disintegration mode, that the rest energy of σ -meson is primarily shared among only a few member of nucleons and the residual part remains in a similar to the excited levels appeared in low energy reactions.

§ 1. Introduction

Since the discovery of two mesons, many authors have investigated the σ -stars in detail. Through the works of Perkins¹⁾ and Menon, Muirhead and Rochat²⁾, we know that the general features of stars from heavy nuclei can satisfactorily be understood by the evaporation theory. While, for σ -stars from light nuclei, our knowledge is left rather obscure.

We have observed 93 light stars out of total 200 σ -stars in Eastman NTA emulsion, which was supplied by the Radiation Laboratory of Berkeley. The branching ratio for various disintegration modes, and the energy distribution of emitted particles were measured. The emission of Be^8 nuclei is examined for ten favourable cases among carbon stars and for further nine examples found in another plate, Ilford C2. A possible interpretation will be given, referring to experimental data on other reactions of light nuclei.

§ 2. Classification of σ -stars

Various methods for the identification of light and heavy stars were proposed by Perkins¹⁾, Heidemann and Lepince-Ringuet³⁾ and others. Here, we adopt similar criteria as were used by Menon, Muirhead and Rochat²⁾. They are;

- 1) If it is accompanied by a slow electron, it is a heavy star.
- 2) If it has a blob, it is a heavy star.
- 3) If it has a recoil track longer than 3μ or two or more short tracks, it is a light star.

4) If it has a particle of sufficiently low energy (proton of $\lesssim 4$ Mev and α -particle of $\lesssim 9$ Mev), it is a light star.

5) If the total charge of a star is definitely $\leq 4e$, it is a heavy star.

Out of total 200 σ -stars with any prong, we were able to identify 93 stars as light and 78 as heavy. Though remaining 29 stars were left undetermined, a considerable part of them could be heavy stars on account of their α - p ratio.

We define the tracks longer than 5μ as prongs, and their prong number distributions are shown in the following table. This distribution is in agreement with that obtained by other authors.

Table 1. Prong Distribution.

No. of prongs	1	2	3	4	5	total	percentage
Light stars	18	30	30	14	1	93	$46.5^{+5.8}_{-5.6}$
Heavy stars	50	20	6	2	0	78	$39.0^{+6.5}_{-5.8}$
Undetermined	12	6	7	3	1	29	$14.5^{+4.9}_{-3.5}$
total	80	56	43	19	2	200	
percentage	40.0 ± 5.8	$28.0^{+5.8}_{-5.3}$	$21.5^{+4.8}_{-4.5}$	$9.5^{+4.0}_{-2.8}$	$1.0^{+2.1}_{-0.6}$		

§ 3. Some characters of light stars

Tracks longer than 10μ are identified by inspection as protons (including deuterons and tritons), α -particles and heavy fragments, whereas some vertical tracks are left undeter-

Table 2. Branching Ratio for Various Disintegration Modes.

Disintegration Mode	Certain	Number of stars Probable	Total	Percentage
$1f$	2	0	2	$2.2^{+4.3}_{-1.3}$
$1p, 1f$	17	0	17	$18.3^{+7.4}_{-5.6}$
$1a, 1f$	10	1	11	$11.8^{+6.9}_{-4.3}$
$2p, 1f$	12	0	12	$12.9^{+7.0}_{-4.6}$
$1p, 2a$	9	6	15	$16.1^{+7.2}_{-5.2}$
$2p, 2a$	4	5	9	$9.7^{+6.2}_{-4.1}$
$3a$	0	7	7	$7.5^{+6.0}_{-3.2}$
$1p, 3a$	3	4	7	$7.5^{+6.0}_{-3.2}$
$3p, 1a$	3	3	6	$6.5^{+5.7}_{-3.0}$
$4p, 1a$	1	0	1	$1.1^{+3.8}_{-0.5}$
Undetermined			6	

mined. Among tracks shorter than 10μ they are attributed to heavy fragments, as far as the conservation of charge compels it. Other short tracks are tentatively assigned as "probable" α 's.

A) The classification of disintegration modes of 93 light stars are given in Table 2. p , α and f denote proton, α -particle and heavy fragment, respectively.

One can suppose that some "probable" α 's are also heavy fragments. Therefore, an unobserved disintegration mode, such as $2f$ may be included in "probable" $1\alpha 1f$, and $1p 1\alpha 1f$ in "probable" $1p 2\alpha$, and so on. While we can show that there are few protons in "probable" α 's, extrapolating the range distribution for protons.

B) The α - p ratio is found to be

$$\alpha/p = 106/103 = 1.03 \pm 0.20,$$

including "probable" α 's. We can in a less ambiguous way obtain the ratio of the number of α -particles and heavy fragments to that of protons, i. e.,

$$(\alpha + f)/p = 148/103 = 1.44 \pm 0.26.$$

These figures should be compared with the results of Menon, Muirhead and Rochat, who obtained $\alpha/p = 263/271$ and $(\alpha + f)/p = 1.55 \pm 0.20$.

C) The range distributions are obtained for stopped protons and α -particles. We get their energy distributions, correcting for the fraction of escaping from the emulsion by geometrical considerations. They are shown in Fig. 1 and 2. The range distribution of heavy fragments are also shown in Fig. 3.

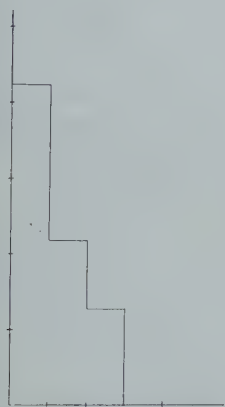


Fig. 1. Energy distribution of protons.



Fig. 2. Energy distribution of α -particles. The broken line indicates the result excluding "probable" α 's. The solid curve is an approximate analytic one, i. e.

$$\sqrt{E} \exp(-E/\epsilon) \text{ with } \epsilon = 4.8 \text{ Mev.}$$

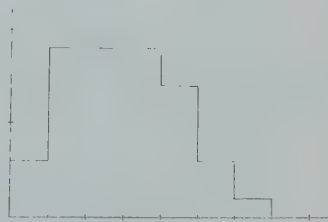


Fig. 3. Range distribution of heavy fragments.

§ 4. Be^8 emission from carbon star

In some case such as $1p2\alpha$ and 3α , we can make identification of the disintegration of particular nuclei. We examined in detail the disintegration of C^{12} nucleus into $1p2\alpha$.

We picked up ten favorable cases, where emitted two α -particles stop in emulsion. We can measure the range, i. e., the energy of two α -particles and the angle between them. Now, we calculate the energies of the relative and the center of mass motion of them. Table 3 shows the results for ten cases.

Table 3. $1p2\alpha$ Stars in NTA Emulsion.

Star No.	Energy of α -particles (Mev)		Cosine of angle between them	Energy of center of mass motion (Mev)	Energy of relative motion (Mev)
186	4.0	3.9	0.995	7.8	0.1
70	8.95	6.3	0.63	12.4	2.85
118	7.4	1.6*	0.45	6.0	2.95
115	5.7	1.4*	0.08	3.8	3.3
65	8.2	3.7	-0.16	5.0	6.9
201	4.4	3.5	-0.91	0.4	7.5
98	9.1	3.5	-0.83	1.6	9.3
55	9.4	5.0	-0.66	2.7	11.7
126	16	2.7	-0.50	5.05	13.6
154	12.8	7.2	-0.88	1.4	18.6

(* These are included in "probable" α 's)

To obtain better statistics, we analyzed further nine $1p2\alpha$ stars out of 200 σ -stars found in Ilford C2 emulsion, and the result is shown in Table 4.

Table 4. $1p2\alpha$ Stars in C2 Emulsion.

Star No.	Energy of α -particles (Mev)		Cosine of angle between them	Energy of center of mass motion (Mev)	Energy of relative motion (Mev)
1095	4.2	4.2	0.99	8.36	0.04
1073	6.0	0.8*	0.20	3.8	3.0
1087	2.5	2.3	-0.73	0.6	4.2
1172	12.7	1.9*	0.08	7.7	6.9
1008	8.6	2.4	-0.57	2.9	8.1
1096	10.2	0.8*	-0.95	2.8	8.2
1040	8.7	1.9*	-0.89	1.7	8.9
1075	9.0	4.4	-0.95	0.7	12.7
1039	10.6	5.4	-0.97	0.7	15.3

(* These are included in "probable" α 's)

If one assumes that the two α -particles are emitted in the form of unstable Be^8 nuclei,

the energy of relative motion should be identified to a some known energy level of Be^8 nuclei. To see whether it be true or not, we made a possible correspondence, as shown in Fig. 4. We can find, in this figure, a well-known ground level and first excited one at 3 Mev.

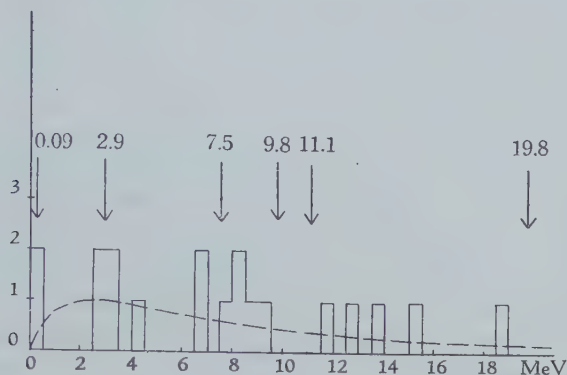


Fig. 4. Distribution of energy of relative motion of two α 's in carbon stars. The arrows indicate the known energy levels which disintegrate into two α -particles⁽¹⁾. The broken curve is the calculated distribution with the assumption of no correlation between two α 's. The energy spectrum of α -particles is assumed as

$$\sqrt{E} \exp(-E/\epsilon) \quad \text{with } \epsilon = 4.8 \text{ Mev,}$$

which is shown in Fig. 2.

Our measured values on the energy of relative motion have an error of at most 0.5 Mev in the most unfavourable cases, while only a few lowest excited energy levels of Be^8 nuclei are unambiguously established from experiments on artificial reactions. Therefore, the above assumption does not seem contradict with the experiments, especially in cases of the small energy of relative motion.

The energy of the center of mass motion should be compared with the kinetic energy of stable heavy fragments. The similarity of both energy distributions makes the assumption more plausible.

§ 5. Discussions

Light σ -stars have very different features from those of heavy ones. The remarkable points of the former case are;

- 1) Large prong number,
- 2) Large α - p ratio,
- 3) Low kinetic energy of emitted α -particles and protons,
- 4) Frequent emission of stable as well as unstable heavy fragments.

One may assume that most part of rest energy of σ -meson is distributed among all nucleons and an equilibrium state is reached in some instance in the nucleus. This hypothesis, combining with the evaporation theory, can satisfactorily explain the main features of heavy σ -stars. But, when it is applied to light nuclei, we at once encounter serious difficulties. For example, the expected mean energy of emitted particles is much

smaller than the observed one.

Therefore, we have to think that a σ -meson is absorbed by some small cluster of nucleons in nuclei, which Menon, Muirhead and Rochat named as "primary process"²⁾. This primary process results in the emission of a few high energy nucleons, which easily escape from a nucleus due to its transparency. Then the rest part of the nucleus receives only a small disturbance, and it can be supposed to be similar to the excited states reached by low energy reactions.

The above consideration can schematically be understood in terms of the α -particle model for light nuclei. In the case C^{12} , a σ -meson is absorbed by one α -cluster in the nucleus, which is completely disrupted by this primary interaction. Due to this disturbance, the rest two α -clusters are left behind in a state similar to an excited Be^8 . Subsequently it disintegrates into two α -particles or proton and Li^7 , etc.

Menon, Muirhead and Rochat have found a few cases out of thousands stars, where the total rest energy seems to be completely disparted throughout the nucleus³⁾, though we have observed no such cases. Thus we may think that the aforesaid is the main disintegration mode of meson-absorbing nuclei.

It is regrettable that our knowledge on other high energy reactions of light nuclei is more obscure than this σ -star case. Gardner and Peterson observed the prong number distribution of stars induced by deuterons of 35~190 Mev.⁵⁾ We can suppose that these stars are mostly originated from light nuclei. They showed that the prong distribution does not vary with bombarding energy and is similar to that of light σ -stars.

Following our point of view, various high energy disintegration of light nuclei should have similar prong distribution, α - p ratio and energy distribution. The difference of kind of agent should appear only in primary interaction, i. e. in fast protons of stars.

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On the Renormalization Theory of the Interaction of Electrons and Photons

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Using the transformations of Lagrangian, the connection among existing renormalization theories of the interaction of electrons and photons is examined.

§ 1. Introduction

The S-matrix in a system composed of electrons and photons was investigated by several authors¹⁾ and various proposals were made by them to obtain finite results free from any divergence. Dyson²⁾ formulated the calculation of S-matrix in a power series of e by dropping divergent terms which appeared due to sub-integrations*. This corresponds to introduce counter terms into the interaction Hamiltonian as we shall show later. In his paper it was also shown an alternative possibility to eliminate divergent terms by replacing e in the S-matrix with the renormalized charge e_1 . Hereafter we shall call these two methods by Dyson's first and second formulation respectively. Gupta³⁾ and Kamefuchi** also proposed other formulations which differ in the counter terms with Dyson's formulations. As it was proved in each papers, all these methods give the same S-matrix free from divergency. But, here, we shall show the equivalence of them more simply by considering transformation of Lagrangian. Throughout this paper we use the concepts of reducible, irreducible, proper and improper graphs introduced by Dyson¹⁾ and of subintegration and true-divergence introduced by Salam.⁴⁾ In § 2 we determine the magnitudes of divergent constants which appear in the calculation of S-matrix and construct the total Lagrangian which gives the divergent free S-matrix. In § 3, by bringing the Lagrangian into different forms through transformations of field variables we prove the equivalence of different formalisms of the previous authors.

§ 2. Calculation of divergent constants

We shall calculate the divergent quantities to be added to the usual interaction Hamiltonian for making the S-matrix finite. As a standard formulation, we adopt Dyson's first one and calculate them. But this choice is only conventional and the divergent quantities appearing in other formulations can be deduced very easily from them as it will be shown

* We use the natural unit $\hbar=c=1$.

** A private communication by him.

in § 3. There are three types of divergence appearing in the S-matrix, that is, vertex type, electron self-energy type and photon self-energy type. Calculations are made separately for them and very simply by using Salam's method.⁶⁾

(i) *Divergence of vertex type*

We divide Feynman's diagrams of the vertex type into classes $V_0, V_1, \dots, V_n, \dots$.

V_0 : a single vertex graph without radiative corrections. Fig. 1.

V_1 : the family of vertex graphs obtained by inserting all possible irreducible graphs in a diagram A_1 of Fig. 2.

V_n : the family of vertex graphs obtained by inserting all possible irreducible graphs in diagrams A_1, A_2, \dots, A_n of Fig. 3.

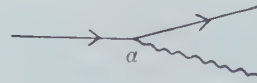


Fig. 1



Fig. 2



Fig. 3

The rule for calculating the vertex part of S-matrix is as follows:

$$\left\{ \begin{array}{l} \text{a vertex gives a factor } \gamma_\mu, \\ \text{Each vertex other than a vertex gives a factor } \Gamma'_{\mu 1}, \\ \text{Each internal electron line gives a factor } S'_{F1}, \\ \text{Each internal photon line gives a factor } D'_{F1}. \end{array} \right. \quad (\text{I})$$

$\Gamma'_{\mu 1}$, S'_{F1} and D'_{F1} are the finite quantities used in Dyson's paper⁷⁾ and to be determined successively in a power series of e as we shall show later. Following Salam's method we calculate the contribution of each class of vertex graphs to the S-matrix and the results are

$$\begin{aligned} \Gamma_\mu[0] &= \gamma_\mu, \\ \Gamma_\mu[1] &= \Gamma'_{\mu 1}[1] + L_1 \gamma_\mu, \\ \Gamma_\mu[2] &= \Gamma'_{\mu 1}[2] + L_1 \Gamma'_{\mu 1}[1] + L_2 \gamma_\mu, \\ &\dots \dots \dots \\ \Gamma_\mu[n] &= \Gamma'_{\mu 1}[n] + L_1 \Gamma'_{\mu 1}[n-1] + L_2 \Gamma'_{\mu 1}[n-2] + \dots + L_n \gamma_\mu, \\ &\dots \dots \dots \end{aligned} \quad (1)$$

where $L_n \gamma_\mu$ is the true divergence of a class V_n and $\Gamma'_{\mu 1}[n]$ is the finite part of the contribution of V_n .

Adding those equations for different classes, we obtain

$$\Gamma_\mu = \Gamma'_{\mu 1} + L \cdot \Gamma_\mu, \quad (2)$$

where

$$I'_{\mu 1} = \gamma_\mu + \sum_{n=1}^{\infty} I'_{\mu 1}[n], \quad I'_\mu = \sum_{n=0}^{\infty} I'_\mu[n]. \quad (3)$$

To eliminate the divergent part $L \cdot I'_\mu$ from the right-hand side of the equation (2), we introduce a counter term

$$\delta H = i\bar{\psi} \cdot L A_\mu \bar{\psi} \gamma_\mu \psi \quad (4)$$

into the interaction Hamiltonian density. Then, the contribution of vertex graphs in all have the finite expression $I'_{\mu 1}$.

(ii) *Divergence of electron self-energy type*

We divide Feynman's graphs of the electron self-energy type into classes $W_1, W_2, \dots, W_n, \dots$.

W_1 : a single electron self-energy graph indicated in Fig. 4.

W_2 : the family of electron self-energy graphs obtained by inserting all possible irreducible graphs in a diagram A_1 of Fig. 5.

W_n : the family of electron self-energy graphs obtained by inserting all possible irreducible graphs in diagrams A_1, A_2, \dots, A_{n-1} of Fig. 6.

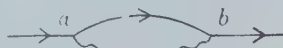


Fig. 4



Fig. 5



Fig. 6

In this case, corresponding to the rules (I),

$$\left\{ \begin{array}{l} \text{a or b vertex gives a factor } \gamma_\mu. \\ \text{Each vertex other than a and b gives a factor } I'_{\mu 1}. \\ \text{Each internal electron line gives a factor } S'_{F1}. \\ \text{Each internal photon line gives a factor } D'_{F1}. \end{array} \right. \quad (II)$$

Then, calculating by Salam's method, the contribution \sum_n of each class W_n to the S-matrix becomes

$$\begin{aligned} \sum_n &= L_{a,1} \sum_{n-1} + L_{a,2} \sum_{n-2} + \dots + L_{a,n-1} \sum_1 \\ &\quad + L_{b,1} \sum_{n-1} + L_{b,2} \sum_{n-2} + \dots + L_{b,n-1} \sum_1 \\ &\quad - \{ L_{a,1} L_{b,1} \sum_{n-2} + L_{a,1} L_{b,2} \sum_{n-3} + \dots \} + \sum_n^* \end{aligned} \quad (5)$$

where $L_{a,i}$ and $L_{b,i}$ are just the same with L_i and the index a or b of them indicates the vertex point which relates with this divergency. Adding \sum_n' s with different n , we have from the equation (5)

$$\Sigma = 2L \cdot \Sigma - L^2 \cdot \Sigma + \Sigma^*,$$

where Σ or Σ^* is the sum of Σ_n or Σ_n^* with respect to n .

The relation between Σ and Σ^* may be written as

$$\Sigma^* = (1 - L)^2 \Sigma. \tag{7}$$

As Σ^* contains the true divergence of W-graphs which occurs due to the integration of all basic variables as a whole, it may be written as

$$\Sigma^* = A + B(\gamma_\mu t_\mu - im) + (\gamma_\mu t_\mu - im) S_C(t). \tag{8}$$

A and B are constants which diverge linearly and logarithmically respectively. By the additional Hamiltonian density (4) the electron self-energy part is free from the L divergence, but to eliminate A and B divergencies we must introduce

$$\delta H = -\delta m \bar{\psi} \psi + \frac{B}{2\pi} \bar{\psi} (\gamma_\mu \partial_\mu - m) \psi \quad \left(\delta m = \frac{A}{2\pi i} \right) \tag{9}$$

into the interaction Hamiltonian density. This procedure gives the finite result $(\gamma_\mu t_\mu - im) S_C(t)$ for the self-energy part of an electron with a momentum t_μ which contains only radiative corrections due to proper graphs. The finite factor $S'_{F1}(t)$ for each internal electron line is given by the equation

$$S'_{F1}(t) = S_F(t) + \frac{1}{2\pi} S_C(t) S'_{F1}(t), \tag{10}$$

which takes account of the radiative corrections due to improper graphs.

(iii) *Divergence of photon self-energy type*

Classifying Feynman's graphs of photon self-energy type like (ii) and using the rules (II), we obtain similar relations as (6) and (7). H^* corresponding to Σ^* yet includes the true divergence of these graphs and has the form

$$H^* = C \cdot k_\mu^2 + k_\mu^2 D_C(k), \tag{11}$$

where the invariancy of H^* to the gauge transformation is assumed. C is a divergent constant and $D_C(k)$ is a finite expression of k_μ .

The additional Hamiltonian density to be required for cancelling this divergent part is

$$\delta H = \frac{1}{4} \frac{C}{2\pi i} F_{\mu\nu} F_{\mu\nu}. \tag{12}$$

The finite factor D'_{F1} for each internal photon line which includes radiative corrections due to improper graphs is given by

$$D'_{F1}(k) = D_F(k) + \frac{1}{2\pi i} D_C(k) D'_{F1}(k). \tag{13}$$

Thus we have determined the additional terms (4), (9) and (12) to the interaction Hamiltonian density for obtaining the divergent free S-matrix.

As we have stated in the rules (I) and (II), we need the knowledge of $\Gamma_{\mu 1}$, S'_{F1}

and D'_{F1} before calculation of these divergent terms. But in our perturbational calculation, $I'_{\mu 1}$, S'_{F1} and D'_{F1} determined to order ϵ^n will suffice to determine them to order ϵ^{n+1} . So $I'_{\mu 1}$, S'_{F1} and D'_{F1} may be calculated by a process of successive approximation, starting from zero-order values γ_μ , S_F and D_F .

§ 3. Formulations of divergent free S-matrix

In § 2 we have shown that the S-matrix can be made free from any divergency by introducing additive terms (4), (9) and (12) to the interaction Hamiltonian density. Exactly speaking, we must also add normal dependent terms because the time derivatives of field variables A_μ and ψ appear in the expressions (9) and (12). To avoid this complexity, we shall write them as the additive terms to the Lagrangian density in the Heisenberg representation. They become

$$\delta L = -ieLA_\mu\bar{\psi}\gamma_\mu\psi + \partial m\bar{\psi}\psi - \frac{B}{2\pi}\bar{\psi}(\gamma_\mu\partial_\mu - m)\psi - \frac{1}{4}\frac{C}{2\pi i}F_{\mu\nu}F_{\mu\nu} \quad (14)$$

without any normal dependent terms,* while the original one is

$$L^\sim = L_0 + L_1, \quad (15)$$

where

$$L_0 = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}\left(\frac{\partial A_\mu}{\partial x_\mu}\right)^2 - \bar{\psi}(\gamma_\mu\partial_\mu - m)\psi, \quad (16)$$

$$L_1 = ieA_\mu\bar{\psi}\gamma_\mu\psi. \quad (17)$$

Adding L^\sim and δL , we obtain the total Lagrangian density L

$$L = -Z_3\left\{\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\frac{\partial A_\mu}{\partial x_\mu}\right)^2\right\} - Z_2\bar{\psi}(\gamma_\mu\partial_\mu - m)\psi + \partial m\bar{\psi}\psi + ieZ_1A_\mu\bar{\psi}\gamma_\mu\psi, \quad (A)$$

where

$$\begin{cases} Z_1 = 1 - L, \\ Z_2 = 1 + \frac{B}{2\pi}, \\ Z_3 = 1 + \frac{C}{2\pi i}. \end{cases} \quad (18)$$

Note that L is not exactly equal to the sum of L^\sim and δL , but the difference

$$(Z_3 - 1)\frac{1}{2}\left(\frac{\partial A_\mu}{\partial x_\mu}\right)^2 \quad (19)$$

will disappear by the Lorentz condition for the wave function and have no effect to the

* This point was investigated extensively by Prof. Y. Nambu. See his paper Prog. Theor. Phys. Vol. 7 (1952), 131.

results. Regarding L_0 as the free Lagrangian and transforming the representation to the interaction one, the resultant Hamiltonian guarantees the finiteness of S-matrix which is evident from the above discussion.

If the divergent constants Z_1 , Z_2 , Z_3 and δm are properly estimated, the total Lagrangian L must be invariant to the Gauge transformation

$$\begin{aligned} \phi &\rightarrow \exp[-i\epsilon A]\phi, & \bar{\psi} &\rightarrow \exp[i\epsilon A]\bar{\psi}, \\ A_\mu &\rightarrow A_\mu - \frac{\partial A}{\partial x_\mu}, \end{aligned} \quad (20)$$

where

$$\square A = 0. \quad (21)$$

That is, the additional terms to the Lagrangian (A) caused by the transformation (21)

$$i\epsilon(Z_1 - Z_2) \frac{\partial A}{\partial x_\mu} \bar{\psi} \gamma_\mu \psi - Z_3 \left\{ \frac{\partial}{\partial x_\mu} \left(A_\nu + \frac{1}{2} \frac{\partial A}{\partial x_\nu} \right) \frac{\partial^2 A}{\partial x_\mu \partial x_\nu} + \left(A_\nu + \frac{1}{2} \frac{\partial A}{\partial x_\nu} \right) \frac{\partial}{\partial x_\nu} \square A \right\} \quad (22)$$

must vanish or reduce to a four dimensional divergent quantity. The second term satisfies this requirement because it only differs in the factor Z_3 from the equation in Schwinger's paper.⁸⁾ Then we have the identity

$$Z_1 = Z_2, \quad (23)$$

which was proved by Ward⁹⁾ before.

In the expression (A) we change the normalization of the field variables as follows:

$$\begin{cases} Z_2^{1/2} \phi \rightarrow \phi, & Z_2^{1/2} \bar{\psi} \rightarrow \bar{\psi}, \\ Z_3^{1/2} A_\mu \rightarrow A_\mu. \end{cases} \quad (24)$$

The transformed Lagrangian density becomes

$$L = L_0 + \delta m \cdot Z_2^{-1} \bar{\psi} \psi + i\epsilon Z_1 Z_2^{-1} Z_3^{-1/2} A_\mu \bar{\psi} \gamma_\mu \psi, \quad (B)$$

or by the identity (24) it reduces to

$$L = L_0 + \delta m \cdot Z_2^{-1} \bar{\psi} \psi + i\epsilon Z_3^{-1/2} A_\mu \bar{\psi} \gamma_\mu \psi. \quad (B')$$

Regarding L_0 as the free Lagrangian and going to the interaction representation, we get the interaction Hamiltonian density

$$H = -i\epsilon Z_3^{-1/2} A_\mu \bar{\psi} \gamma_\mu \psi - \delta m Z_2^{-1} \bar{\psi} \psi. \quad (25)$$

This formulation corresponds to Dyson's second one and also gives the finite expression in ϵ for S-matrix. That is, if we calculate the S-matrix with the ordinary interaction Hamiltonian density

$$H = -i\epsilon_1 A_\mu \bar{\psi} \gamma_\mu \psi \quad (26)$$

subtracting only the mass renormalization term, the resulted S-matrix have the divergent expression in a power series of ϵ_1 . But regarding ϵ_1 as the divergent quantity expressed by

$$e_1 = e Z_3^{-1/2} \quad (27)$$

and rearranging the S-matrix in a power series of e , we have the divergent free S-matrix. The equivalence of Dyson's two methods was proved in a different way by him, while here it was done through the transformation of Lagrangian.

If we change only the normalization of ψ and $\bar{\psi}$ in the expression (A)

$$Z_2^{1/2} \psi \rightarrow \psi, \quad Z_2^{1/2} \bar{\psi} \rightarrow \bar{\psi}, \quad (28)$$

it follows

$$L = L_0 - (Z_3 - 1) \left\{ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} \left(\frac{\partial A_\mu}{\partial x_\mu} \right)^2 \right\} + \delta m \cdot Z_2^{-1} \bar{\psi} \psi + i e A_\mu \bar{\psi} \gamma_\mu \psi. \quad (C)$$

Regarding L_0 as the free Lagrangian and going to the interaction representation, we have Gupta's formalism whose favourable point is the appearance of charge e with no divergent constant due to the identity (23).

If we take the first and the second terms of the right hand side of the equation (C) as the free Lagrangian instead of L_0 and go to the interaction representation, we have Kamefuchi's Hamiltonian

$$H = -i e A_\mu \bar{\psi} \gamma_\mu \psi - \delta m \cdot Z_2^{-1} \bar{\psi} \psi, \quad (D)$$

where the commutation relations between the field variables are

$$Z_3 [A_\mu(x), A_\nu(x')] = i \delta_{\mu\nu} D(x - x'). \quad (29)$$

To obtain the ordinary commutation relations between the field variables we have only to change the normalization of A_μ

$$Z_3^{1/2} A_\mu \rightarrow A_\mu \quad (30)$$

and then his method becomes identical with the formalism (B) or (B').

We have discussed the different formalisms (A), (B), (C) and (D). They can be led from the same Lagrangian (A) and have the same commutation relations between the field variables in their interaction representation, that is, each formalism is related to others by a canonical transformation. So we can easily show that there exists a suitable unitary transformation $U(t)$ for each two formalisms, by which the S-matrices of them are related with each other as follows:

$$S_1 = U^\dagger(\infty) S_2 U(-\infty), \quad (31)$$

where S_1 and S_2 are the matrix of these two formalisms respectively.

As the difference of these two formalisms at $t = \infty$ and $t = -\infty$ is in the normalization of the field variables, $U^\dagger(\infty)$ and $U(-\infty)$ become I except a phase factor if the wave functions of initial and final states are properly chosen. Thus we have

$$S_1 = S_2 \quad (32)$$

and the divergent free character of the S-matrices in the different formalisms are proved.

Our discussion have started from Dyson's first formalism, but, of course, we can start

from another formalism and calculate the divergent constants by this.

Moreover, we can construct new formalisms other than those which were discussed above. For example, by the transformation

$$Z_3^{1/2} A_\mu \rightarrow A_\mu, \quad (33)$$

the total Lagrangian density (A) changes into

$$L_1 = L_0 - (Z_2 - 1) \bar{\psi}(\gamma_\mu \partial_\mu - m)\psi + \partial_m \bar{\psi} \psi + ie Z_1 Z_3^{-1/2} A_\mu \bar{\psi} \gamma_\mu \psi. \quad (E)$$

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On Relativistic Theory of Rotating Disk

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From group-theoretical point of view, using the composition law of velocities in special relativity, a relativistic theory of rotating disk is established by an axiomatic method. In this theory, first of all, transformation formula from galilean frame to disk frame is obtained and then, basing upon this, the velocity distribution and the intrinsic geometry of the disk are determined. The distribution satisfies the linear law in the limit of $\rho\omega/c \rightarrow 0$, although it is not the case in general. The geometry is not euclidean and its gaussian curvature is negative in spite of the fact that the proper circumference of any circle with center 0 (the center of rotation), is obtained in euclidean manner.

§ 1. Introduction

Since the appearance of the special theory of relativity in 1905 various relativity-theoretical investigations concerning the problem of the rotating disk have been made by many authors i.e. Ehrenfest¹⁾, Einstein and Infeld²⁾, Eddington³⁾, Berenda⁴⁾, Hill⁵⁾, Rosen⁶⁾, Clark⁷⁾, Kurşunoglu⁸⁾, etc. The problems treated by these authors can be classified into the following two main categories though closely connected with each other:

(I) *The problem of determining the geometry of the rotating disk* In connection with this, Berenda clarified that the 'intrinsic' geometry should be distinguished from the 'relative' one⁴⁾.

(II) *The problem of defining the uniform rotation of the disk* This problem has been investigated by generalizing relativistically some characteristic properties concerning the rotation of classical rigid body or incompressible fluid. In connection with this problem the law of velocity distribution of uniformly rotating disk is especially important.

In the present stage, no decisive answer to these problems seems to exist. Since we are particularly interested in the problem of the velocity distribution and the contents of this paper are closely connected with it, we shall explain it somewhat in detail. Taking, now, Minkowski space-time M , its line element in a cylindrical coordinate system is written as

$$ds^2 = -d\rho^2 - \rho^2 d\phi^2 - dz^2 + c^2 dt^2, \quad (1.1)$$

where c is the light velocity. Let D be a disk rotating uniformly about z -axis in (ρ, ϕ) -plane, and T be the transformation from the non-rotating galilean system (1.1), (galilean frame), to the reference system at rest relative to D i.e. rotating reference system, (disk

frame). Eddington, Berenda, Rosen, Clark, etc. adopted, as the equation for transformation T, the ordinary relation

$$\rho' = \rho, \quad \phi' = \phi - \omega t, \quad z' = z, \quad t' = t. \quad (1.2)$$

This implies that they admit tacitly the linear speed-distance relation for rotating disk i.e. the linear law of the velocity distribution. Although a theoretical foundation of this 'linear law' was given by Rosen, Hill opposed to Rosen's point of view and gave a suggestion that the final solution of the problem will be given by some group-theoretical considerations.⁹⁾

In this paper, adopting the group-theoretical point of view suggested by Hill, we shall first try to give a new definition of uniform rotation and shall deduce a new transformation equation for T in place of (1.2). Then by making use of this equation, we shall introduce the velocity distribution law of our theory, shall give some properties of our 'uniform rotation', shall determine the four-dimensional line element of our rotating system and the intrinsic geometry of the rotating disk, and shall study about their properties.

§ 2. Uniform rotation

We shall use, in the following, the notation $U(\omega)$ for the phrase 'uniform rotation with constant angular velocity ω ' for brevity's sake. Let D be a disk (or system of points or 'fluid') rotating about z -axis in (ρ, ϕ) -plane in M. Now in order to introduce a new definition for $U(\omega)$ of D, we start with the following group-theoretical assumption:

(A₁) $U(\omega)$'s constitute a group \mathfrak{G} with a continuous parameter ω . The unit element of \mathfrak{G} is given by $U(0)$.

From this assumption we know that if D_2 makes $U(\omega)$ relative to D_1 (i.e. galilean disk) and D_3 makes $U(\bar{\omega})$ relative to D_2 , then D_3 makes $U(\bar{\bar{\omega}})$ relative to D_1 . As will be seen in § 3, $\bar{\bar{\omega}}$ is given by

$$\bar{\bar{\omega}} = \omega + \bar{\omega}, \quad (2.1)$$

and accordingly the inverse element of $U(\omega)$ is given by $U(-\omega)$.

Let any system of rectangular axes of D_1 contain Ox and Oy and the polar coordinates with initial line Ox be (ρ, ϕ) . We assume that D_2 makes $U(\omega)$ relative to D_1 and D_3 makes $U(\bar{\omega})$ relative to D_2 . Take any point $A_1(x, y)$ except O on D_1 , and let two points on D_2 and D_3 which coincide with A_1 at a certain instant be A_2 and A_3 respectively. If we denote the three-dimensional velocity by $(v^1, v^2, 0)$ and its speed by v , we have

$$v^1 = -v \sin \phi = -y \xi, \quad v^2 = v \cos \phi = x \xi, \quad (2.2)$$

where $\xi = v/\rho$ is the 'angular velocity' of A_2 .

Concerning this ξ , we put the assumption:

(A₁₁) ξ is a function of ρ and ω , and for sufficiently small $\rho\omega/c$ it coincides with ω neglecting higher order infinitesimals.

From this assumption, v is also a function of ρ and ω and each particle on the circumference of any circle with center O rotates at the same speed $v(\rho, \omega)$. Furthermore in case of infinitesimal $U(\omega)$ the linear law holds approximately for finite ρ .

Now, if we denote the quantities v, ξ, \dots of A_3 relative to D_2 and D_1 by $\bar{v}, \bar{\xi}, \dots$ and $\bar{v}, \bar{\xi}, \dots$ respectively, similar relations to (2.2) also hold for these quantities. Then, at the instant of the coincidence of A_1, A_2 and A_3 , A_3 moves uniformly at speeds \bar{v} and \bar{v} relative to D_1 and D_2 respectively and A_2 moves uniformly at speed v relative to D_1 in the same direction with A_3 . Hence we assume that:

(A_{III}) \bar{v} is obtained from v and \bar{v} according to the composition law in special relativity, i.e.,

$$\bar{v} = (v + \bar{v}) / (1 + v\bar{v}/c^2). \quad (2.3)$$

This assumption is consistent with the investigation of Metz concerning Sagnac's experiment¹⁰. From (2.3) we have the following law of composition of angular velocities:

$$\bar{\xi} = (\xi + \bar{\xi}) / (1 + \rho^2 \xi \bar{\xi} / c^2). \quad (2.4)$$

§ 3. Transformation equation

As stated in § 1, Eddington, Berenda, etc. used (1.2) as the transformation equation between the reference systems (ρ, ϕ, z, t) of D_1 and (ρ', ϕ', z', t') of D_2 . In this case from

$$\bar{\xi} = d\phi/dt, \quad \bar{\xi} = d\phi'/dt', \quad (3.1)$$

we have

$$\bar{\xi} = \omega + \bar{\xi}, \quad (3.2)$$

for the motion of A_3 , and therefore it is evident that (2.4) can not hold except for $c = \infty$.

In this section, on the basis of (2.4), we look for a new transformation equation in place of (1.2). For this purpose, as a first trial, we put the following further assumption:

$$(A_{IV}) \quad \rho' = \rho, \quad z' = z. \quad (3.3)$$

Though this seems to be somewhat artificial, it is natural from the standpoint of special relativity because both directions of ρ and z are perpendicular to that of motion.

Now, in the first place, we treat the case of infinitesimal transformation assuming ω to be small. Putting

$$x'^i = x^i + \omega \eta^i, \quad (x^1, \dots, x^4 = \rho, \phi, z, t), \quad (3.4)$$

we have from (3.3)

$$\eta^1 = \eta^3 = 0, \quad (3.5)$$

and from (3.1)

$$\bar{\xi} = \left\{ \bar{\xi} + \omega \left(\bar{\xi} \frac{\partial \eta^0}{\partial \phi} + \frac{\partial \eta^0}{\partial t} \right) \right\} / \left\{ 1 + \omega \left(\bar{\xi} \frac{\partial \eta^4}{\partial \phi} + \frac{\partial \eta^4}{\partial t} \right) \right\}. \quad (3.6)$$

On the other hand, from (A₁₁) and (2.4) we have

$$\bar{\xi} = (\bar{\xi} - \omega) / (1 - \rho^2 \bar{\xi} \omega / c^2), \quad (3.7)$$

from which we obtain

$$\frac{\partial \eta^2}{\partial \phi} = 0, \quad \frac{\partial \eta^2}{\partial t} = -1; \quad \frac{\partial \eta^4}{\partial \phi} = -\frac{\rho^2}{c^2}, \quad \frac{\partial \eta^4}{\partial t} = 0. \quad (3.8)$$

Integrating these equations and taking $t' = \phi' = 0$ for $t = \phi = 0$ initially, we have

$$\eta^2 = -t, \quad \eta^4 = -\rho^2 \phi / c^2. \quad (3.9)$$

Accordingly the symbol of the infinitesimal operator of this transformation is

$$X = -\{t \partial_\phi + (\rho^2 \phi / c^2) \partial_t\}. \quad (3.10)$$

The finite form of this transformation is given by integrating

$$d\phi' / d\omega = -t', \quad dt' / d\omega = -\rho^2 \phi' / c^2, \quad (3.11)$$

under the condition that $\phi' = \phi$ and $t' = t$ for $\omega = 0$. This becomes

$$T(\omega) : \begin{cases} \rho' = \rho, & z' = z, \\ \phi' = \{\phi - (ct/\rho) \tanh \lambda\} \cosh \lambda, \\ t' = \{t - (\rho\phi/c) \tanh \lambda\} \cosh \lambda, & (\lambda = \rho\omega/c). \end{cases} \quad (3.12)$$

Evidently $T(\omega)$'s constitute a group with parameter ω and if we put $T(\bar{\omega})T(\omega) = T(\bar{\omega})$, $\bar{\omega}$ is given by (2.1). When $\lambda \rightarrow 0$, $T(\omega)$ reduces to (1.2).

Here if we put

$$\rho\phi = \Phi, \quad ict = T, \quad \lambda = iA, \quad (3.13)$$

(3.12) becomes

$$\rho' = \rho, \quad z' = z, \quad \Phi' = \Phi \cos A - T \sin A, \quad T' = \Phi \sin A + T \cos A. \quad (3.14)$$

In other words, $T(\omega)$ can be regarded as a rotation in the (Φ, T) -plane. This situation corresponds to the fact that the ordinary Lorentz transformation is a rotation in the (x, ict) -plane. Mathematically, however, $T(\omega)$ differs from Lorentz transformation in that the 'angle' A of rotation is not a constant but a function of ρ .

As will be made clear in § 6, Minkowski metric is transformed into non-static form by $T(\omega)$. On the other hand, in his research concerning the new transformation equation which may take the place of (1.2), Rosen assumed that the transformed metric to be static⁽⁶⁾. This is the reason why our $T(\omega)$ did not appear in his research.

§ 4. Velocity distribution

(3.12) shows that the disk D_2 makes $U(\omega)$ relative to D_1 . Hence the expression

$$\phi = (ct/\rho) \tanh \lambda + \alpha / \cosh \lambda, \quad (4.1)$$

which is obtained from (3.12) by putting $\phi' = \text{const.} = \alpha$, gives $U(\omega)$ -motion of a particle on the circumference of any circle with center O and radius ρ . Hence the particles on the OX -axis at $t=0$ come to be distributed on a curve defined by (4.1) at time t . This 'spiral' differs from the ordinary logarithmic one in that the number of rolls is finite.

From (4.1) we have

$$\xi = (c/\rho) \tanh \lambda, \quad \text{hence} \quad v = c \tanh \lambda, \quad (4.2)$$

which gives the velocity distribution law of our $U(\omega)$. Therefore when λ is small the linear speed-distance relation holds for our $U(\omega)$ approximately and when $\rho \rightarrow \infty$ we have $v \rightarrow c$. This shows that our $U(\omega)$ satisfies Hill's 'a priori criteria'.

From (4.2) and (3.12), we can rewrite $T(\omega)$ in the form

$$\rho' = \rho, \quad z' = z, \quad \phi' = a(\phi - \xi t), \quad t' = a(t - \rho^2 \xi \phi / c^2), \quad (3.3)$$

$$\text{where} \quad a \equiv (1 - \rho^2 \xi^2 / c^2)^{-1/2}. \quad (4.4)$$

This corresponds to the ordinary form of Lorentz transformation.

By the above consideration our velocity distribution law is made clear without touching directly on the physical properties, e.g., rigidity or incompressibility of the disk.

§ 5. Some properties of our $U(\omega)$

In this section we shall study some properties of our $U(\omega)$ by comparing it with those of Hill and Rosen.

(a) *Comparison with $U(\omega)$ of Hill⁽⁵⁾* Let D_2 be a disk rotating arbitrarily with center O relative to the galilean frame D_1 and B be any point of D_2 with $OB = \rho$. Further taking B as origin, let (x', y', z') be the coordinates of the inertial system momentarily at rest relative to D_2 , y' -axis being Oy , at the instant when B passes through y -axis of D_1 . Then three-dimensional velocity components of any point of D_2 with respect to this inertial system are given by

$$v^{x'} = -y' \left[\frac{\xi + \rho d\xi/d\rho}{1 - v^2/c^2} \right]_0, \quad v^{y'} = x' \left[\frac{\xi}{1 - v^2/c^2} \right]_0, \quad v^{z'} = 0, \quad (5.1)$$

as shown by Hill using Lorentz transformation, where bracketed expressions denote their values at B . Adopting Hill's nomenclatures we shall call $\mathcal{Q}' = \frac{1}{2} \text{curl}' \mathbf{v}'$, $V' \equiv \text{div}' \mathbf{v}'$ and $\gamma' \equiv \frac{1}{2} (\partial v^{x'}/\partial y' + \partial v^{y'}/\partial x')$ 'relative angular velocity', 'apparent rate of volume dilatation' and 'apparent rate of shearing motion' respectively. Then as is easily shown, we have the following table contrasting our $U(\omega)$ with that of Hill:

our $U(\omega)$	Hill's $U(\omega)$
velocity distribution ;	
$v = c \tanh(\rho\omega/c)$	$= -\frac{icJ_1(i2\rho\omega/c)}{J_0(i2\rho\omega/c)}$

relative angular velocity ;

$$\begin{aligned}\mathcal{Q}' &= \frac{1}{2} \left\{ \xi (1 - \rho^2 \xi^2 / c^2)^{-1} + \omega \right\} &= \omega \\ &= \frac{1}{2} \left\{ \frac{c}{2\rho} \sinh \frac{2\rho\omega}{c} + \omega \right\} &[\text{Hill's defining equation of } U(\omega)]\end{aligned}$$

apparent rate of volume dilatation ;

$$I'' = 0^* \quad = 0$$

apparent rate of shearing motion ;

$$\gamma' = \frac{1}{2} \left\{ \frac{c}{2\rho} \sinh \frac{2\rho\omega}{c} - \omega \right\} = \frac{v}{\rho} \left(1 - \frac{v^2}{c^2} \right)^{-1} - \omega, \quad (5.2)$$

where J_0 and J_1 are usual Bessel functions.

Accordingly, when $\rho\omega/c \rightarrow 0$, the corresponding four quantities of both theories coincide with each other by neglecting higher order infinitesimals. In the case of $\rho\omega \rightarrow \infty$, however, the values of \mathcal{Q}' and γ' in both theories do not coincide. In fact, it holds that $\mathcal{Q}' \rightarrow \infty$ and $\gamma' \rightarrow \infty$ in our case and $\mathcal{Q}' = \omega$ and $\gamma' \rightarrow \omega$ in Hill's case.

(b) *Comparison with $U(\omega)$ of Rosen⁶⁾* Rosen defined the relativistic rotation of rigid body by using the four-dimensional tensor equation

$$2\dot{p}_{ij} \equiv \nabla_j u_i + \nabla_i u_j - u^s u_i \nabla_s u_j - u^s u_j \nabla_s u_i = 0, \quad (u^i = dx^i/ds), \quad (5.3)$$

where \dot{p}_{ij} is obtained by generalizing the classical three-dimensional strain tensor i.e. $(\partial v^a/\partial x^b + \partial v^b/\partial x^a)$, ($a, b = 1, 2, 3$), and deduced the linear velocity-distance relation. Now, we shall calculate the components of this generalized strain tensor with respect to our $U(\omega)$ omitting z-components for brevity's sake. By using Minkowski metric $ds^2 = -dx^2 - dy^2 + c^2 dt^2$, we can easily obtain

$$\dot{p}_{11} = -\dot{p}_{22} = xy\alpha, \quad \dot{p}_{12} = \frac{1}{2}(y^2 - x^2)\alpha, \quad \dot{p}_{14} = \frac{1}{2}xb, \quad \dot{p}_{24} = \frac{1}{2}yb, \quad \dot{p}_{44} = 0, \quad (5.4)$$

where

$$\begin{cases} \alpha = (\xi'/c\rho) (1 - v^2/c^2)^{-3/2} = -2\gamma'(\cosh \lambda)/c\rho^2, \\ b = (\rho\xi\xi'/c) (1 - v^2/c^2)^{-3/2} = -\gamma'(\sinh 2\lambda)/\rho, \quad (\xi' = d\xi/d\rho), \end{cases} \quad (5.5)$$

and γ' is the quantity given in (5.2). Hence for our $U(\omega)$ Rosen's relation $\dot{p}_{44} = 0$ does not hold in general though it does in the case of $\rho\omega/c \rightarrow 0$. Especially when $\rho\omega \rightarrow \infty$ both α and b become infinite.

He also showed that we can study the properties of rotating motion by using four-dimensional tensor $\mathcal{Q}_{ij} = \nabla_{[j} u_{i]}$ which is a generalization of the classical angular velocity tensor. This tensor becomes

$$\begin{aligned}\mathcal{Q}_{12} &= (\sinh \lambda + \lambda \cosh \lambda)/2\rho, \quad \mathcal{Q}_{14} = -xI', \quad \mathcal{Q}_{24} = -yI', \\ \text{other } \mathcal{Q}_{ij} &= 0, \quad \text{where } I' = (\omega/4\rho) \sinh 2\lambda, \end{aligned} \quad (5.6)$$

* This relation holds for any $\xi(\rho)$ as is easily seen from (5.1).

in our case. Hence when $\lambda \rightarrow 0$, F and Ω_{12} are proportional to usual centripetal acceleration and ω respectively. In a similar way we can deal with the other quantities treated by him.

By the above consideration we know that our $U(\omega)$ differs from those of Hill and Rosen but it coincides with them when $\rho\omega/c \rightarrow 0$.

§ 6. Geometry on the rotating disk

In this section we shall determine the intrinsic geometry in our theory, shall compare it with that of Berenda and shall finally discuss the relation between the experimental test of the general relativity proposed by him and our theory.

Now, operating $T(\omega)$ on the Minkowski metric in polar form:

$$ds^2 = -d\rho^2 - \rho^2 d\phi^2 + c^2 dt^2, \quad (6.1)$$

$$\text{we obtain} \quad ds^2 = -(1+P)d\rho^2 - \rho^2 d\phi^2 + c^2 dt^2 + 2Qd\rho d\phi + 2Sd\rho dt, \quad (6.2)$$

$$\text{where} \quad P = (c^2 Y / \rho^2 - 4\omega\phi t) \sinh^2 \lambda - (c\omega/\rho) (t^2 + \rho^2 \phi^2 / c^2) \sinh 2\lambda + \omega^2 Y, \\ Q = \rho\phi \sinh^2 \lambda + \frac{1}{2} ct \sinh 2\lambda - \omega\rho t, \quad (Y \equiv t^2 - \rho^2 \phi^2 / c^2), \quad (6.3)$$

$$S = (c^2 t / \rho) \sinh^2 \lambda + \frac{1}{2} c\phi \sinh 2\lambda + \omega\rho\phi,$$

z -components being omitted. This (6.2) is the *four-dimensional metric of the rotating system defined by our $U(\omega)$* . If we operate $T(\omega)$ on (6.2), we have the same form as (6.2) in which ω is replaced by $(\omega + \bar{\omega})$ since $T(\bar{\omega})T(\omega) = T(\omega + \bar{\omega})$. Such a group property was strongly required by Hill as stated in § 1.

Adopting Berenda's assertion,⁽⁴⁾ we shall take the length of the vector standing at right angles to the world lines of any two points fixed on the disk as the spatial element of the rotating disk. Then from

$$ds^2 = g_{ab} dx^a dx^b - (g_{a4} dx^a)^2 / g_{44}, \quad (a, b = 1, 2), \quad (6.4)$$

and

$$dx^4 = -g_{a4} dx^a / g_{44} = -(S/c^2) d\rho, \quad (6.5)$$

we have

$$\begin{aligned} -ds^2 &= dl^2 = (1 + P + S^2/c^2) d\rho^2 + \rho^2 d\phi^2 - 2Qd\rho d\phi \\ &= (1 + Q^2/\rho^2) d\rho^2 + \rho^2 d\phi^2 - 2Qd\rho d\phi. \end{aligned} \quad (6.6)$$

This (6.6) is the *line element which determines the intrinsic geometry of our rotating disk*. In this geometry the proper length C_0 of the circumference of a circle with center O^*

* On the circumference of this circle it holds that $ds^2 = -\rho^2 d\phi^2 + c^2 dt^2$ and $T(\omega)$ constitutes the transformation group which keeps the form of this line element invariant together with $t' = t + \text{const.}$ and $\phi' = \phi + \text{const.}$

is given by $C_0 = \rho \int_0^{2\pi} d\phi = 2\pi\rho$. Hence C_0 is obtained by the euclidean law. Nevertheless the geometry of (6.6) is non-euclidean since its gaussian curvature is, as is easily seen, given by

$$K(\omega) = K(-\omega) = -(\sinh 2\lambda)(4\lambda + \sinh 2\lambda)/4\rho^2 \leq 0. \quad (6.7)$$

Now we shall compare these results with Berenda's ones. Using (1.2) in place of $T(\omega)$, by the same method as the above, he obtained

$$(dl^2)_B = d\rho^2 + (1 - \rho^2\omega^2/c^2)^{-1}\rho^2 d\phi^2, \quad (6.8)$$

where index B means Berenda. From this he deduced

$$(C_0)_B = 2\pi\rho(1 - v^2/c^2)^{-1} \geq 2\pi\rho, \quad (6.9)$$

$$\text{and} \quad (K)_B = -(3\omega^2/c^2)(1 - \omega^2\rho^2/c^2)^{-2} \leq 0, \quad (6.10)$$

in place of our C_0 and (6.7). Hence his geometry is also non-euclidean, and $(C_0)_B$ does not satisfy the euclidean law contrary to our case. $(K)_B$ differs from our K but both are non-positive. As a matter of course, however, both geometries coincide with each other in the limiting case of $\rho\omega/c \rightarrow 0$.

Next we shall study the experimental test of general relativity by use of cyclotron from the standpoint of our $U(\omega)$. Since the problem of rotating disk is not necessarily identical with that of rotating particle and moreover it has not been made clear as yet whether the formula $I = I_0 \exp(-\mu t)$ used by Berenda is invariant under any motion, it seems to the writer that Berenda's theory is not necessarily correct. But his conclusion is very interesting, so we shall apply his method tentatively to our $U(\omega)$.

Basing upon Tolman's theory concerning the behaviour of clocks in general relativity⁽¹¹⁾ and using the transformation law (1.2), Berenda deduced the following relation:

$$\frac{\partial t_1^0}{\sqrt{(1 - \rho_1^2\omega^2/c^2)}} = \frac{\partial t_2^0}{\sqrt{(1 - \rho_2^2\omega^2/c^2)}} = \partial t_3^0, \quad (6.11)$$

where ∂t_1^0 and ∂t_2^0 are proper periods of clocks at rest relative to the disk with distances ρ_1 and ρ_2 from O respectively, and ∂t_3^0 is that of a clock placed anywhere in non-rotating galilean plane. From this relation he concluded that in general, clocks on the disk will run slow relative to those placed on the galilean plane and used this result in determining the change of radio-activity of the radio-active element rotating in a cyclotron. To deduce (6.11), he used the four-dimensional static line-element which is obtained from (6.1) by the transformation (1.2). Since our line element (6.2) is non-static, however, we can not use his reasoning as it is. But as is easily shown, we can deduce (6.11) directly from the original Minkowski metric (6.1) by using the linear law $d\phi_1/dt = d\phi_2/dt = \omega$. By applying this simple method to our $U(\omega)$ using (6.1) and $d\phi_a/dt = \xi_a$ ($a=1, 2$), we obtain

$$\frac{\partial t_1^0}{\sqrt{(1 - \rho_1^2\xi_1^2/c^2)}} = \frac{\partial t_2^0}{\sqrt{(1 - \rho_2^2\xi_2^2/c^2)}} = \partial t_3^0, \quad \left(\xi_a = \frac{c}{\rho_a} \tanh \frac{\rho_a \omega}{c} \right), \quad (6.12)$$

in place of (6.11). Hence as to the change of the radio-activity of particle rotating in

a cyclotron, our theory also leads to the similar conclusion as Berenda's and both differ from each other by the small quantity whose order is at least four with respect to $\rho\omega/c$.

§ 7. Conclusion

Basing upon the group-theoretical idea suggested by Hill and assuming the composition law of velocities in special relativity, we established a new relativistic theory of rotating disk by an axiomatic method. In this new theory, in the first place, the transformation formula $T(\omega)$ is obtained and using this the velocity distribution and the intrinsic geometry of the disk are deduced. When $\rho\omega/c$ is very small the velocity distribution satisfies the linear law approximately, but it is not the case in general. In the limit of $\rho \rightarrow \infty$ the velocity of particles on the disk coincides with c , the light velocity. The geometry of the disk is also of a new type and the proper circumference C_0 of a circle with center O is given by euclidean formula $2\pi\rho$ in spite of the fact that the geometry is non-euclidean.

Its gaussian curvature K is a non-positive even function of ω , decreases monotonic with ω^2 and become minus infinity when $\omega^2 \rightarrow \infty$. If we apply tentatively Berenda's proposal concerning the experimental test of general relativity to our theory we obtain similar conclusion as Berenda's.

The problem of the relativistic investigation of the rotating disk is very complicated and difficult as is stated in the introduction. Of course the new theory given in this paper is imperfect and we can never say that it gives a perfect solution of the problem. The writer thinks, however, that it gives a new type of method for attacking the problem.

The writer is grateful to Professor Y. Mimura, director of the Institute for Theoretical Physics, Hiroshima University, for his helpful discussions.

Appendix

Our transformation $T(\omega)$ is somewhat complicated as compared with the classical transformation (1.2). But we can simplify its form by taking some special coordinate systems. Hence it will be advantageous to use these coordinate systems in order to investigate the mathematical properties of $T(\omega)$. Here we shall give two kinds of such coordinate systems.

(a) In the first place, we shall show that there exists a coordinate system in which our $T(\omega)$ given by (3.12) takes the classical form (1.2). If we denote the coordinate variables in this system by $(\bar{\rho}, \bar{\phi}, \bar{t})$, a necessary and sufficient condition that this system have the property stated above is that the vector η^i given by (3.5) and (3.9) take the form $(0, -\bar{t}, 0)$ in this system. (z -components are omitted for brevity's sake.) Hence the transformation equation

$$x^i \rightarrow \bar{X}^i = X^i(x^j), \quad \det(\partial X^i / \partial \bar{X}^j) \neq 0, \quad (1)$$

is given by solving

$$\bar{\eta}^i = -\left(t \frac{\partial}{\partial \phi} + \frac{\rho^2 \phi}{c^2} \frac{\partial}{\partial t}\right) \bar{X}^i = -t \partial_{\bar{t}}^i. \quad (2)$$

An example of such a transformation, i.e., a special solution of (2) is given by

$$\rho = \bar{\rho}, \quad \phi = \frac{c\sqrt{Y}}{2\rho} \log \frac{c\bar{t} + \rho\bar{\phi}}{c\bar{t} - \rho\bar{\phi}}, \quad \bar{t} = \sqrt{Y}. \quad (3)$$

Therefore, in the coordinate system $(\bar{\rho}, \bar{\phi}, \bar{t})$ given by (3), our $T(\omega)$ is expressed by

$$\bar{\rho}' = \bar{\rho}, \quad \bar{\phi}' = \bar{\phi} - \omega \bar{t}, \quad \bar{t}' = \bar{t}, \quad (4)$$

which is the same form as (1.2). When $c \rightarrow \infty$ the transformation (3) becomes the identical transformation.

We can also prove that the ordinary rotation about $z(=\bar{z})$ -axis in this coordinate system, i.e.,

$$\bar{\phi}' = \bar{\phi} - \phi_0, \quad (\phi_0 \text{ is a const.}), \quad (5)$$

corresponds to the transformation

$$\begin{cases} \rho' = \rho, & \phi' = \{\phi + (c\ell/\rho) \tanh \mu\} \cosh \mu, \\ t' = \{t + (\rho\phi/c) \tanh \Gamma\} \cosh \mu, & (\mu = -\rho\phi_0/c\sqrt{Y}), \end{cases} \quad (6)$$

in X^i -system.

(b) In the same way, by a transformation obtained by solving

$$\bar{\eta}^i = -\left(t \frac{\partial}{\partial \phi} + \frac{\rho^2 \phi}{c^2} \frac{\partial}{\partial t}\right) \bar{X}^i = -\delta_2^i, \quad (7)$$

we get a coordinate system in which our $T(\omega)$ is represented by (5). An example of such a transformation is given by

$$\bar{\rho} = \rho, \quad \bar{\phi} = \frac{c}{2\rho} \log \frac{c\ell + \rho\phi}{c\ell - \rho\phi}, \quad \bar{t} = \frac{c}{2\rho} \sqrt{Y}. \quad (8)$$

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On the Structure of the Interaction of the Elementary Particles, I

—The Renormalizability of the Interactions*—

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The applicability of the renormalization theory was investigated for the general types of the local interactions. It was found that the interaction having the coupling constant of the dimension L^{η} could be renormalized if the conditions $\eta \leq 0$ were valid. It is remarkable that this condition is closely related to Heisenberg's classification of the interactions.

§ 1. Introduction

In the recent years, considerable progress in the theory of the elementary particles was achieved by the development of the renormalization theory.¹⁾ Especially, the quantum electrodynamics of the electron has had great success not only in circumventing the well known divergence difficulties but also in explaining the Lamb shift of the hydrogen atom and the subtle anomaly in the magnetic moment of the electron. In spite of this, as we have stressed in the previous paper²⁾, the renormalization theory should be regarded as an *abstract* formalism, behind which *concrete* structures of the elementary particles lie hidden. So long as the renormalization theory is successful, it is unnecessary to expose the detailed features of such hidden parameters, but as soon as the defects become obvious we must seriously look for them.

As is well known, the renormalization theory has encountered grave difficulties in the problems of the mesons. Not only do calculations still yield divergent results, but even if finite, we often obtain results which are quantitatively inconsistent with experimental data. For instance, none of the four types of the meson theory has yet been able to explain the anomalous magnetic moment of the nucleon.

Nevertheless, it would be too early to conclude that the renormalization theory failed in treating such problems. In fact, *if we apply the philosophy of the renormalization to the utmost, any sorts of experiments can always be accounted for by introducing new kinds of interactions as the counter terms of the original interactions.* For example, the magnetic anomaly of the nucleon can be explained by assuming the existence of the magnetic moment of Pauli type.

In the present stage of the theoretical advance, there is no criterion deciding *what*

* A preliminary report of this work was made at the REKS meeting at Nagoya University, February 6, 1951 (See S. Sakata, H. Umezawa and S. Kamefuchi, Phys. Rev. **84** (1951), 154).

kinds of interactions are the primary interactions of the elementary particles, and what kinds of interactions are the consequences of the others. It is therefore quite legitimate at present to assume the existence of the Pauli moment as the primary interaction. However, if it were so, the great successes of the renormalization theory in the quantum electrodynamics would be also regarded as merely accidental, because we can not understand the reason why the Pauli moment of the electron vanishes identically.

Thus, it is required that the future theory of the elementary particles should account for *the structure of the interactions*, that is, the qualitative differences as well as the intimate relationships between various kinds of interactions. However, in the present stage, we must content ourselves with a more formal or phenomenological classification of interactions. Although the renormalization theory treats originally all kinds of interactions on the equal footing, it has become clear recently^{3, 6)} that they may be classified into two groups as follows: (a) those interactions which can be renormalized by assuming the coexistence of a finite number of interactions belonging to the same group, and (b) those interactions which require the further introduction of infinitely many interaction terms having successively higher derivatives as well as larger numbers of the field quantities. The electromagnetic interaction of the electron,¹⁾ scalar interaction of the scalar meson²⁾ and so on are examples of the former group, while the electromagnetic interaction of the vector meson,⁵⁾ the direct interaction between spinor particles⁶⁾ and so on belong to the latter.

In the case when all the interactions realized in the nature belong to the first group, the renormalization theory will form a closed system in the framework of the present quantum field theory. On the contrary, in the case when there exist some interactions belonging to the second group, the renormalization theory will exceed the limit of its applicability. As the assembly of infinite number of interactions having successively higher derivatives is equivalent to a non-local interaction, we must then take into account the structure of the elementary particles consistently.

From these considerations it seems to be very important to classify all the interactions into two groups and to investigate whether the interactions of the second kind really exist in the nature or not. In this paper we shall treat the first subject, whereas the second subject will be discussed in the following paper. The renormalizability of the general types of the local interactions is studied in detail in § 2. As we shall show in § 3, this condition can be written in a very simple form

$$\eta \leq 0, \quad (1.1)$$

where η has such a physical meaning that the dimension of the coupling constant is written as (length) $^{\eta}$.

Now, in connection with this condition (1.1), which indicates the limit of applicability of the renormalization theory, it is of interest to recall Heisenberg's classification of interactions⁷⁾ into the first ($\eta=0$) and the second kind ($\eta>0$). As is well known, his classification was also related to the limit of the applicability of the quantum field theory—though the previous formulation was adopted. In this respect, our results may be regarded

* We use natural units throughout with $\hbar=c=1$.

as the revision of Heisenberg's work in the light of the new formulation.

§ 2. Renormalizability of the interactions

In this section we shall investigate the conditions for the renormalizability of the most general type of interactions. Here, the interactions are said to be renormalizable if, by introducing a finite number of local interactions as the counter terms into the original interactions we can obtain a closed theory which does not contain any singularity.

i) Suppose that there are m charged fields $U^\alpha(x)$ and n neutral fields $V^\beta(x)$ interacting with each other. They have generally several components, which will be denoted as $U_i^\alpha(x)$ or $V_j^\beta(x)$ when necessary. In the following, we shall adopt the interaction representation throughout. So that the field quantities satisfy the equations of motion for free fields*:

$$\begin{aligned} \{\square - (x^\alpha)^2\} U^\alpha &= 0, \\ \{\square - (x^\beta)^2\} V^\beta &= 0, \end{aligned} \quad (2.1)$$

with some subsidiary conditions, where x^α and x^β denote masses of respective fields. The commutation relations are given by

$$\begin{aligned} [U_i^\alpha(x), U_{i'}^{\alpha'}(x')]_{\pm} &= i\delta_{\alpha\alpha'} \Delta_{ii'}^\alpha(x-x'), \\ [V_j^\beta(x), V_{j'}^{\beta'}(x')]_{\pm} &= i\delta_{\beta\beta'} \Delta_{jj'}^\beta(x-x'), \end{aligned} \quad (2.2)^*$$

where commutator or anti-commutator is taken according as the fields have integer or half-integer spins. $\Delta_{ii'}^\alpha(x)$ may be written generally in the form:

$$\Delta_{ii'}^\alpha(x) = \Gamma_{ii'}^\alpha(x^\alpha, i\partial_\mu) \Delta(x), \quad (2.3)$$

where $\Gamma_{ii'}^\alpha(x^\alpha, i\partial_\mu)$ is a polynomial of ∂^α -th degree with respect to the derivative operators $\partial_\mu = \partial/\partial x_\mu$.

The interaction of the system of fields can be described by the Tomonaga-equation

$$i \frac{\delta \Psi(\sigma)}{\delta \sigma(x)} = H(x) \Psi(\sigma), \quad (2.4)$$

where $\Psi(\sigma)$ denotes the state vector on a surface σ , while H is the interaction. By integrating this equation, Dyson¹⁾ has shown that the S-matrix, which combines the state vector on the surface in the remote past with that of the distant future is given by

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n \int dx_1 \cdots \int dx_n P[H(x_1), \dots, H(x_n)]. \quad (2.5)$$

Now we assume that the most general type of the interactions is given by

$$H = \sum_l H_l, \quad (2.6)$$

with

$$H_l = f_l H_{\alpha} \{ (\partial^{a_{i_1}} U_{i_1}^{\alpha}) \cdots (\partial^{a_{i_k}} U_{i_k}^{\alpha}) \}^*$$

* The field quantities are to be understood as those expressed in irreducible representations. This must be borne in mind especially in the application of the conditions (I) and (II) given below.

$$\times \{ (\partial^{\alpha}_{j_1} U^{\alpha}_{j_1}) \dots (\partial^{\alpha}_{j_\mu} U^{\alpha}_{j_\mu}) \} \\ \times H_{\beta} \{ (\partial^{\beta}_{k_1} V^{\beta}_{k_1}) \dots (\partial^{\beta}_{k_\nu} V^{\beta}_{k_\nu}) \}, \quad (2.7)$$

where f_i 's denote the coupling constants and $\partial'' U^{\alpha}$ stands for $\partial'' U^{\alpha} / \partial x_{\rho_1} \partial x_{\rho_2} \dots \partial x_{\rho_n}$. As λ and μ depend on α and ν on β , they will be specified as λ^{α} , μ^{α} , ν^{β} when necessary. The detailed form of H is determined by various formal requirements such as Lorentz invariance, Hermitian property, charge symmetry, conservations of charge and spin, etc., but it need not be specified for the purpose of this paragraph. Though H contains in general the normal dependent terms, they may be neglected here altogether.

It would be very difficult or even impossible to derive such a general interaction as (2.6) and (2.7) from the canonical formalism, because it contains higher derivatives of the field quantities. Nevertheless, we assume here, from the stand point of the correspondence principle, that as a consequence of some procedure the S-matrix can be computed by (2.5) with H given by (2.6) and (2.7). In the following, it will be called the *interaction in a broad sense*.*

ii) By extending Dyson's analysis of the S-matrix,¹⁾ we can now investigate the condition for the renormalizability of the interaction term (2.7). Let us first consider the contributions to the S-matrix from each of the field quantities involved in H_i . If a virtual quantum of the α -field passes between x' and x'' , the matrix element will contain in the integrand a factor

$$\langle P(\partial'^{\alpha}_i U^{\alpha}_i(x'), \partial''^{\alpha}_j U^{\alpha}_j(x'')) \rangle_0 = i \partial'^{\alpha}_i \partial''^{\alpha}_j \mathcal{A}^{\alpha}_{ijF}(x' - x'') \quad , \quad (2.8)**$$

where $\mathcal{A}^{\alpha}_{ijF}(x' - x'')$ is Feynman's propagation function for α -field derived from $\mathcal{J}^{\alpha}_{ij}$. By using the representation in the momentum space,

$$\mathcal{A}^{\alpha}_{ijF}(x) = \text{const.} \int d^4 k \frac{\Gamma^{\alpha}_{ij}(x^{\alpha}, k_{\mu})}{k_{\mu}^2 + (x^{\alpha})^2} e^{-ik_{\mu} x^{\mu}}. \quad (2.9)**$$

As $\Gamma^{\alpha}_{ij}(x^{\alpha}, k_{\mu})$ is a polynomial of b^{α} -th degree with respect to k_{μ} , (2.8) contributes to the matrix element in the momentum space a factor of the highest order $k^{\alpha_i + \alpha_j + b^{\alpha} - 2}$ for the large value of k . In other words, $\partial'^{\alpha}_i U^{\alpha}$ in H_i contributes to the matrix element a factor of the highest order $k^{\alpha_i + b^{\alpha} - 1}$. In the same way, we can see that the contribution from $\partial^{\beta}_k V^{\beta}$ is given by a factor of the order $k^{\beta_k + b^{\beta} - 1}$.

For the convenience of the following discussion, we shall now introduce several quantities which characterize the interaction term H_i :

* Such a form of generalized S-matrix is obtained by the direct quantization method of Yang and Feldman, Phys. Rev. **79** (1950) 972. See also Z. Koba, Prog. Theor. Phys. **5** (1950), 696.

The improved and more general formulation of this quantization method will be developed in the forthcoming paper (Y. Takahashi and H. Umezawa, Prog. Theor. Phys. in press).

** In (2.8) and (2.9) the surface dependent terms coming from the differentiations of $\epsilon(x)$ have been neglected. Including these terms, however, does not make higher the degree of momentum and so the following arguments suffer no alteration. Usually, the surface dependent terms are of no direct physical meaning.

$$A_i = \sum_{\alpha} (a_{i1}^{\alpha} + \dots + a_{i\lambda}^{\alpha}) + \sum_{\alpha} (a_{j1}^{\alpha} + \dots + a_{j\mu}^{\alpha}) + \sum_{\beta} (a_{k1}^{\beta} + \dots + a_{k\nu}^{\beta}), \quad (2.10)$$

which is the total sum of the degrees of the derivation operators in H_i ,

$$B_i = \sum_{\alpha} b^{\alpha} \lambda^{\alpha} + \sum_{\alpha} b^{\alpha} \mu^{\alpha} + \sum_{\beta} b^{\beta} \nu^{\beta}, \quad (2.11)$$

which is the total sum of b 's of field quantities appeared in H_i ,

$$C_i = \sum_{\alpha} (\lambda^{\alpha} + \mu^{\alpha}) + \sum_{\beta} \nu^{\beta}, \quad (2.12)$$

which is the number of the field quantities in H_i and finally

$$K_i = 4 - A_i - B_i/2 - C_i. \quad (2.13)$$

iii) In the following discussion, we shall denote, for the sake of simplicity, charged as well as neutral fields by the superscripts α 's.

Now, let us consider a Feynman's graph having L external lines, I internal lines and n vertices. Among these we assume that L^{α} lines refer to the field α and n_i vertices belong to the interaction term H_i . Then we have

$$\begin{aligned} \sum_i L^{\alpha} &= L, \\ \sum_i n_i &= n. \end{aligned} \quad (2.14)$$

Taking account of the fact that C_i lines start from the vertex H_i and one internal line runs between two vertices (or returns to its starting vertex), we find immediately

$$2I + \sum_i L^{\alpha} = \sum_i n_i C_i. \quad (2.15)$$

By performing the integrations over internal momenta (i. e., contracting the appropriate operators), this graph could, in general, be brought into a form of a vertex, which resembles to a primary interaction giving rise to the corresponding transition. Since the free operators U^{α} 's (external lines) carry the derivation operators of the degree α^{α} , the Feynman's graph for the interaction of such a general type should be specified as $G(E^{\alpha}, N^{\alpha})$, where $N^{\alpha} = \sum_i a_i^{\alpha}$. Furthermore, as a result of the integration over internal momenta, $G(E^{\alpha}, N^{\alpha})$ splits into several independent graphs, denoted by $G(E^{\alpha}, N^{\alpha}, M^{\alpha})$, which have further derivation operators on U^{α} 's with the total number $M = 0, 1, 2, \dots$ *. For each of these $G(E^{\alpha}, N^{\alpha}, M^{\alpha})$ we may now introduce the characteristic constants in accordance with the definitions (2.10) to (2.13);

$$\begin{aligned} A_k &= \sum_{\alpha} (N^{\alpha} + M^{\alpha}), \\ B_k &= \sum_{\alpha} b^{\alpha} E^{\alpha}, \end{aligned} \quad (2.16)$$

* Such a decomposition of $G(E^{\alpha}, N^{\alpha})$ into $G(E^{\alpha}, N^{\alpha}, M^{\alpha})$ is obtained when the use is made of the expansion of the integrand

$$R(p, p^i) = R(p^0, p^i) + (p - p^0)_{\mu} \frac{\partial R}{\partial p_{\mu}}(p^0, p^i) + \frac{1}{2} (p - p^0)_{\mu} (p - p^0)_{\nu} \frac{\partial^2 R}{\partial p_{\mu} \partial p_{\nu}}(p^0, p^i) + \dots,$$

where p and p^i are the external and the internal momenta, respectively and p^0 is a constant free particle momentum.

$$C_k = \sum_{\alpha} I_k^{\alpha},$$

$$K_k = 4 - A_k - B_k/2 - C_k.$$

As a first step to the study of renormalizability, let us consider the connected graphs with primitive divergence. It was shown in (ii) that $\delta^{a_i} U_i^{\alpha}$ in H_l contributes to the matrix element in the momentum space a factor $k^{a_i + b^{\alpha}/2 - 1}$ for the large value of the momentum k . Therefore, H_l will contribute the following power of momentum in the high energy region:

$$A_l + \sum_{\alpha} (b^{\alpha}/2 - 1) \lambda^{\alpha} + \sum_{\beta} (b^{\beta}/2 - 1) \mu^{\beta} + \sum_{\gamma} (b^{\gamma}/2 - 1) \nu^{\gamma} \\ = A_l + 1/2 \cdot B_l - C_l. \quad (2.17)$$

As $G(E^{\alpha}, N^{\alpha}, M^{\alpha})$ has n_l vertices which belong to H_l , the highest power of the internal momentum in the matrix element is found to be

$$N = \sum_l n_l (A_l + 1/2 \cdot B_l - C_l) - (A_k + 1/2 \cdot B_k - C_k). \quad (2.18)$$

Further contribution to the integrand of the S-matrix comes from the volume element of the momentum space. Among the $4I$ internal momenta $4(\sum_l n_l - 1)$ variables are eliminated by the conservation law of the energy-momentum, so that the contribution becomes in terms of the power of the internal momentum:

$$N' = 4(I - \sum_l n_l + 1). \quad (2.19)^*$$

Thus the graph $G(E^{\alpha}, N^{\alpha}, M^{\alpha})$ under consideration is divergent if

$$N + N' \geq 0, \quad (2.20)$$

where it is to be understood that $N + N' = 0, 1, 2, \dots$ means logarithmic, linear, quadratic, ... divergences, respectively. By using (2.15), (2.16), (2.18) and (2.19) we find

$$N + N' = K_k - \sum_l n_l K_l. \quad (2.21)$$

So that the condition (2.20) becomes

$$K_k \geq \sum_l n_l K_l. \quad (2.22)$$

It should be noted that the left hand side of this inequality is determined only by the properties of external lines.

From (2.22), we find the following remarkable facts:

a) If the condition

$$K_l \geq 0 \quad (\text{for all } l) \quad (I) \quad (2.23)$$

is satisfied, the number of the primitive divergent graphs $G(E^{\alpha}, N^{\alpha}, M^{\alpha})$ remains always

* In general, $I \geq \sum_l n_l - 1$. We shall restrict ourselves, in the following, to the graphs for which $I > \sum_l n_l - 1$. The graphs for which $I = \sum_l n_l - 1$ always converge, since available δ -functions eliminate all the internal momenta. The graphs consisting from 2-vertices only are examples of the latter case.

to be finite.* In fact, the possible values of E^α , N^α , M^α for diverging graph are restricted by the relation

$$4 \geq \sum_{\alpha} (E^\alpha + N^\alpha + M^\alpha), \quad (2 \cdot 23)$$

and the order of divergence decreases according as the values of E^α , N^α and M^α increase. Furthermore, no matter how many interaction terms we may introduce, the number of primitive divergent graphs will not exceed a definite number j so far as (I) is satisfied. The number j depends only on the number of the fields participating in the interaction under consideration.

b) If there is at least one H_i with $K_i < 0$, any $G(E^\alpha, N^\alpha, M^\alpha)$ will always diverge for a sufficiently large value of n_i (i. e., in the higher order approximation of the perturbation calculation). Moreover, there arises infinitely high degree of divergence in the graphs $G(E^\alpha, N^\alpha, M^\alpha)$.** Accordingly, we have infinite number of primitive divergences.

iv) Now, we shall consider the question whether the divergences arising from diverging $G(E^\alpha, N^\alpha, M^\alpha)$ can be removed by introducing the appropriate counter terms. The following two cases will be treated separately:

- a) the case in which the condition (I) is satisfied;
- b) the case in which the condition (I) is not satisfied at least by one of the interaction terms.

First, let us consider the case (a). In order to cancel the divergences, we must introduce new primary interaction terms $\sum H_{i'}$ as the counter terms which have the same forms as those of the terms obtained by contracting primitively diverging $G(E^\alpha, N^\alpha, M^\alpha)$ terms and have coupling constants with infinite values. As the characteristic constants $K_{i'}$'s of the counter terms $H_{i'}$'s are respectively identical with K_i 's of primitively diverging graphs $G(E^\alpha, N^\alpha, M^\alpha)$, it is evident that $K_{i'}$'s also satisfy condition (I):

$$K_{i'} = K_i \geq \sum_i K_i n_i \geq 0. \quad (2 \cdot 24)$$

Therefore, according to the remarks given in the previous section, there appear no new primitive divergences even if we introduce the counter terms.*** Thus, as shown below, it is possible to renormalize all the divergences by suitably choosing j coupling constants.

Here, we shall only sketch the outline of the renormalization procedure. In general, corresponding to a diverging graph there exists a diverging integral over the internal momenta p_1, p_2, p_3, \dots , which, when employing the general method proposed by Salam,⁽⁸⁾ is further subdivided into a set of "subintegrations" p_1, p_2, p_3, \dots ; $p_1 p_2, p_1 p_3, \dots$; $p_1 p_2 p_3, \dots$.

* In his own theory of S-matrix, which starts with the consideration of unitarity and causality, Stueckelberg has also found the corresponding condition in quantum electrodynamics; A. Petermann and E. C. C. Stueckelberg, Phys. Rev. **82** (1951), 548; Helv. Phys. Acta **24** (1951), 317.

** The highest degree of divergence in the graph $G(E^\alpha, N^\alpha, M^\alpha)$ is given by (2.21) unless there appear particular circumstances.

*** At first sight it seems unnecessary to examine the condition (2.24) for counter terms, since it may be expected that they play only the role of compensation of divergences and do not appear by themselves. But, this is incorrect. In general, there frequently occur the cases in which counter terms appear separately and directly contribute to the further divergences (for example, 'final Møller part' in the self-energy graph of meson).

As is evident from the above consideration, each diverging term arising from the subintegrations p_1, p_2, p_3, \dots is always of the same operator form as any of the counter terms H_U 's. Hence, these divergences are all cancelled by the counter terms. Next, we shall consider the subintegrations $p_1 p_2, p_1 p_3, \dots$. As the degree of divergence for a particular subintegration is, in this case, unaffected by subtractions corresponding to other subintegration and so there appear no new type of divergences after this manipulation, we see that divergences coming from the subintegrations $p_1 p_2, p_1 p_3, \dots$ are again cancelled by the corresponding counter terms. Divergences due to the subintegrations $p_1 p_2 p_3, \dots$ are eliminated in a similar way, and so on. Proceeding successively in this way and finally removing the divergences of the final integration $p_1 p_2 p_3, \dots$, we can obtain the unique and finite result, being independent of the choice of the "basic variables" p_1, p_2, p_3, \dots .

The values of f coupling constants of the counter terms are determined as follows. Let f_U be the coupling constant of the counter term H_U corresponding to the diverging $G(E^\alpha, N^\alpha, M^\alpha)$ graphs ($N^\alpha + M^\alpha$: fixed.), f_U is as usual expanded in the power of the coupling constants of H_U 's: $f_U = \sum_n f_U^{(n)}$.^{*} The determination of f_U is thus reduced to those of $f_U^{(n)}$'s, which are carried through step by step from the lowest order in the following way: $f_U^{(1)}$ is put equal to the negative of the sum of the divergent factors of lowest order $G(E^\alpha, N^\alpha, M^\alpha)$ ($N^\alpha + M^\alpha$: fixed) and in general $f_U^{(n)}$ is put equal to the negative of the sum of the divergent factors of all diverging $G(E^\alpha, N^\alpha, M^\alpha)$ graphs ($N^\alpha + M^\alpha$: fixed) of the order (n) , from which all the internal divergences (coming from all the subintegrations except the final one) have been already subtracted by suitably choosing the lower order part of the coupling constants $f_U^{(m)}$ ($m < n$), in other words $f_U^{(n)}$ is equal to the negative of the sum of the diverging factors coming from the final integration of $G(E^\alpha, N^\alpha, M^\alpha)$ graph.

In this way, by the consistent use of the counter terms** we can obtain the unique and absolutely convergent matrix elements for all scattering processes.

Next, we shall consider the case (b). In this case, as was mentioned in the previous section, any $G(E^\alpha, N^\alpha, M^\alpha)$ diverges if we take a sufficiently large value for n_i . In order to remove these divergences in the same manner as in the case (a), it is necessary to introduce infinitely many counter terms with higher derivatives. Among these counter terms there are certainly terms with negative K'_i , which give rise to further divergences, so that we have to introduce successively new counter terms one after another.

Such a serious situation would be avoided, if the infinitely many divergences could be eliminated simultaneously by introducing a finite number of counter terms. But, the following considerations exclude this possibility, too. In general, to cancel a diverging term the coupling constants of the counter terms must satisfy an algebraic equation. As we proceed to the higher order, new diverging terms, which are mutually independent, appear one after another, so that the number of equations to be satisfied increases unlimitedly.

* In general (n) implies, as it were, the degree of complexity of the construction of the graph.

** The subtraction method with the use of counter terms is easily shown to be equivalent to the conventional renormalization of mass and charge. This problem has been also considered by G. Takeda in the private communication and will be discussed in general in our forthcoming paper, Prog. Theor. Phys. in press.

It is obvious that such infinitely many simultaneous equations for finite number of variables have no solution except for trivial ones. Therefore it is impossible to remove all the divergences by introducing only a finite number of local interactions.

Summarizing the results obtained above, we may conclude as follows: When the interaction satisfies the condition (I), all the divergences may be eliminated by introducing a finite number of local interactions, that is, the renormalization theory succeeds to construct a consistent closed theory. On the contrary, if there is at least one interaction term for which this condition is not satisfied, the renormalization theory will break down completely.

§ 3. The "Universal Length" as the limit of the renormalization theory

Hitherto, the limit of validity of the quantum theory of fields was discussed by many authors.⁹⁾ It has been generally believed that the main defects of the present theory came from the ignorance of the finite size of the elementary particles, and that the future theory would be established only by taking into account the existence of the so called "universal length" consistently.¹⁰⁾ As far as we deal with free fields, it may be permitted to ignore the finite size of elementary particles, because the universal length finds expression solely in the form of their masses. But, as soon as the interactions are introduced, we shall be confronted in general with the problem of the structure of the elementary particles.

In order to discuss this problem in detail, Heisenberg⁷⁾ had classified, for some years ago, the interactions into two groups:

- i) the interaction of the first kind, which contains a dimensionless coupling constant Z ,
- ii) the interaction of the second kind, which contains a coupling constant of dimension (length) $^\eta$ with $\eta > 0$.

If $Z \ll 1$, the interaction of the first kind may always be regarded as a small perturbation, and, moreover, as it does not contain the universal length, we may certainly ignore the structure of the elementary particles. On the contrary, the interaction of the second kind can not be considered as a small perturbation at least in the high energy region, as can easily be shown by the dimensional consideration, *the ratio of the $(n+1)$ -th order approximation to the n -th order one in the perturbation calculation involves a factor $(l/\lambda)^\eta$ for small λ* , where l means a characteristic length contained in the coupling constant and λ denotes the wave length. In such a case, the reaction of the self field becomes very effective and the structure of the elementary particles could not be ignored.

Nevertheless, these arguments had not been fully justified, before the renormalization theory was developed. Though the interaction of the electron with the electromagnetic fields belongs to the first kind, it leads to the well known divergence difficulties, as long as we adopt the previous formulation. But, the situation was entirely changed now. In fact, it has become clear recently, as was ingeniously shown by Dyson,^{1,11)} that a non-singular quantum electrodynamics could be established without touching on the problem of the structure of the elementary particles.

Now, it seems to be very interesting to compare Heisenberg's classification of the interactions⁷⁾ with that of ours performed in § 2. For this purpose, we shall first determine the dimension of the coupling constant f_l appeared in the general type of the interaction H_l . As we have adopted the natural unit, f_l has a dimension of $(\text{length})^{\eta_l}$. If we assume that the normalization of the field quantities $U^\alpha(x)$ is so performed that L_{ij}^α in the right hand side of the commutation relations (2.2) becomes in the following form:

$$L_{ij}^\alpha = a_{ij}^{\alpha(0)} \partial^{b^\alpha} + a_{ij}^{\alpha(1)} x^\alpha \partial^{b^\alpha-1} + \cdots + a_{ij}^{\alpha(b^\alpha)} (x^\alpha)^{b^\alpha} \quad (3.1)$$

with $a_{ij}^{\alpha(0)}$, $a_{ij}^{\alpha(1)}$, \cdots being dimensionless constants, then

$$\partial^{a_i} U^\alpha \sim_c \lambda^{-\{a_i + b^\alpha/2 + 1\}} \quad \text{for small } \lambda, \quad (3.2)$$

where c denotes a dimensionless constant. As H_l has a dimension of $(\text{length})^{-1}$, we can easily obtain from (2.7), (3.2)

$$\eta_l = A_l + B_l/2 + C_l - 4 = -K_l. \quad (3.3)$$

Observing this result, we see immediately that the classification made in § 2 coincides exactly with that of Heisenberg,⁷⁾ if the interactions with $\eta_l \leq 0$, which were not considered by him, are also classified into those of the first kind. By using (3.4), we may write (I) in the following form:

$$\eta_l \leq 0 \quad (\text{for all } l), \quad (\text{II})$$

which gives a physical basis for the validity of the renormalization theory.*

In fact, we may easily understand from (II) the reason why the renormalization theory succeeds for the interactions of the first kind. It is certainly due to the fact that the structure of the elementary particles does not play an essential role in these cases. In order to get a closed theory, the coexistence of finite number of interactions belonging to the first kind is often required. As an example of such a case, we may mention the interaction between the nucleon and the scalar neutral meson $V^s(x)$ via scalar coupling which requires to introduce another interaction having the form $f \int d^3x (\eta_l = -1)$.⁴⁾ It should, however, be noted, as was shown in § 2, that the introduction of the interaction of the second kind had never been required for the closure of the theory which involves merely the interaction of the first kind. This point is in striking contrast to the conclusion obtained by Heisenberg,⁷⁾ who emphasized the inevitable appearance of the interaction of the second kind in company with that of the first kind, provided the field has a non-vanishing mass.**

* It should be noted that before η_l is determined all the field quantities in H_l are to be so normalized that (3.1) is valid.

** Heisenberg has pointed out that in the quantum electrodynamics there appears, for example, the term of the scattering of photon by photon, which belongs to the interaction of the second kind and is closely related to the applicability of the theory. However, this term is in itself of the form of non-local interaction and apparently contains interactions of the second kind as a part. In the renormalization theory it lies certainly within the limit of the applicability, because it is not required to introduce the primary interaction belonging to the second kind.

On the other hand, if there is an interaction of the second kind, it will be required that there exist simultaneously infinite number of interactions which belong also to those of the second kind. Such an assembly of interactions is equivalent to a non-local interaction corresponding to an extended model of the elementary particles. So that, we should then take into account the structure of the elementary particles from the beginning.

Thus, one of the most important problem to be solved in the future would be the question *whether interactions of the second kind really exist in the nature or not*. If it becomes clear that they do not exist at all, a non-singular and closed quantum field theory¹²⁾ will be established by means of the renormalizations of masses and coupling constants, and the well known divergence difficulties will be hidden behind these procedures. On the contrary, if the existence of the interaction of the second kind is found, we can not remain in the framework of the renormalization theory. In order to answer to this question, we shall investigate, in the following paper, the types of interactions which may be realized for various sorts of elementary particles.

Appendix

A remark on the general type of interactions

In § 2, we discussed the renormalizability of the interactions by taking out each element H_i of the most general type of interactions separately. However, the actual interactions are the linear combinations of these elements and may be written in the form which is obtained by inserting $D_i^\alpha(\partial_\mu)U_i, \dots$, in place of $\partial^{a_i}U_i^\alpha$, where $D_i^\alpha(\partial_\mu)$'s denote rational functions of ∂_μ as well as x^2 .

The renormalizability condition for such general interactions is also given by (I), provided the definition of a_i^α is suitably changed. Usually a_i^α is equal to the highest degree of ∂ , involved in $D_i^\alpha(\partial)$, but there are some exceptional cases in which this rule does not hold. In such cases, it is convenient to define a_i^α as one half of the difference between the highest degrees of derivation operators operated on the Δ -functions in $D_i^\alpha(\partial_\mu) \times D_i^\alpha(\partial'_\mu)\Delta_{ii}^\alpha$ and of Δ_{ii}^α . We shall call this a_i^α the *true degree* of $D_i^\alpha(\partial_\mu)U^\alpha$.

As an example of these exceptional cases, we shall consider the following operator operating on the field $U_{\mu_1\mu_2}\dots$ with integral spin and non-vanishing mass :

$$D(\partial_\mu) = \partial_{\mu\nu;\mu'} = \partial_\mu \partial_{\nu\nu'} - \partial_\nu \partial_{\mu'\mu},$$

which is the four dimensional rotation and satisfies the relation

$$\partial_{\mu\nu;\mu'}\partial_{\mu'} = \partial_\mu\partial_\nu - \partial_\nu\partial_\mu = 0.$$

Though the highest degree of this operator is 1, the true degree is found to be zero, because Δ_{ij}^α corresponding to this field contains a factor¹³⁾

$$R(\mu', \nu') = \left(\partial_{\mu'\nu'} - \frac{1}{x^2} \partial_{\mu'} \partial_{\nu'} \right)$$

so that $D_i^\alpha(\partial_\mu)D_i^\alpha(\partial_\mu)$ does not raise effectively the degree of the derivative of the delta

function.*

These situations are, in general, due to the following facts: For a given tensor field $U_{\mu_1 \mu_2 \dots \mu_n}(x)$ with rank n and non-vanishing mass, we can introduce the following n quantities $U_{[\mu_1, \nu_1] \mu_2 \dots \mu_n}^{(1)}, U_{[\mu_1, \nu_1] [\mu_2, \nu_2] \mu_3 \dots \mu_n}^{(2)}, \dots, U_{[\mu_1, \nu_1] \dots [\mu_n, \nu_n]}^{(n)}$ defined by¹³⁾

$$U_{[\mu_1, \nu_1] \mu_2 \dots \mu_n}^{(1)} = \frac{\partial}{\partial x_{\mu_1}} U_{\nu_1 \mu_2 \dots \mu_n} - \frac{\partial}{\partial x_{\nu_1}} U_{\mu_1 \mu_2 \dots \mu_n} = \partial_{\mu_1 \nu_1} U_{\mu_2 \dots \mu_n},$$

$$U_{[\mu_1, \nu_1] \dots [\mu_n, \nu_n]}^{(n)} = \partial_{\mu_1 \nu_1} \dots \partial_{\mu_n \nu_n} U_{[\mu_1, \nu_1] [\mu_2, \nu_2] \dots [\mu_{n-1}, \nu_{n-1}]}^{(n-1)}.$$

As these quantities $U^{(\alpha)}$'s satisfy the same wave equations and subsidiary conditions as those of the original field $U \dots$, the degree of the highest derivatives in their commutation relations are equal to those of U . We obtain, therefore, that $\alpha_i^a = 0$ for $\partial_{\mu\nu} U$.

In terms of the spinor notation, including the fields of both integer as well as half integer spin, these circumstances are also stated as follows: Let $\varphi_{r\alpha_1 \dots \alpha_n}^{t_1 \dots t_n}(x)$ and $\chi_{r\alpha_1 \dots \alpha_n}^{t_1 \dots t_n}(x)$ be the field quantities with non-vanishing mass. They satisfy the wave equations:

$$\partial_{rs} \varphi_{\alpha_1 \dots \alpha_n}^{t_1 \dots t_n} = i \kappa \chi_{r\alpha_1 \dots \alpha_n}^{t_1 \dots t_n},$$

where ∂_{rs} is the derivation operator in the spinor notation. From $\varphi_{\alpha_1 \dots \alpha_n}^{t_1 \dots t_n}$ and $\chi_{\alpha_1 \dots \alpha_n}^{t_1 \dots t_n}$, the field quantities $\varphi_{\alpha_1 \dots \alpha_n}^{(\alpha) t_1 \dots t_n}$ and $\chi_{\alpha_1 \dots \alpha_n}^{(\alpha) t_1 \dots t_n}$ are introduced in the following way,¹³⁾

$$\varphi_{\alpha_2 \dots \alpha_n}^{(1) s u_1 t_1 \dots t_n} = \frac{i}{\kappa} \partial_{u_1 \alpha_1} \varphi_{\alpha_2 \dots \alpha_n}^{s t_1 \dots t_n},$$

$$\varphi_{\alpha_{a+1} \dots \alpha_n}^{(\alpha) s u_1 \dots u_a t_1 \dots t_n} = \frac{i}{\kappa} \partial_{u_a \alpha_a} \varphi_{\alpha_{a+1} \dots \alpha_n}^{s u_1 \dots u_{a-1} t_1 \dots t_n},$$

$$\chi_{\alpha_{a+1} \dots \alpha_n}^{(n+1) s u_1 \dots u_a t_1 \dots t_n} = \varphi_{\alpha_{a+1} \dots \alpha_n}^{(n) s u_1 \dots u_a t_1 \dots t_n}.$$

Then, $(\varphi_{\alpha_1 \dots \alpha_n}^{(\alpha) t_1 \dots t_n}, \chi_{\alpha_1 \dots \alpha_n}^{(\alpha) t_1 \dots t_n})$ describes the fields satisfying the same wave equations as those of $(\varphi_{\alpha_1 \dots \alpha_n}^{t_1 \dots t_n}, \chi_{\alpha_1 \dots \alpha_n}^{t_1 \dots t_n})$ and the degree of the highest derivatives in the commutation relations of $(\varphi_{\alpha_1 \dots \alpha_n}^{(\alpha) t_1 \dots t_n}, \chi_{\alpha_1 \dots \alpha_n}^{(\alpha) t_1 \dots t_n})$ are the same as those of $(\varphi_{\alpha_1 \dots \alpha_n}^{t_1 \dots t_n}, \chi_{\alpha_1 \dots \alpha_n}^{t_1 \dots t_n})$. Therefore, the true degree of the operator $\partial_{rs} \varphi_{\alpha_1 \dots \alpha_n}^{t_1 \dots t_n}$ is found to be zero.

In conclusion, it should be noted that the above-mentioned arguments concerning the true degree give an example which shows that the special structure of the interactions could weaken the divergence of the interaction.

* This situation illustrates the well known fact that in the case of vector field the degree of highest power of the momentum in the Fourier amplitude of $K_{\mu\nu} (\equiv \partial_\mu U_\nu - \partial_\nu U_\mu)$ is same as that of U_μ .

Note added in proof:

The characteristic difference between interactions of the first kind and those of the second kind may also be seen in the energy dependency of the cross section. In the former case the power of external energies in the leading term at the high energy region is left unchanged whatever order of approximation we may take into account. In the latter case, however, the degree of external energies will be increased unimittedly when the higher order effects are considered. These circumstances are easily understood either from the dimensional consideration, or for the former case, from the procedure of the renormalization. Hence, we may say as follows: The damping effects of the field reaction do not give any appreciable effect in the theory having interaction of the 1st kind only as long as the coupling constants are very small, while they lead to drastic effects in the theory containing interactions of the 2nd kind even if the coupling constants are very small. The conclusion for the former theory, though already well known by the results of the lowest order approximation, is guaranteed for the first time after the inherent divergences have been removed by the renormalization method. In this connection, it should also be interesting to notice a possibility that the strong singularity in the latter theory may be altogether compensated by the correspondingly strong damping effects in a way recently suggested by Feynman, Hu (Phys. Rev. **80** (1950), 1109) and Thirring (Z. f. Naturf. **6a** (1951), 462).

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Radiative Correction to Decay Processes, III*—Forbidden Beta-Transitions caused by Radiation Processes—*

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Forbidden beta-transitions caused by the radiative correction are investigated on the basis of the effective Hamiltonian which we derived, as presented in the previous paper, using the covariant formalism. The ultraviolet divergence inherent in the virtual photon processes has been removed with the aid of the regulator method. The results are compared with those described by other authors with the use of the non-covariant perturbation theory.

§ 1. Introduction

In the ordinary theory of beta-decay developed by Fermi, Konopinski and Uhlenbeck,¹⁾ forbidden transitions are considered as due firstly to the presence of the relativistic part of the interaction Hamiltonian and secondly to the retardation of lepton waves in the interior of the nucleus. However, it has recently been pointed out by Longmire,²⁾ Lipkin³⁾ and Merzbacher⁴⁾ that radiative corrections also give another cause for forbidden beta-transitions. This is interesting in connection with the fact that there remain a few exceptional nuclei for which the transition schemes have not yet been unambiguously established on the basis of the Fermi theory.

Of the radiative corrections, Longmire studied the first-order process in which a gamma-ray with a continuous spectrum was emitted from the nucleus simultaneously with a beta-ray, while the latter two authors took up the second-order process in which a virtual photon was exchanged between the beta-electron and the nucleus. According to Merzbacher, the transition rate for the latter process is at least 10^3 times as great as that for the former process, and the cause of this behavior is ascribed to the large number of intermediate states available to the virtual photon. As is well known, however, integrals over the momentum space occupied by virtual photons usually diverge at the high-frequency limit. The cause of convergence of Merzbacher's result seems to lie in the fact that he enumerated only those events in which the beta-emission and the gamma-emission (or gamma-reabsorption) are brought about separately by two different nucleons in the nucleus, omitting the

process in which a single nucleon plays the double rôle of the beta-emitter and the gamma-emitter (or gamma-reabsorber). The effective Hamiltonian responsible for this process has been calculated before by the present authors and their coworkers chiefly with a view to analyzing the behavior of divergence. As was reported in the previous paper,⁵⁾ after the mass renormalization has been performed there still remains in the effective Hamiltonian a divergent term which, being different in transformation property, can not be amalgamated into the original Hamiltonian. The present paper is concerned firstly with the removal of this divergence difficulty by means of the regulator,⁶⁾ and secondly with a discussion of forbidden transitions caused by the radiative correction on the basis of the effective Hamiltonian thus obtained.

§ 2. Regulated effective Hamiltonian

The regulation of the second order effective Hamiltonian $I_F^{(2)}(x)$ can be attained at once by the following replacement in the expressions for $\mathcal{L}^{(2)}$, $Z^{(3)}$ and $K(p, \gamma; P, I')$ given by (3.16) and (2.24)* of II:

$$\left. \begin{aligned} \log(2k_{\max}/m) &\rightarrow 1/2 + (1/2) \sum_{i \neq 0} C_i \log(m^2/x_i^2), \\ \log(2k_{\max}/M) &\rightarrow 1/2 + (1/2) \sum_{i \neq 0} C_i \log(M^2/x_i^2), \end{aligned} \right\} \quad (2.1)$$

where C_i and x_i ($i=1, 2, 3, \dots$) represent the weights and masses, respectively, of the auxiliary photon (or more properly speaking, neutral vector meson) fields, which are assumed to satisfy the following conditions:

$$\left. \begin{aligned} \sum_{i=0} C_i &= 0, \quad (C_0=1), \\ \sum_{i \neq 0} C_i/x_i^2 &\rightarrow 0 \quad \text{as} \quad x_i^2 \rightarrow \infty. \end{aligned} \right\} \quad (2.2)$$

$V_F^{(2)}(x)$, thus regulated, can be greatly simplified by neglecting small terms of the order of (m/M) into the form

$$\begin{aligned} V_F^{(2)}(x) &= g(d/4\pi) \bar{\Psi}(x) \bar{\varphi}(x) \beta B \{ \zeta(pP) + \eta \sigma \Sigma \\ &\quad + (1/2)(\gamma I') (i\gamma P/M) \} \phi(x) \Psi(x) + \text{conj.}, \end{aligned} \quad (2.3)$$

with

$$\left. \begin{aligned} \zeta(pP) &= [\log(m/2k_{\min}) + \log(M/2k_{\min})] (2 - pPF_0) \\ &\quad + 1 + \log(M/m) - pPF_0 - pPG - MmF_0(i\gamma P/M), \\ \eta &= (1/4) \log(M/m) - 3/8 - (1/4) \sum_{i \neq 0} C_i \log(mM/x_i^2). \end{aligned} \right\} \quad (2.4)$$

It is to be noted here that, as was the case with Steinberger in his calculation of meson decay,⁷⁾ the regulator leaves its trace, in the final results, in the form of $\sum_{i \neq 0} C_i \log(mM/x_i^2)$ whose value we can not determine uniquely.

In the approximation in which the proton momentum P is neglected, the first term of (2.3), which is responsible for cancellation of the infra-red divergence inherent in the inner bremsstrahlung, does not give rise to any modification of the selection rules for

nuclear transitions, whereas the remaining two terms cause forbidden transitions of the type hitherto ignored, since the operators $\gamma I'$ and $\sigma \Sigma'$ when multiplied into βB produce various forms of interaction as shown in Table I.⁽⁸⁾ The forbidden transitions due to these two terms will be discussed in the next section.

§ 3. Forbidden transitions arising from $V_F^{(2)}(x)$

As has been remarked by Longmire and Merzbacher, forbidden transitions resulting from the radiative correction are obscured by the ordinary direct transitions except when the latter are absolutely forbidden. If we confine ourselves to the approximation in which the proton momentum and the retardation effects of lepton waves are both neglected but the relativistic part of $\beta B \sigma \Sigma'$ is retained, the absolutely forbidden transitions caused by $I_F^{(2)}(x)$ are restricted to the following four cases: (S) transition with spin change $1 \leftrightarrow 0$ and parity change no, through scalar interaction; (V) $0 \rightarrow 0$ yes for vector interaction; (A) $0 \rightarrow 0$ no for axial vector interaction; (P) $1 \leftrightarrow 0$ yes for pseudo-scalar interaction. The probability that a beta-electron is emitted with energy between W and $W+dW$ in the unit of mc^2 is given by the well-known allowed formula

$$P^{(2)}(W)dW = (g^2/2\pi^3) (W^2-1)^{1/2} W (W_0-W)^2 dW \times (a/4\pi)^2 \eta^2 \xi^2 |\mathfrak{M}|^2, \quad (3.1)$$

with $\xi=2$, $\mathfrak{M}=\beta\sigma$ for (S) $1 \leftrightarrow 0$ no; $\xi=6$, $\mathfrak{M}=\gamma_5$ for (V) $0 \rightarrow 0$ yes; $\xi=6$, $\mathfrak{M}=1$ for (A) $0 \rightarrow 0$ no; and $\xi=2$, $\mathfrak{M}=\beta\alpha$ for (P) $1 \leftrightarrow 0$ yes. In deriving (3.1), the plane wave solutions have been substituted for $\psi(x)$ and $\varphi(x)$. The effect of the nuclear charge can be expressed simply by multiplying the Fermi factor $F(Z, W)(1+s)/2$ into (3.1), so far as the transitions without parity change are concerned. For the transitions (V) $0 \rightarrow 0$ yes and (P) $1 \leftrightarrow 0$ yes, however, the first-order effects of the proton momentum and the retardation of lepton waves must be taken into account. The nuclear matrix element $|\mathfrak{M}|^2$ in (3.1) is then replaced by a correction factor which contains $|\mathfrak{f}\sigma \cdot \mathbf{r}|$ and $|\mathfrak{f}\sigma \cdot \mathbf{P}|$ besides $|\mathfrak{f}\gamma_5|$ for (V) $0 \rightarrow 0$ yes transition; or $|\mathfrak{f}\sigma \times \mathbf{r}|$ and $|\mathfrak{f}\sigma \times \mathbf{P}|$ besides $|\mathfrak{f}\beta\alpha|$ for (P) $1 \leftrightarrow 0$ yes transition. In the approximation of this order, absolutely forbidden transitions of the following types are also expected to occur: (S) $0 \rightarrow 0$ yes, $|\mathfrak{f}\mathfrak{M}|^2 = |\mathfrak{f}\sigma \cdot \mathbf{r}|^2 \cdot A''$; (S) $2 \leftrightarrow 0$ yes, $|\mathfrak{f}\mathfrak{M}|^2 = \sum_{ij} |B_{ij}|^2 \cdot 3a$; (P) $0 \rightarrow 0$ no, $|\mathfrak{f}\mathfrak{M}|^2 = |\mathfrak{f}\beta\alpha \cdot \mathbf{r}|^2 \cdot A''$; and (P) $2 \leftrightarrow 0$ no, $|\mathfrak{f}\mathfrak{M}|^2 = \sum_{ij} |A_{ij}|^2 \cdot 3a + \sum_{ij} |T_{ij}|^2 \cdot (1/12)[3D_+ - c] - [\sum_{ij} T_{ij} A_{ij}^* + c.c.] \cdot \frac{E}{2}$.⁽⁹⁾ If an account is taken of the higher-order effects of the proton momentum and the retardation of lepton waves, a variety of absolutely forbidden transitions will further be found, but we shall not enter into this problem.

Let us now examine the competition in transition probability between the forbidden transition due to the radiative correction and the ordinary forbidden transition in the case where both obey the same selection rules. According to the theory of Konopinski and Uhlenbeck, the probability for the n th forbidden transition is characterized by a factor of the order of $(v/c)^{2(n-1)} (W_0 R)^{2n} (I=0, 1, \dots, n)$. On the other hand, the probability for

the forbidden transition caused by the second-order radiative correction bears a factor of the order of α^2 . Hence it is expected that the forbidden transition due to $V_F^{(2)}(x)$ will be weak as compared with the ordinary first forbidden transition, but it will be a match for the second forbidden transition. Of the second forbidden transitions, those for which the radiative correction can not be ignored are: (V) $1 \longleftrightarrow 0$ no; and (T) $0 \rightarrow 0$ no. Neglecting the effect of the nuclear charge, we have the following beta-spectrum for these transitions:

$$P(W)dW = (g^2/2\pi^3) W(W^2-1)^{1/2} (W_0-W)^2 dW \times C_2, \quad (3.2)$$

with

$$C_2 = |\{ \boldsymbol{\alpha} \times \mathbf{r} \}^2 A'_-(Z=0) - (u/4\pi) (6\eta+1) (1/3) (W_0-2W+W^{-1}) \\ \times \{ (\{ \boldsymbol{\alpha} \times \mathbf{r} \}^* \cdot (\{ \boldsymbol{\sigma} \} + c.c.) + (u/4\pi)^2 (6\eta+1)^2 |\{ \boldsymbol{\sigma} \}|^2 \\ \text{for (V) } 1 \longleftrightarrow 0 \text{ no} \quad (3.3)$$

and

$$C_2 = |\{ \beta \boldsymbol{u} \cdot \mathbf{r} \}^2 A''_-(Z=0) + (u/4\pi) \{ 4\eta (W_0-2W+W^{-1}) + (W_0-W) (2W)^{-1} \} \\ \times \{ (\{ \beta \boldsymbol{u} \cdot \mathbf{r} \}^* (\{ 1 \} + c.c.) + |\{ 1 \}|^2 (u/4\pi)^2 (36\eta) (4\eta-1+W^{-1}) \\ \text{for (T) } 0 \rightarrow 0 \text{ no.} \quad (3.4)$$

From these expressions it is inferred that, for light nuclei for which A'_- or A''_- is of the order of unity, the transition caused by the radiative correction will prevail over the ordinary direct transition, if we grant that the arbitrary constant η is also of the order of unity. (The assumption $\sum_{i \neq j} C_i \log(mM/x_i^2) = 0$ leads to $\eta \approx 1.5$). For heavy nuclei, however, the radiative correction will be negligible, since A'_- or A''_- then takes on a large value ($\sim (\alpha Z/2R)^2$).

§ 4. Forbidden transitions caused by $V_F^{(1)}(x)$

With the Lorentz condition in mind, the first-order effective Hamiltonian given by (2.9) of II may be rewritten into the form

$$V_F^{(1)}(x) = (4\pi u)^{1/2} g \bar{\psi}(x) \bar{\varphi}(x) \{ (Pk)^{-1} (PA) - (pk)^{-1} (pA) - (1/4) (pk)^{-1} \hat{\gamma}_{\mu\nu} \hat{F}_{\mu\nu} \} \\ \cdot \psi(x) \Psi(x) + (4\pi u)^{1/2} g \bar{\psi}(x) \bar{\varphi}(x) (1/4) (Pk)^{-1} \sum_{\mu\nu} \hat{F}_{\mu\nu} \psi(x) \Psi(x) + \text{conj.}, \quad (4.1)$$

where $\hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ represents the electromagnetic field strengths. The first term of this expression accounts for the inner bremsstrahlung associated with the sudden acceleration of the beta-electron.¹⁰⁾ It does not give rise to any modification of the selection rules, so far as we confine ourselves to the approximation in which the proton momentum \mathbf{P} is neglected. The second term, which causes forbidden transitions, has a small magnitude of the order of m/M as compared with the first term. This is probably natural, if we consider that the acceleration of the recoil proton is smaller by a factor of about m/M than that of the beta-electron.

It is to be noted that the operator $(Pk)^{-1}(PA)$ contributes to the inner bremsstrahlung emitted from the beta-electron, although it depends apparently on the energy-momentum of the proton but not on that of the lepton field. The reason for this paradoxical behavior is that we have carried out the calculations without eliminating the scalar and the longitudinal parts of the electromagnetic field.

The forbidden transition caused by the second term of (4.1) is similar in shape of nuclear matrix to that arising from the second term of (2.3). In the approximation such as that made in deriving (3.1), the probability that a beta-electron with energy between W and $W+dW$ and a photon with energy between k and $k+dk$ in the unit of mc^2 are simultaneously emitted is given by

$$P^{(1)}(W, k) dW dk = (g^2/2\pi)^3 W(W^2-1)^{1/2} (W_0 - W - k)^2 dW (dk/k) \\ \times (k^2/12M^2) (u/4\pi) (\xi/2) |\mathfrak{M}|^2, \quad (4.2)$$

with $\xi=2$ and $\mathfrak{M}=\beta\sigma$ for (S) $1 \longleftrightarrow 0$ no; $\xi=6$ and $\mathfrak{M}=\gamma_5$ for (V) $0 \rightarrow 0$ yes; $\xi=6$ and $\mathfrak{M}=1$ for (A) $0 \rightarrow 0$ no; $\xi=2$ and $\mathfrak{M}=\beta a$ for (P) $1 \longleftrightarrow 0$ yes; $\xi=6$ and $\mathfrak{M}=\sigma$ for (V) $1 \longleftrightarrow 0$ no; and $\xi=12$ and $\mathfrak{M}=\beta$ for (T) $0 \rightarrow 0$ no. Of these transitions, the first four are absolutely forbidden in the ordinary theory, while the selection rules for the last two are satisfied also by the ordinary second forbidden transitions.

Integrated with respect to k from 0 up to $(W_0 - W)$, (4.2) becomes

$$P^{(1)}(W) dW = (g^2/2\pi^3) W(W_0 - W)^2 (W^2 - 1)^{1/2} dW \\ \times (u/4\pi) [(W_0 - W)/12M]^2 (\xi/2) |\mathfrak{M}|^2. \quad (4.3)$$

Taking the ratio of (4.3) to (3.1), we have

$$P^{(1)}(W)/P^{(2)}(W) = [(W_0 - W)/12M]^2 (4\pi/u) (2/\xi) \eta^{-2}, \quad (4.4)$$

from which it is observed that the forbidden transition originating in the first order radiative correction is roughly 10^{-5} times as weak as that arising from the second order radiation process, should we grant that the arbitrary constant η were of the order of unity.

§ 5. Discussions

First let us compare our results given in § 3 with those of Merzbacher. As an example we take the absolutely forbidden transition, (S) $1 \longleftrightarrow 0$ no, for which the probability is evaluated by Merzbacher as follows:

$$P^{(M)}(W) dW = (g^2/2\pi^3) W(W^2-1)^{1/2} (W_0 - W)^2 dW \\ \times (u/4\pi)^2 (1/2)^2 |\int (\mathbf{R}/R) \times \beta \mathbf{a}|^2, \quad (5.1)$$

where \mathbf{R} denotes a vector drawn from the beta-decaying nucleon up to a proton that acts as an emitter or a reabsorber of a virtual photon. On comparing (5.1) with (3.1), we see that both have an allowed spectrum, and that in (3.1) 2η and $\beta\sigma$ take the places of $1/2$ and $(\mathbf{R}/R) \times \beta \mathbf{a}$, respectively, of (5.1). Since $|\int \beta\sigma|^2 \approx 1$ and $|\int (\mathbf{R}/R) \times \beta \mathbf{a}|^2 \approx (v/c)^2 \approx 10^{-2}$, it turns out that, of the second order radiative corrections, the

process described by $V_F^{(2)}(x)$ would be roughly 100 times as frequent as the Merzbacher detour process, granted that η were of the order of unity. The ft -value estimated by Merzbacher on the basis of (5.1) is of the order of 10^{11} or 10^{12} . If (3.1) with $\eta \approx 1$ is used instead of (5.1), it can be reduced to 10^9 , which exactly agrees with the observed value for the transition $C^{14} \rightarrow N^{14}$. However, as has been remarked by Merzbacher, this agreement does not lead us to the conclusion that the discrepancy between the spectrum shape and the half-life of C^{14} can be settled by taking account of radiative correction, because the scalar interaction alone can not explain the allowedness of the beta-transitions of the Gamow-Teller type such as $He^6 \rightarrow Li^6$, $F^{18} \rightarrow O^{18}$, and $Al^{26} \rightarrow Mg^{26}$.

Next, we compare our results for the first order radiative correction given in §4 with those of Longmire, who, using the non-covariant perturbation theory, first calculated the half-lives of nuclei decaying by simultaneous beta-gamma-emission. For the beta-gamma-spectrum he gives

$$P^{(L)}(W, k) dW dk = I_0 W (W^2 - 1)^{1/2} (W_0 - W - k) dW (dk/k) k^4 (W_1 - k)^{-2}, \quad (5.2)$$

where I_0 is a constant which depends on W_0 and W_1 but not on W and k , $W_1 (> W_0)$ denoting the energy difference between the first excited, intermediate level and the ground level of the final nucleus. (The gamma-emission is treated as succeeding the beta-emission. If the order of emissions is reversed, $(W_1 - k)^{-2}$ should be replaced by $(W_2 + k)^{-2}$, where $W_2 (> 0)$ denotes the energy needed in order to excite the initial nucleus from the ground level to the first excited, intermediate level.) Our spectrum (4.2) disagrees with (5.2) by a factor $k^2 (W_1 - k)^{-2}$. This is due to the fact that our calculation is based on the covariant formalism, which at the present stage can not afford to take into consideration the effect of the nuclear binding on the intermediate states.

Our estimate of the second-order radiative correction depends essentially on the arbitrary constant η . There seems to be no criterion to determine this constant theoretically. We note here only that it takes a value $(1/4) \log(M/m) - 3/8 \approx 1.5$, if we put $\sum_{i=1}^N C_i \log(mM/x_i^2) = 0$, which corresponds to Steinberger's fourth condition.⁷ With $g \approx 5 \times 10^{-12} (mc^2) (\hbar/mc)^3$ and $\eta \approx 1.5$, the ft -value for a forbidden transition caused by $V_F^{(2)}(x)$ is estimated to be roughly

$$ft = 3 \times 10^8 / \xi^2 |\mathcal{M}|^2, \quad (5.3)$$

which lies in the region between the first and the second forbidden groups in the histogram of ft -values.¹¹⁾

According to the results of beta-analyses accumulated by many investigators the tensor interaction seems to be the most promising of all the five forms. If this form of interaction is assumed, there are at present no absolutely forbidden transitions, and the possibility that the second-order radiative correction is detected by experiment will be expected only in its competition with the second or higher forbidden transitions. Of the second forbidden transition, the only case in which the radiative correction can not be ignored is the (T) $0 \rightarrow 0$ no transition of light nuclei, for which the result is given by (3.4). From the considerations based on Mayer's nuclear shell model,¹²⁾ however, transitions of this type seem

unlikely to occur between ground levels of light nuclei.

Table I. Modification of interaction form caused by radiative correction.

	βB	$\beta B(\gamma\Gamma)$	$\beta B\sigma\Sigma$
S	1	$\gamma\Gamma$	$2(\sigma\Sigma/2)$
V	$\gamma\Gamma$	$4-2(\sigma\Sigma/2)$	$-6\gamma\Gamma-6\gamma_5\Gamma_5\gamma\Gamma$
T	$\sigma\Sigma/2$	$-3\gamma\Gamma-3\gamma_5\Gamma_5\gamma\Gamma$	$12-8(\sigma\Sigma/2)+12\gamma_5\Gamma_5$
A	$-\gamma_5\Gamma_5\gamma\Gamma$	$2(\sigma\Sigma/2)-4\gamma_5\Gamma_5$	$6\gamma\Gamma+6\gamma_5\Gamma_5\gamma\Gamma$
P	$\gamma_5\Gamma_5$	$\gamma_5\Gamma_5\gamma\Gamma$	$2(\sigma\Sigma/2)$

Finally, we make a remark on the beta-spectrum of RaE. Very recently Petschek and Marshak¹³ have shown that this spectrum can be explained, consistently with the parity prediction of the Mayer shell, as due to the $0 \rightarrow 0$ yes transition by assuming the presence of the pseudo-scalar interaction besides the usually accepted tensor interaction. The value 13 ± 1 , which must be assumed for $\Gamma = -i\lambda_p \int \gamma_5 / \lambda_T (\int \sigma \cdot \mathbf{r})$ in order for the linear combination of the two forms of interaction to give a satisfactory fit of the RaE beta-spectrum, implies that the weight λ_p of the pseudo-scalar interaction is several times as great as that λ_T of the tensor interaction. As may be seen from (2.3) and Table I, the pseudo-scalar interaction appears of itself as a result of the radiative correction, even if we assume the tensor interaction alone but not the coexistence of the pseudo-scalar interaction from the beginning. However, its ratio to the original tensor interaction being $12\gamma_5(\alpha/4\pi)$, the pseudo-scalar interaction arising from the radiative correction will be too weak to account for the phenomenologically required value of λ_p/λ_T , though γ_5 is arbitrary. In connection with the fact that there has not yet been found any experimental evidence to indicate that the scalar interaction should not be added to the (TP) combination, we wish to note here that the divergence difficulty could be removed at least so far as the second-order radiative corrections are concerned by the procedure of renormalization of the Fermi constant without the use of regulator, if the linear combination should happen to satisfy the condition $\lambda_S = \lambda_P = 3\lambda_T$ or $\lambda_S = \lambda_P = -\lambda_T$.¹⁴⁾

Notes and References

*) Erratum: On the right-hand side of (2.24) of II, the first two lines should be read

$$\begin{aligned} & (a/4\pi) \{ \log(2k_{\max}/m) + \log(2k_{\max}/M) + 1/2 + (1/4)(M^2 - m^2)F_1 \\ & - (1/4)(M^2 + m^2 + 2\not{P})F_0 \} (1/4) \gamma_\mu \gamma_\nu \Gamma_\mu \Gamma_\nu \\ & - (a/4\pi) \{ [\log(m/2k_{\min}) + \log(M/2k_{\min}) + 2] F_0 + G \} \not{P}. \end{aligned}$$

- 1) E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60** (1941), 308; E. Konopinski, Rev. Mod. Phys. **15** (1943), 209; C. S. Wu, Rev. Mod. Phys. **22** (1950), 386.
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- 9) Konopinski's notations are used.
- 10) J. K. Knipp and G. Uhlenbeck, Physica **3** (1936), 425; F. Bloch, Phys. Rev. **50** (1936), 272; C. S. Wang-Chang and D. L. Falkoff, Phys. Rev. **76** (1949), 365.
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- 12) M. G. Mayer, Phys. Rev. **78** (1950), 16, 22.
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- 14) Refer to Eq. (4.8), (iib) or (iic) of II, and reference 8.

On the Renormalization in Quantum Electrodynamics

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The new renormalization procedure in quantum electrodynamics is presented in this paper. With this procedure, the finite S-matrix can be automatically obtained by use of the modified commutation relations and propagation functions and, therefore, without introducing the counter terms of mass- and charge-types. The various significant features of this new renormalization procedure are also discussed.

§ 1. Introduction and summary

As is well known, in the theory of the S-matrix in quantum electrodynamics there appear four infinite constants (∂x , Z_1 , Z_2 and Z_3).¹⁾ The contributions of Z_1 and Z_2 , however, cancel each other in virtue of the Ward's identity $Z_1 = Z_2$,²⁾ and so the infinities remaining in the final expression of S-matrix are those of ∂x and Z_3 , which are regarded as the induced mass and charge due to the reaction of the self-field.

In the usual renormalization theory these two infinities are removed by means of the counter terms of mass and charge. In the present paper the new renormalization method is presented, in which the introduction of such counter terms is unnecessary.

As is well known, the diverging term ∂x of the mass type of the electron can be eliminated by means of a canonical transformation, which leads to the result of changing the mass x_0 into $x_0 + \partial x$. Therefore, it is expected that without introducing the counter terms of mass type we can eliminate the diverging term by using the commutation relation of the electron field with the mechanical mass $x_0 = x - \partial x$ (and so the propagation function $S_F(x_0)$) and regarding x as the finite, observable mass.*

The charge renormalization term Z_3 changes the original charge e into the observable charge e_1 . This effect amounts to regarding the vacuum as the polarizable medium with the field independent polarizability. Therefore, this type of divergence may be removed by starting with the Lagrangian function of the electromagnetic field in the medium having the constant polarizability. As is evident from the argument of covariance, the Lagrangian function of the electromagnetic field in such a medium has the same form as the usual one excepting the constant numerical factor, and so the commutation relations of electromagnetic potentials A_μ 's is modified by the inverse ratio of this constant factor (cf. § 2. 1). Thus, we can expect that the infinite term can be removed by using such a commutation relation and without using the counter term as was introduced by Gupta.³⁾

* This expectation was previously pointed out in connection with the treatment of the meson cloud: H. Umezawa, Y. Takahashi and S. Kamefuchi, *Phys. Rev.* **85** (1952), 505.

In fact, we can prove in the following section that the infinities ∂x and Z_3 can be removed by means of the modified commutation relations (and so the propagation functions) of the electron and the electromagnetic fields.

The characteristic features of this new renormalization procedure are as follows: First, in this form of the theory Dyson's opinion⁴⁾ that the renormalization theory is the non-singular form which is equivalent to the usual singular form of the quantum electrodynamics is given a firm base, because we start with the Lagrangian function which is equal and thus equivalent to the usual one and then develop the theory in a different direction by use of different operators and representation. Second, although our discussion is developed by means of power series expansion in the interaction constant, the present formalism may be frequently advantageous to develop the renormalization procedure in the calculations which are not necessarily based on the perturbation approximation of the weak coupling, because we do not introduce the counter terms of the mass and charge, but consistently use the commutation relations with modified parameters,* by which all the infinite terms are eliminated automatically. In this connection, when remembering that the propagation functions with radiative corrections, S'_F , D'_F (see below) are equivalent to the respective Green functions as defined by Schwinger,⁵⁾ it seems very interesting that all the procedures of mass and charge renormalizations can be expressed in terms of electron and photon Green functions. Third, in this form of the theory the interpretation is given that the "vacuum" is a sort of "matter" which has the paramagnetic property (permeability $\kappa_m = (c^2 - c_1^2)/c_1^2 > 0$). Hence, the charge renormalization prescription is considered as equivalent to treating the vacuum as the matter.

§ 2. The renormalization method

1. As usual, the Lagrangian function of the system of the electron and electromagnetic fields is given by

$$L = -\frac{1}{4}F_{\mu\nu}^*F_{\mu\nu}^* - \frac{1}{2}\left(\frac{\partial A_\mu^*}{\partial x_\mu}\right)^2 - \left(\bar{\psi}_0\gamma_\mu\frac{\partial}{\partial x_\mu}\psi_0 + \kappa_0\bar{\psi}_0\psi_0\right) + ieA_\mu^*\bar{\psi}_0\gamma_\mu\psi_0, \quad (1)**$$

where quantities with the asterisk mean the unrenormalized quantities of the electromagnetic field and ψ_0 is the field operator of the electron field with the (mechanical) mass κ_0 .

Now, we shall introduce the observable (and so renormalized)

$$\text{quantities as follows: } A_\mu = \theta A_\mu^*, \quad F_{\mu\nu} = \theta F_{\mu\nu}^*, \quad \theta = e/e_1, \quad (2)$$

where e_1 is considered as the finite observable (and so renormalized) charge. Then, in terms of these observable quantities (1) can be rewritten as follows:

* For example, the strong coupling theory is based on those commutation relations and canonical transformation which pick out the effect of the bound field.

** We use natural unit $\hbar=c=1$ throughout this paper.

and

$$L = L_R + L_D + ie_1 A_\mu \bar{\psi}_0 \gamma_\mu \psi_0, \quad (3)$$

$$L_R = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} (1-f) - \frac{1}{2} \left(\frac{\partial A_\mu}{\partial x_\mu} \right)^2 (1-f), \quad (4)$$

$$f = -1 - \theta^{-2},$$

where L_D is the third term in (1) (the Lagrangian function of free electron field).

In contrast with the Gupta's renormalization theory³⁾ in which the term $(1/2) f(\partial A_\mu / \partial x_\mu)^2$ is omitted and the term $(1/4) f F_{\mu\nu} F_{\mu\nu}$ is regarded as the interaction term, we proceed in the following way: we regard L_R as the Lagrangian function of the free electromagnetic field and use the interaction representation in which the interaction is given by the last term in (3). Thus, in this representation we have the following commutation relations:

$$\begin{aligned} [A_\mu(x), A_\nu(x')] &= \frac{i \delta_{\mu\nu}}{1-f} D(x-x') \\ \{\psi_{0\alpha}(x), \bar{\psi}_{0\beta}(x')\} &= \frac{1}{i} S_{\alpha\beta}(x-x', x_0 = x - \delta x), \end{aligned} \quad (5)^*$$

where x is regarded as the finite observable mass of the electron.

The following discussions are developed in the momentum space. The propagation functions of photon with the momentum l and electron with the momentum t in momentum representation are given by

$$\begin{aligned} D_F &= \frac{1}{1-f} D_F(l), \\ S_F(t, x_0) &= \frac{S_F(x)}{1 + 2\pi i \frac{\partial S_F(x)}{\partial x}}, \end{aligned} \quad (6)^{**}$$

where $D_F(l)$ is the ordinary Feynman function of photon and $S_F(x)$ is the abbreviation of $S_F(t, x)$. The second equation of (6) is easily obtained directly or by means of the relations

$$\begin{aligned} \frac{\partial^n S_F(x)}{\partial x^n} &= n! (2\pi i)^n \{S_F(x)\}^{n+1}, \\ S_F(x_0) &= \sum_{n=0}^{\infty} \frac{(-\delta x)^n}{n!} \left\{ \frac{\partial^n S_F(x)}{\partial x^n} \right\}. \end{aligned}$$

It is shown in the following paragraphs that we can obtain the finite S-matrix by suitably choosing infinite values for two adjustable parameters δx and f respectively.

2. Following Dyson's program¹⁾, we use the true operators I'_μ , S'_F and D'_F which correspond to the vertex, electron and photon self-energy parts respectively and assume that

* We use the usual abbreviations $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$.

** This relation holds also in the configuration space in the sense of matrix multiplication.

$$\begin{aligned}\Gamma_\mu &= Z_1 \Gamma_{\mu 1}(e_1), \\ S'_F &= Z_2 S'_{F1}(e_1), \\ D'_F &= (Z_1 Z_2^{-1})^2 D'_{F1}(e_1),\end{aligned}\quad (7)$$

where Z_1 and Z_2 are the suitable infinite constants which are to be determined in such a way as to make $\Gamma_{\mu 1}(e_1)$, $S'_{F1}(e_1)$ and $D'_{F1}(e_1)$ finite.

On the other hand, S'_F and D'_F satisfy following integral equations:

$$\begin{aligned}S'_F &= S_F(x_0) + S_F(x_0) \Sigma^* S'_F, \\ D'_F &= \frac{1}{1-f} D_F + \frac{1}{1-f} D_F \Pi^* D'_F,\end{aligned}\quad (8)$$

where Σ^* and Π^* are the operators of the proper electron and photon self-energy parts respectively and further it is to be understood that all quantities on both sides refer to the same momentum. The solutions of (8) are formally written as

$$\begin{aligned}S'_F &= \frac{1}{1 - S_F(x_0) \Sigma^*} S_F(x_0) = \frac{1}{1 + (2\pi i) \partial_x S_F(x) - S_F(x) \Sigma^*} S_F(x), \\ D'_F &= \frac{1}{1 - \frac{1}{1-f} D_F \Pi^*} \frac{1}{1-f} D_F = \frac{1}{(1-f) - D_F \Pi^*} D_F,\end{aligned}\quad (9)$$

respectively. Further $D_F \Pi^*$ can be written in the following form by considering the general structure of proper photon self-energy part,*

$$\begin{aligned}D_F \Pi^* &= e_1^2 D_F \int \gamma_\mu S'_F \Gamma'_\mu S'_F d^4 p = e_1^2 D_F Z_1^{-1} Z_2^2 \int \gamma_\mu S'_{F1}(e_1) \Gamma'_{\mu 1}(e_1) S'_{F1}(e_1) d^4 p \\ &= \frac{1}{2\pi i} Z_1^{-1} Z_2^2 (C(e_1) + D_C(e_1)),\end{aligned}\quad (10)$$

where $C(e_1)$ is the logarithmically diverging constant and $D_C(e_1)$ is the finite function. Z_1^{-1} in the last expression of (10) is the contribution of the b -divergence.¹⁾

Similarly, $S_F(x) \Sigma^*$ can be written in the form

$$\begin{aligned}S_F(x) \Sigma^* &= e_1^2 S_F(x) \int \Gamma'_\mu S'_F \gamma'_\mu D'_F d^4 p = e_1^2 S_F(x) Z_1 Z_2^{-1} \int \Gamma'_{\mu 1} S'_{F1} \gamma'_\mu D'_{F1} d^4 p \\ &= Z_2^{-1} \left(A(e_1) S_F(x) + \frac{1}{2\pi} B(e_1) + \frac{1}{2\pi} S_C(e_1) \right),\end{aligned}\quad (11)$$

where both $A(e_1)$ and $B(e_1)$ are logarithmically diverging constants. It is to be noticed that $A(e_1)$ and $B(e_1)$ are defined by the expansion coefficients of Σ^* which are developed not in $(t_\mu \gamma_\mu - ix_0)$ but in $(t_\mu \gamma_\mu - ix)$, that is;

$$\Sigma^* = A(e_1) + B(e_1) (t_\mu \gamma_\mu - ix) + (t_\mu \gamma_\mu - ix) S_C(e_1).$$

* The final integration of $D_F \Pi^*$ is obtained by replacing two electron lines and one vertex by $S_{F'}$ and Γ_μ respectively in the expression of the irreducible photon self-energy graph with two vertices.

When we substitute (10) and (11) into (9) and take into account the following conditions

$$\lim_{\epsilon_1 \rightarrow 0} \begin{cases} Z_1, Z_2 \rightarrow 1, \\ C(e_1), D_c(e_1) \rightarrow 0, \\ S'_F \rightarrow S_F, D'_F \rightarrow D_F, \end{cases}$$

we see that (9) takes the form (7) for the following values of two parameters δx and f :

$$1-f(\equiv e_1^2/e^2) = (Z_1^{-1}Z_2)^2 \left(1 + \frac{1}{2\pi i} C(e_1)\right), \quad (12)$$

$$\delta x = \frac{1}{2\pi i} Z_2^{-1} A(e_1), \quad (13)$$

$$\left(Z_2 = 1 + \frac{1}{2\pi} B(e_1)\right).$$

These relations between various constants ($e, e_1, \delta x, Z_1, Z_2$ and f) are in agreement with Dyson's¹⁾ results.

Next, we shall consider the radiative correction of external lines. When we remember the relation (6) and the alternative integral equation of S'_F analogous to (8), the modification of external electron line can be written as follows;

$$\begin{aligned} \psi'(t) &= \psi_0(t) + S'_F(x_0) \Sigma^* \psi_0(t) = S'_F S_F^{-1}(x_0) \psi_0 \\ &= S'_F S_F^{-1}(x) (1 + (2\pi i) \delta x S_F(x)) \psi_0. \end{aligned} \quad (14)$$

Now, as was done when deriving the relation (11) we designate the electron state in terms of the *dressed* particle state, that is, as the free operator representing the external electron we does not use the operator $\psi_0(t)$ given by the equation $(t\gamma - ix_0)\psi_0(t) = 0, t^2 + x_0^2 = 0$, but use the operator $\psi(t)$ which satisfies $(t\gamma - ix)\psi(t) = 0, t^2 + x^2 = 0$. The required operator is given by

$$\psi(t) = \{1 + (2\pi i) \delta x S_F(x)\} \psi_0(t), \quad (15)$$

because as is easily seen $(t\gamma - ix)\psi(t) = 0$. From (14) and (15), we find

$$\psi'(t) = S'_F S_F^{-1}(x) \psi(t). \quad (14')$$

When considering that $S_c(e_1)\psi = 0$ since $S_c(e_1)$ in (11) is proportional to $(t\gamma - ix)$, we get $S'_F \psi(t) = Z_2 S_F(x) \psi(t)$. Substituting this expression into (14') we have

$$\psi'(t) = Z_2 \psi(t). \quad (14'')$$

But, as was pointed out by Dyson¹⁾, the numerical factor on the right hand side of (14'') is indeterminate between 1 and Z_2 according to the order in which the equation of motion is used. This factor is to be rather determined from the unitarity of the S-matrix and in this case we must take $Z_2^{1/2}$ instead of Z_2 :

$$\psi'(t) = Z_2^{1/2} \psi(t). \quad (14''')$$

From the physical point of view, this superfluous factor $Z_3^{1/2}$ is to be attributed to the effect of the source which has created the quanta of the external line.

The situations are the same for the external photon lines and we must take from the similar consideration ;

$$A'_\mu(l) = (Z_1 Z_2^{-1}) A_\mu(l). \quad (16)$$

In the case of external potentials ($l^2 \neq 0$), we must also take

$$A'_\mu(l) = (2\pi i) (Z_1 Z_2^{-1}) D'_{F'}(e_1) l^2 A_\mu(l^2) \quad (16')$$

by adjusting the magnitude of practical unit for measuring the potential.

From (7), (14'''), (16) and (16') we see that infinite factors Z_1^{-1} , Z_2 and $(Z_1 Z_2^{-1})$ are attached to each vertex as the contributions from vertex, electron and photon lines respectively and they compensate one another. Thus, we obtain the finite S-matrix.

Moreover, when we take into account the Ward's identity $Z_1 = Z_2$, the discussions are simplified and (7) shows that $D'_{F'}$ becomes finite in itself at each stage of the perturbation approximation. This finiteness of D'_i can also directly established by means of mathematical induction.

In the following we shall give some disconnected remarks.

3. First, we shall consider the application of our method to the system of other charged particles and electromagnetic field.

For the theory in which the mass type terms are eliminated by the canonical transformations, the mass renormalization will be performed in the similar way as above.

Our method of the charge renormalization will be useful, for the other quantum electrodynamics; too. Furthermore, the discussions shall be simplified if we use the Ward's identity which can be easily generalized from the general viewpoint of gauge invariance of the theory. In virtue of this identity various infinite constants Z_1, Z_2, Z_4, \dots , which appear in connection with various kinds of vertices, compensate one another in the expression of S-matrix and finally the renormalized charge e_1 will be given by $e_1 = Z_3^{1/2} e_{\text{bare}}$.

4. In conclusion, it is noteworthy that by regarding (4) as the free field Lagrangian of electromagnetic field we have modified the photon propagation function. The Lagrangian function (4) gives the field equations of electromagnetic field in the medium with the field independent polarizability. Since A_μ 's give the 6-vector \mathfrak{G} , $\mathfrak{B} (= F_{\mu\nu})$ as usual and \mathfrak{D} and \mathfrak{H} are defined by $\partial L_R / \partial \mathfrak{G}$ and $-\partial L_R / \partial \mathfrak{B}$ respectively, the equations of motion described by the Lagrangian function (3) are written in the usual form $\text{div } \mathfrak{D} = \rho$, $\text{rot } \mathfrak{H} = \partial \mathfrak{D} / \partial t + \mathfrak{J}$, $\text{rot } \mathfrak{G} = -\partial \mathfrak{B} / \partial t$, $\text{div } \mathfrak{B} = 0$. Hence, we find that the "vacuum" is equivalent to the "matter" having the polarizabilities :

* As is well known, there appear in general divergences which can not be removed by the renormalization. But, our method will be generally useful for the renormalization term corresponding to the electromagnetic charge.

** The problem of the observable electric charge of elementary particles in the renormalizable field theory will be discussed in detail in our forthcoming paper, Prog. Theor. Phys. in press.

$$\begin{aligned} \text{dielectric susceptibility} \quad \chi &= -f = (e_1^2 - e^2)/e^2, \\ \text{permiability} \quad \kappa_m &= f/(1-f) = (e^2 - e_1^2)/e_1^2. \end{aligned} \quad (17)$$

When we remember the result $(e_1^2 - e^2)/e^2 < 0$ obtained in our previous paper,⁷⁾ we see that

$$\chi < 0, \quad \kappa_m > 0, \quad (18)$$

that is, the vacuum shows the paramagnetic properties.

The fact that the effect of the charge renormalization term can be interpreted as corresponding to the polarizability of the vacuum has already been known in the theory of Dirac-Heisenberg-Weisskopf,³⁾ where the electromagnetic field was treated as C-number. Our results show that such an interpretation is also possible in the completely quantum mechanical treatment of the system. It is a very interesting fact that the charge renormalization procedure corresponds to regarding the "vacuum" as the polarizable "matter", that is, giving a sort of material structure to the vacuum.

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On the Density Matrix in Hartree-Field, I

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Part I The density matrix of many particle system is examined in the light of the theory of symmetric group, and is brought to reduced forms for convenient practical manipulations.

Part II. The density matrix of many electron system is computed eliminating spin variables. In this case, we have a sort of the theorem of large numbers with respect to the total spin quantum number.

§ 1. Introduction

It was W. Kefink¹⁾ who has shown that the density matrix of the many particle system may have an elegant outlook when viewed from the theory of symmetric group. However, his reasoning appears to lack logical consequence in some points, which we want to remedy in this paper. We consider the energy eigen-values of one particle system as given, and the system of particles may have any symmetry property. As we want to neglect all the exchange energy, the total energy is the sum of energies of all the particles. In short, we treat the problem from the point of view of Hartree's self-consistent theory, not of Hartree-Fock's theory.

Our treatment rests on the fact that the first-order perturbation theory leads automatically to the classification of wave-functions belonging to one term-system aimed at in Hartree's theory of self-consistent field. Summing up the results for all the term-systems, we arrive at exactly the same formula as Kofink has obtained. Furthermore, Kofink's "principal theorem" can be obtained by a direct integration with respect to coordinates. We have explicitly proved the theorem for the cases of one, two and three explicit coordinates.

§ 2. Density function for an irreducible representation of symmetric group

Let $\psi_1(q_1), \psi_2(q_2) \dots$ be wave functions of the self-consistent method. They are wave-functions of single particles with no interaction and can be treated as unperturbed functions. The coordinates q_i of the i -th particle may well imply the spin variable.

An irreducible representation of symmetric group corresponds to a partition of the integer n :

$$n = \lambda_1 + \lambda_2 + \dots + \lambda_k,$$

and we shall denote it by (λ) .

$$n_1\varepsilon_1 + n_2\varepsilon_2 + \cdots + n_n\varepsilon_n = E_\nu \quad (1)$$

is the energy of a non-perturbed term-system. The wave function which belongs to this term or to the irreducible representation (λ) is

$$\Phi_\nu^{(\lambda)}(q_1 q_2 \cdots q_n) = \sum_P C_P^{(\lambda)} \psi_1(Pq_1) \psi_2(Pq_2) \cdots \psi_n(Pq_n) = \sum_P C_P^{(\lambda)} \varphi(Pq), \quad (2)$$

where P denotes the permutation of the coordinates, and $C_P^{(\lambda)}$ is a coefficient which has to be determined by the properties of transformation of wave functions.

The density function is

$$W^{(\lambda)}(q_1 q_2 \cdots q_n) = \sum_\nu \exp(-E_\nu \beta) |\Phi_\nu^{(\lambda)}(q_1 q_2 \cdots q_n)|^2, \quad (3)$$

where the summation for ν is to be performed over all the term-systems, and $\beta = 1/kT$.

In the first place, we consider one term-system (1). As the permutation of coordinates which belong to one and the same energy ε_i , makes $\varphi(l')$ equal to $\varphi(E)$, such permutations constitute a subgroup g whose order is

$$n_1! n_2! \cdots n_\nu! = h_g.$$

Then we classify the permutation group by this subgroup and resolve it into lefthanded cosets.

$$G = g + R_1 g + \cdots + R_a g, \quad (a = n!/h_g).$$

R_i is a typical element of $R_i g$, and $\varphi(R_i)$ has a common value for all the elements of the coset $R_i g$, and γ denotes an element of g . Let us assume a zero-order wave function to be

$$\Psi_\kappa = \frac{1}{h_g} \sum_R \sum_\tau \delta^*(R\gamma) \varphi(R), \quad (4)$$

and perform the first order perturbation. The secular equation is

$$\begin{vmatrix} J(E) - \Delta\varepsilon & J(R_2) & J(R_3) & \cdots & J(R_a) \\ J(R_2^{-1}R_1) & J(E) - \Delta\varepsilon & J(R_2^{-1}R_3) & \cdots & J(R_2^{-1}R_a) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J(R_2^{-1}R_1) & \cdots & \cdots & \cdots & J(E) - \Delta\varepsilon \end{vmatrix} = 0, \quad (5)$$

where

$$J(S^{-1}R) = J(SR) = \int \varphi^*(S) H' \varphi(R) d\tau = \int \varphi^*(E) H' \varphi(S^{-1}R) d\tau,$$

and H' is a perturbing potential. We shall solve this equation by following the method which is expounded by Wigner²⁾, Delbrück³⁾ and Heitler⁴⁾.

Firstly, we consider a f -dimensional unitary representation of the symmetric group. The calculation which follows was done by Wigner for another purpose, in order to know how many f folded energy levels exist. When the unperturbed term-system has on

degeneracy, the answer for this problem is simply f . But the possible degeneracy makes the calculation difficult, and this number is not necessarily equal to f . The following calculation will be made on the line of Delbrück's paper.

We build up a group-matrix of f dimension from the representation in question and set up the secular equation

$$\frac{1}{h_g} \sum_k \sum_{\tau} \sum_{j=1}^f a_{kj}(R\gamma) b^j J(R\gamma) - \epsilon b^k = 0, \quad (k=1 \cdots f). \quad (6)$$

Since this representation is not always irreducible for the subgroup g , we want to reduce it by finding the number of identical representations contained in it. By means of the orthogonal relations of characters, this number is

$$f' = \frac{1}{h_g} \sum_{\tau} \chi(\gamma). \quad (7)$$

Changing columns and rows properly, the representation matrix takes the form fig. 1.1. where f' l's stand along on the diagonal line, and the hatched part belongs to the other irreducible representations. Any element of every coset is easily reduced by means of this figure. When $a_{ik}(R\gamma)$ is the matrix element of $R\gamma$, then it holds

$$a_{ik}(R\gamma) = \sum_{l=1}^{f'} a_{il}(R) a_{lk}(\gamma).$$

For $k \leq f'$, $a_{lk}(\gamma) = \delta_{lk}$ and consequently, $a_{ik}(R\gamma) = a_{ik}(R)$.
For $k > f'$,

$$a_{ik}(R\gamma) = \sum_{l=f'+1}^f a_{il}(R) a_{lk}(\gamma).$$

$\sum a_{lk}(\gamma)$ is zero, for $k > f'$, because $a_{lk}(\gamma)$ is identically zero for $l \leq f'$, $k > f'$, and further for $l > f'$, $a_{lk}(\gamma)$ is an element of none-identical representation, so that the summation over g is zero by virtue of the well-known relation in the matrix elements of an irreducible representation. Accordingly we know that the representation matrix of every element of one and the same coset has a common part of f' columns and rows as indicated in fig. 1. 2.

In these circumstances, we need only f' equations from the system for the purpose of obtaining energies. Let the energies be ϵ_m ($m=1, 2, \cdots f'$), and the corresponding vectors b_m^i ($j, m=1, 2, \cdots f'$), which satisfy the following orthogonal relations.

$$\sum_m b_m^{i*} b_m^j = \delta_{ij}, \quad \sum_i b_m^{i*} b_{m'}^i = \delta_{mm'}.$$

It will be easily seen that ϵ_m satisfies the secular equation (5). If we put

$$b^*(R\gamma) = \sum_{j=1}^{f'} a_{mj}(R\gamma) b^j. \quad (8)$$

The S -th equation of (5) runs now

$$\frac{1}{h_g} \left(\sum_R \sum_{\tau} b^*(R\gamma) J(S^{-1}R) - \varepsilon \sum_{\tau'} b^*(S\gamma') \right) = 0. \quad (9)$$

Substituting (8) into (9), we have

$$\begin{aligned} & \frac{1}{h_g} \left(\sum_R \sum_{\tau} \sum_{j=1}^{f'} a_{\kappa j}(R\gamma) b^j J(S^{-1}R) - \varepsilon \sum_{\tau'} \sum_{l=1}^{f'} a_{\kappa l}(S\gamma') b^l \right) \\ &= \frac{1}{h_g} \sum_{\tau} \sum_{l=1}^f a_{\kappa l}(S\gamma') \left(\frac{1}{h_g} \sum_R \sum_{\tau} \sum_{j=1}^f a_{lj}(\gamma'^{-1} S^{-1} R\gamma) b^j J(\gamma'^{-1} S^{-1} R\gamma) - \varepsilon b^l \right) = 0 \end{aligned}$$

because of (6).

It must be noticed that the summation over j or l may be extended to f instead of f' . Because, for example,

$$\begin{aligned} \sum_{\tau'} \sum_{l=1}^{f'} a_{\kappa l}(S\gamma') b^l &= \sum_{\tau'} \sum_{l=1}^{f'} \sum_{j=1}^f a_{\kappa j}(S) a_{jl}(\gamma') b^l \\ &= \sum_{j=1}^f \sum_{l=1}^{f'} a_{\kappa j}(S) b^l \sum_{\tau'} a_{jl}(\gamma') \\ &= \sum_{j=1}^f \left(\sum_{l=1}^f a_{\kappa j}(S) b^l \sum_{\tau'} a_{jl}(\gamma') - \sum_{l=f'+1}^f a_{\kappa j}(S) b^l \sum_{\tau'} a_{jl}(\gamma') \right), \end{aligned}$$

and the latter is zero, for $1 \leq j \leq f$ and $f' < l \leq f$, (summation over "Stellenzelle" of a non-identical representation is always zero.)

By this fact, we have seen that these $f' \varepsilon_m^s$ are contained in the term $n_1 \varepsilon_1 + n_2 \varepsilon_2 + \dots + n_v \varepsilon_v$.

When we perform the same process with respect to an other f_1 -dimensional representation of the symmetric group, we shall find the existence f_1' levels, and in this way all the term-systems will be obtained. If f' turns out to be zero, that is, if we find no identical representation, then it tells us non-existence of this energy level.

The f wave functions which belong to the fixed f -dimensional irreducible representation and to different eigen-values are orthogonal to each other,

$$\begin{aligned} \Psi_{\kappa m} &= \frac{1}{h_g} \sum_R \sum_{\tau} \sum_{j=1}^{f'} a_{\kappa j}(R\gamma) b_m^j \varphi(R), \quad (\kappa = 1, 2, \dots, f), \\ \int \Phi_{\kappa' m'}^* \Phi_{\kappa m} d\tau &= \frac{1}{h_g^2} \sum_R \sum_{\tau} \sum_{\tau'} \sum_{j=1}^{f'} \sum_{j'=1}^{f'} a_{j\kappa}(\gamma'^{-1} R^{-1}) a_{\kappa' j'}(S\gamma') b_m^{j*} b_{m'}^{j'} \int \varphi^*(R) \varphi(R) d\tau \\ &= \frac{1}{h_g^2} \sum_R \sum_{\tau} \sum_{\tau'} \sum_{j=1}^{f'} \sum_{j'=1}^{f'} a_{j\kappa}(\gamma'^{-1} R^{-1}) a_{\kappa' j'}(R\gamma') b_m^{j*} b_{m'}^{j'} \\ &= \frac{1}{h_g} \sum_R \sum_{\tau} \sum_{j=1}^{f'} \sum_{j'=1}^{f'} a_{j\kappa}(\gamma'^{-1} R^{-1}) a_{\kappa' j'}(R\gamma) b_m^{j*} b_{m'}^{j'}. \end{aligned}$$

Putting $R\gamma = P$, the last expression runs

$$\begin{aligned} \frac{1}{h_g} \sum_P \sum_{j=1}^{f'} \sum_{i=1}^{f'} a_{j\kappa}(P^{-1}) a_{\kappa i}(P) b_m^{j*} b_{m'}^i &= \frac{n!}{h_g f} \delta_{\kappa\kappa'} \delta_{ij} \sum_{j=1}^{f'} b_m^{j*} b_{m'}^i \\ &= \frac{n!}{h_g f} \delta_{\kappa\kappa'} \delta_{mm'}. \end{aligned}$$

These f wave functions undergoes unitary transformation, as follows :

$$\begin{aligned} S\Psi_\kappa &= \frac{1}{h_g} \sum_K \sum_{\gamma=1}^{f'} \sum_{j=1}^{f'} a_{\kappa j}(R\gamma) b_m^j \varphi(SR) \\ &= \frac{1}{h_g} \sum_K \sum_{\gamma=1}^{f'} \sum_{j=1}^{f'} \sum_{i=1}^f a_{\kappa i}(S^{-1}) a_{ij}(SR\gamma) b_m^j \varphi(SR) \\ &= \sum_{i=1}^f a_{\kappa i}(S^{-1}) \Psi_i \\ &= \sum_{i=1}^f a_{i\kappa}(S) \Psi_i. \end{aligned}$$

Consequently the density built up by these wave functions belonging the same energy ϵ_m is unitary invariant, and we shall find it equal to $n!/h_g$, when it is integrated with respect to all the coordinates. This value does not depend on m , i.e. is common for the different eigenvalues and we can normalize the wave functions once for all, by multiplying each wave function by $\sqrt{h_g/n!}$. In the second place we build up the total density of the term-system, as follows,

$$\sum_{m=1}^{f'} \sum_{n=1}^f \Psi_{\kappa m}^* \Psi_{\kappa m} = \frac{1}{n! h_g} \sum_m \sum_{\kappa} \sum_K \sum_S \sum_{\gamma=1}^{f'} \sum_{j=1}^{f'} \sum_{i=1}^f a_{j\kappa}(r^{-1}R^{-1}) a_{\kappa i}(S\gamma') b_m^{j*} b_n^i \varphi^*(R) \varphi(S).$$

On putting $R\gamma=P$, $S\gamma'=Q$, this becomes

$$\begin{aligned} \frac{1}{n! h_g} \sum_m \sum_{\kappa} \sum_P \sum_Q \sum_{j=1}^{f'} \sum_{i=1}^{f'} a_{j\kappa}(P^{-1}) a_{\kappa i}(Q) b_m^{j*} b_n^i \varphi^*(P) \varphi(Q) \\ = \frac{1}{n! h_g} \sum_P \sum_Q \sum_{i=1}^{f'} a_{ii}(P^{-1}Q) \varphi^*(P) \varphi(Q), \end{aligned}$$

where the summation over i may be extended to f by the same reason as the preceding discussion. Thus it becomes

$$\begin{aligned} \frac{1}{n! h_g} \sum_P \sum_Q \chi(P^{-1}Q) \varphi^*(P) \varphi(Q) \\ = \frac{1}{n! h_g} \sum_P \sum_Q \chi(P) \varphi^*(Q) \varphi(RP) \\ = \frac{1}{n! h_g} \sum_P \sum_Q \chi(P) \varphi^*(P|Q^{-1}) \varphi(P|Q^{-1}). \end{aligned} \quad (10)$$

The fore letters in wave functions stand for permutations of the coordinates, and the

rear letters these of the energy indices. We can obtain the similar results as (13), for all conceivable term-systems, in which only the factor h_{ν} may be different. Finally we attach to the density the Boltzmann factor $\exp\{-E_{\nu}\beta\}$, where E_{ν} is

$$n_1\varepsilon_1 + n_2\varepsilon_2 + \dots + n_{\nu}\varepsilon_{\nu}, \quad (11)$$

and sum up the results for all the term-systems. Then we arrive at an expression, which is equivalent to the sum with respect to P over the group of the following expression

$$\sum_{\mu} \exp\{-\varepsilon_{\mu}\beta\} \cdot \psi_{\mu}^*(q_1) \psi_{\mu}(Pq_1) \sum_{\nu} \exp\{-\varepsilon_{\nu}\beta\} \cdot \psi_{\nu}^*(q_2) \psi_{\nu}(Pq_2) \dots, \quad (12)$$

which is exactly the product of n density matrices. Since the characters are class-functions, we can replace individual permutations P by class $u = (u_1, u_2, \dots, u_n)$,

$$1 \cdot u_1 + 2 \cdot u_2 + 3 \cdot u_3 + \dots + n \cdot u_n = n. \quad (13)$$

Now we construct a function, as follows

$$\begin{aligned} & S(q_1 q_1 : \beta) S(q_2 q_2 : \beta) \dots S(q_{a_1} q_{a_1} : \beta) \quad (\text{product of } a_1 \text{ terms}) \\ & \times S(q_{a_1+1} q_{a_1+2} : \beta) S(q_{a_1+2} q_{a_1+1} : \beta) \dots \quad (\text{product of } 2a_2 \text{ terms}) \\ & \dots \\ & \times S(q_{A+1} q_{A+2} : \beta) S(q_{A+2} q_{A+1} : \beta) \dots S(q_B q_{A+1} : \beta) \\ & \quad (\text{product of } \nu u_{\nu} \text{ terms}) \\ & \dots; \end{aligned}$$

where A and B are as follows respectively.

$$A = \sum_{i=1}^{\nu-1} i u_i, \quad B = \sum_{i=1}^{\nu} i u_i.$$

Denoting this function by $\Pi_{(\alpha)}(q_1 q_2 \dots q_n)$, the density gets the form

$$W^{(\lambda)}(q_1 q_2 \dots q_n) = \frac{1}{n!} \sum_{(\alpha)} \chi^{(\alpha)} \sum_{P(\alpha)} \Pi_{(\alpha)}(q_1 q_2 \dots q_n), \quad (14)$$

where $\sum_{P(\alpha)}$ is the summation over all permutations which belong to class (α) . (14) is the diagonal element, but we can easily obtain the non-diagonal element by replacing $\varphi^*(q_1 q_2 \dots q_n)$ by $\varphi^*(q_1' q_2' \dots q_n')$, and this replacement does not invalid at the above calculations. Therefore, we have as a generalization of (14)

$$W^{(\lambda)}(q_1' q_2' \dots q_n', q_1 q_2 \dots q_n) = \frac{1}{n!} \sum_{(\alpha)} \chi^{(\alpha)} \sum_{P(\alpha)} \Pi(q_1' q_2' \dots q_n', q_1 q_2 \dots q_n). \quad (15)$$

(14) or (15) is exactly the same as Kofink's results.

§ 2. Integration of the density function

The result (14) or (15) has a very elegant form, but is hard to deal with.

The expectation value of any physical quantity is represented by

$$\langle \mathfrak{H} \rangle = \int W(q_1 q_2 \cdots q_n, q'_1 q'_2 \cdots q'_n) \mathfrak{H}(q'_1 q'_2 \cdots q'_n, q_1 q_2 \cdots q_n) dq_1 \cdots dq_n dq'_1 \cdots dq'_n.$$

For practical purposes, the density needs only to have one or two explicit coordinates at most, especially for the case in which \mathfrak{H} is the Hamiltonian of a many electron system.

1. One explicit coordinate

We make use of the well-known properties of density matrix

$$\sigma(\nu\beta) = \int S(qq; \beta) dq, \quad \int S(qq'; \nu\beta) S(q'q; \mu\beta) dq' = S(qq; [\nu + \mu]\beta).$$

We integrate (14) over all coordinates with the result:

$$Z^{(\lambda)} = \sum_{\alpha} \chi_{(\alpha)}^{(\lambda)} \frac{\sigma(\beta)^{\alpha_1} \sigma(2\beta)^{\alpha_2} \cdots \sigma(\nu\beta)^{\alpha_\nu}}{a_1! 1^{\alpha_1} a_2! 2^{\alpha_2} \cdots a_\nu! \nu^{\alpha_\nu}}. \quad (16)$$

$Z^{(\lambda)}$ is Schur's simple characteristics. In order to get the density matrix with one explicit coordinate, we must leave q_1 and q'_1 free and take traces with respect to the remaining variables. We first consider any permutation which belongs to the class (α) , (13). If the arbitrarily chosen pair q_1, q'_1 happens to lie in a cycle of length one, then the factor $S(q_1 q'_1; \beta)$ remains unintegrated. If it lies in a cycle of length two, where q_1 is exchanged with q_2 , then the factor $S(q_1 q'_1; \beta) S(q_2 q'_1; \beta)$ gives $S(q_1 q'_1; 2\beta)$ on integration with respect to $q_2 = q'_2$. Similarly if q_1, q'_1 exist in an r -cycle, they give rise to $S(q_1 q'_1; r\beta)$. Having the particles in the r -cycle fixed, there are as many ways of distributing the remaining particles within the same class as the number of non-zero solutions of the equation

$$1 \cdot a_1 + 2 \cdot a_2 + \cdots + r(a_r - 1) + \cdots + na_n = n - r, \quad (17)$$

and this number is given by

$$h_{n-r} = \frac{(n-r)!}{a_1! 1^{\alpha_1} a_2! 2^{\alpha_2} \cdots (a_r - 1)! r^{\alpha_r - 1} \cdots}.$$

On the other hand, since the position of q_1 or q'_1 in an r -cycle is irrelevant for the result, $(r-1)!$ cases give the same result, $S(q_1 q'_1; r\beta)$.

The number of ways in which we choose $n-r$ particles except q_1 , is

$${}_{n-1}C_{n-r}.$$

Hence the number of terms containing the factor $S(q_1 q'_1; r\beta)$ is equal to

$${}_{n-1}C_{n-r} (r-1)! h_{n-r} = \frac{(n-1)!}{a_1! 1^{\alpha_1} \cdots (a_r - 1)! r^{\alpha_r - 1} \cdots} = \frac{r a_r}{n} h_{\alpha}, \quad (18)$$

where

$$h_{\alpha} = \frac{n!}{a_1! 1^{\alpha_1} \cdots a_r! r^{\alpha_r} \cdots a_n! n^{\alpha_n}}.$$

The validity of this way of calculating the number may easily be checked by the

following identity.

$$h_\alpha \sum_{r=1}^n r u_r = h_\alpha.$$

Since the integration over the remaining $n-r$ variables gives rise to the same factor

$$\sigma(\beta)^{\alpha_1} \sigma(2\beta)^{\alpha_2} \dots \sigma(r\beta)^{\alpha_{r-1}} \dots \sigma(n\beta)^{\alpha_n},$$

the resulting expression runs as follows ;

$$S(q_1 q_1 : r\beta) \frac{(n-1)! \sigma(\beta)^{\alpha_1} \dots \sigma(r\beta)^{\alpha_{r-1}} \dots \sigma(n\beta)^{\alpha_n}}{\alpha_1! 1^{\alpha_1} \alpha_2! 2^{\alpha_2} \dots (\alpha_r-1)! r^{\alpha_r-1} \dots}$$

Multiplying by $\chi(u)$ and summing over r , we get

$$\begin{aligned} W^{(\lambda)}(q_1 q_1') &= \frac{1}{n} \sum_r S(q_1 q_1' : r\beta) \sum_\alpha \chi^{(\lambda)}(u) \frac{\sigma(\beta)^{\alpha_1} \dots \sigma(r\beta)^{\alpha_{r-1}} \dots \sigma(n\beta)^{\alpha_n}}{\alpha_1! 1^{\alpha_1} \dots (\alpha_r-1)! r^{\alpha_r-1} \dots} \\ &= \frac{1}{n} \sum_r S(q_1 q_1' : r\beta) \sum_\alpha \chi^{(\lambda)}(u) u_r \frac{\sigma(\beta)^{\alpha_1} \dots \sigma(r\beta)^{\alpha_{r-1}} \dots \sigma(n\beta)^{\alpha_n}}{\alpha_1! 1^{\alpha_1} \dots \alpha_r! r^{\alpha_r-1} \dots} \\ &= \frac{1}{n} \sum_r r S(q_1 q_1' : r\beta) \frac{\partial Z^{(\lambda)}}{\partial \sigma(r\beta)}. \end{aligned}$$

2. The explicit coordinates

In this case we must leave certain pair of variables q_1 and q_2 free. According to the position of this two particles, the following cases will be investigated separately.

1. q_1 and q_2 are in cycles of length r_1 and r_2 respectively, where $r_1 < r_2$.
2. The same as 1, but with $r_1 > r_2$.
3. In two distinct cycles, but of the same length, respectively.
4. Both are in one and the same cycle of length $(r_1 + r_2)$.

Four functions which follow except numerical factors correspond to the above four cases. They are

$$S(q_1 q_1 : r_1 \beta) S(q_2 q_2 : r_2 \beta) \sigma(\beta)^{\alpha_1} \dots \sigma(r_1 \beta)^{\alpha_{r_1-1}} \dots \sigma(r_2 \beta)^{\alpha_{r_2-1}} \dots, \quad (20)$$

$$S(q_1 q_1 : r_2 \beta) S(q_2 q_2 : r_1 \beta) \sigma(\beta)^{\alpha_1} \dots \sigma(r_2 \beta)^{\alpha_{r_2-1}} \dots \sigma(r_1 \beta)^{\alpha_{r_1-1}} \dots, \quad (21)$$

$$S(q_1 q_1 : r\beta) S(q_2 q_2 : r\beta) \sigma(\beta)^{\alpha_1} \dots \sigma(r\beta)^{\alpha_{r-2}} \dots, \quad (22)$$

$$S(q_1 q_2 : r_1 \beta) S(q_2 q_1 : r_2 \beta) \sigma(\beta)^{\alpha_1} \dots \sigma(r\beta)^{\alpha_{r-2}} \dots. \quad (23)$$

Corresponding Diophantine equations are

$$1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \dots + r_1 (\alpha_{r_1} - 1) + \dots + r_2 (\alpha_{r_2} - 1) + \dots = n - (r_1 + r_2), \quad (24)$$

$$1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \dots + r_2 (\alpha_{r_2} - 1) + \dots + r_1 (\alpha_{r_1} - 1) + \dots = n - (r_1 + r_2), \quad (25)$$

$$1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \dots + r (\alpha_r - 2) + \dots = n - 2r, \quad (26)$$

$$1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \dots + (r_1 + r_2) (\alpha_{r_1+r_2} - 1) + \dots = n - (r_1 + r_2). \quad (27)$$

The numbers of solutions of (24) (25) (26) and (27) are respectively

$$\begin{aligned} & \frac{(n-r_1-r_2)!}{\dots (a_{r_1}-1)! r_1^{\alpha_{r_1}-1} \dots (a_{r_2}-1)! r_2^{\alpha_{r_2}-1} \dots}, \\ & \frac{(n-r_1-r_2)!}{\dots (a_{r_2}-1)! r_2^{\alpha_{r_2}-1} \dots (a_{r_1}-1)! r_1^{\alpha_{r_1}-1} \dots}, \\ & \frac{(n-2r)!}{\dots (a_r-2)! r^{\alpha_r-2} \dots}, \\ & \sum_{\substack{r_1+r_2=r \\ r_1, r_2}} \frac{(n-r_1-r_2)!}{\dots (a_{r_1+r_2}-1)! (r_1+r_2)^{\alpha_{r_1+r_2}-1} \dots}. \end{aligned}$$

q_1 and q_2 take positions in the cycles in the following schemes,

$$\begin{array}{ccc} (\underbrace{\dots q_1 \dots}_{r_1}) (\underbrace{\dots q_2 \dots}_{r_2}) & (\underbrace{\dots q_1 \dots}_r) (\underbrace{\dots q_2 \dots}_r) \\ & (\underbrace{q_1 \dots}_{r_1}, \underbrace{q_2 \dots}_{r_2}). \end{array}$$

In these figures, positions of q_1 and q_2 are indifferent, so that $(r_1+r_2-2)!$, $(2r-2)!$ and $(r_1+r_2-2)!$ permutations lead to the same result respectively. Then the number of ways to choose different configuration in each case is ${}_{n-2}C_{n-r_1-r_2}$ corresponding to (24) (25) and (27), and ${}_{n-2}C_{n-2r}$ corresponds to (26). After all, the numerical factors to be attached to (20), (21), (22) and (23) are

$$h_\alpha \frac{r_1 r_2 a_{r_1} a_{r_2}}{n(n-1)}, \quad h_\alpha \frac{r_2 r_1 a_{r_2} a_{r_1}}{n(n-1)}, \quad h_\alpha \frac{r^2 a_r (a_r-1)}{n(n-1)}, \quad (28)$$

and

$$\sum_{\substack{r_1+r_2=r \\ r_1, r_2}} h_\alpha \frac{(r_1+r_2) a_{r_1+r_2}}{n(n-1)}.$$

Summing up the above results over r , we have

$$\begin{aligned} & \frac{h_\alpha}{n(n-1)} \left\{ \sum_{\substack{r_1+r_2=r \\ r_1, r_2}} r_1 r_2 a_{r_1} a_{r_2} + \sum_{r=1}^n r^2 a_r (a_r-1) + \sum_{\substack{r_1+r_2 \leq n \\ r_1, r_2}} (r_1+r_2) a_{r_1+r_2} \right\} \\ & = \frac{h_\alpha}{n(n-1)} \left\{ \left(\sum_{r=1}^n r a_r \right)^2 - \sum_{r=1}^n r^2 a_r + \sum_{\substack{r_1+r_2 \leq n \\ r_1, r_2}} (r_1+r_2) a_{r_1+r_2} \right\}. \end{aligned}$$

By virtue of the relation

$$r(r-1) = \sum_{\substack{r_1+r_2=r \\ r_1, r_2}} (r_1+r_2),$$

the last formula becomes

$$\begin{aligned} & \frac{h_\alpha}{n(n-1)} \left\{ n^2 - \sum_{r=1}^n r^2 a_r + \sum_{r=1}^n r^2 a_r - \sum_{r=1}^n r a_r \right\} \\ & = \frac{h_\alpha}{n(n-1)} (n^2 - n) = h_\alpha. \end{aligned}$$

The above considerations are thereby verified. Combining the results of (20), (21), (22) and (23) with (28), multiplying them by class-character and summing over all classes, we have

$$r_1 r_2 \frac{\partial^2 Z^{(\lambda)}}{\partial \sigma(r_1 \beta) \partial \sigma(r_2 \beta)}, \quad r_2 r_1 \frac{\partial^2 Z^{(\lambda)}}{\partial \sigma(r_2 \beta) \partial \sigma(r_1 \beta)}, \quad r_2^2 \frac{\partial^2 Z^{(\lambda)}}{\partial \sigma(r \beta)^2}$$

and

$$\sum_{r_1 r_2}^{r_1 + r_2 = r} (r_1 + r_2) \frac{\partial Z^{(\lambda)}}{\partial \sigma([r_1 + r_2] \beta)}.$$

As the first three expressions are of uniform character, we may sum them over r_1 and r_2 . Thus we get

$$W^{(\lambda)}(q_1 q_2; q_1' q_2') = \frac{1}{n(n-1)} \sum_{r_1 r_2}^{\leq n} \left\{ r_1 r_2 S(q_1 q_1'; r_1 \beta) S(q_2 q_2'; r_2 \beta) \frac{\partial^2 Z^{(\lambda)}}{\partial \sigma(r_1 \beta) \partial \sigma(r_2 \beta)} \right. \\ \left. + (r_1 + r_2) S(q_1 q_2'; r_1 \beta) S(q_2 q_1'; r_2 \beta) \frac{\partial Z^{(\lambda)}}{\partial \sigma([r_1 + r_2] \beta)} \right\}. \quad (29)$$

Kofink makes use of the fact that (16) is a homogeneous function of

$$\sigma(\beta) \sigma(2\beta)^{1/2} \sigma(3\beta)^{1/3} \dots \sigma(\nu\beta)^{1/\nu} \dots,$$

so that it is possible to apply Euler's theorem. Applying this theorem once, twice and so on, it gets the forms:

$$Z^{(\lambda)} = \frac{1}{n} \sum_{r=1}^n r \sigma(r\beta) \frac{\partial Z^{(\lambda)}}{\partial \sigma(r\beta)},$$

$$Z^{(\lambda)} = \frac{1}{n(n-1)} \sum_{r_1 r_2}^{\leq n} r_1 r_2 \sigma(r_1 \beta) \sigma(r_2 \beta) \frac{\partial^2 Z^{(\lambda)}}{\partial \sigma(r_1 \beta) \partial \sigma(r_2 \beta)}$$

$$+ \sum_{r_1 r_2}^{r_1 + r_2 \leq n} (r_1 + r_2) \sigma([r_1 + r_2] \beta) \frac{\partial Z^{(\lambda)}}{\partial \sigma([r_1 + r_2] \beta)},$$

$$Z^{(\lambda)} = \frac{1}{n(n-1)(n-2)} \sum_{r_1 r_2 r_3}^n r_1 r_2 r_3 \sigma(r_1 \beta) \sigma(r_2 \beta) \sigma(r_3 \beta) \frac{\partial^3 Z^{(\lambda)}}{\partial \sigma(r_1 \beta) \partial \sigma(r_2 \beta) \partial \sigma(r_3 \beta)}$$

$$+ \sum_{r_1=1}^n \sum_{r_2 r_3}^n (r_2 + r_3) r_1 \sigma([r_2 + r_3] \beta) \sigma(r_1 \beta) \frac{\partial^2 Z^{(\lambda)}}{\partial \sigma([r_1 + r_2] \beta) \partial \sigma(r_1 \beta)}$$

$$+ \sum_{r_2=1}^n \sum_{r_1 r_3}^n (r_1 + r_3) r_2 \sigma([r_1 + r_3] \beta) \sigma(r_2 \beta) \frac{\partial^2 Z^{(\lambda)}}{\partial \sigma([r_1 + r_3] \beta) \partial \sigma(r_2 \beta)}$$

$$+ \sum_{r_3=1}^n \sum_{r_1 r_2}^n (r_1 + r_2) r_3 \sigma([r_1 + r_2] \beta) \sigma(r_3 \beta) \frac{\partial^2 Z^{(\lambda)}}{\partial \sigma([r_1 + r_2] \beta) \partial \sigma(r_3 \beta)}$$

$$+ 2! \sum_{r_1 r_2 r_3}^{r_1 + r_2 + r_3 \leq n} (r_1 + r_2 + r_3) \sigma([r_1 + r_2 + r_3] \beta) \frac{\partial Z^{(\lambda)}}{\partial \sigma([r_1 + r_2 + r_3] \beta)}.$$

If in these expressions, we replace partition function σ by S according to the following rule;

$$\begin{aligned}
\sigma(r_1\beta) &\longrightarrow S(q_1q_1:r_1\beta), \\
\sigma(r_2\beta) &\longrightarrow S(q_2q_2:r_2\beta), \\
\sigma(r_3\beta) &\longrightarrow S(q_3q_3:r_3\beta), \\
\sigma([r_1+r_2]\beta) &\longrightarrow S(q_1q_2:r_1\beta)S(q_2q_1:r_2\beta), \\
\sigma([r_2+r_3]\beta) &\longrightarrow S(q_2q_3:r_2\beta)S(q_3q_2:r_3\beta), \\
\sigma([r_1+r_3]\beta) &\longrightarrow S(q_1q_3:r_1\beta)S(q_3q_1:r_3\beta), \\
2!\sigma([r_1+r_2+r_3]\beta) &\longrightarrow S(q_1q_2:r_1\beta)S(q_2q_3:r_2\beta)S(q_3q_1:r_3\beta).
\end{aligned} \tag{30}$$

We can obviously obtain expressions, which are exactly coincident with (19) or (29). Although the transition from right to left in (30) is unique, that from left to right is not unique, so that it requires an exact proof.

Of course we can see this rule to be true for the cases of three or more explicit coordinates. Consequently, the rule (30) will be correct for any case. Formulae (19) and (29) are very convenient to deal with, as only simple characteristics' enter into the formulae when the symmetry property of particle is known.

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Letters to the Editor

On the Velocity Operator in Quantum Mechanics

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In the classical hydrodynamics the velocity operator is given by the current operator $J(x)$ and the density operator $\rho(x)$ in the form

$$V(x) = J(x) / \rho(x). \quad (1)$$

In quantum mechanics, the current operator is represented by

$$J(x) = \frac{\hbar}{2mi} \sum_{\alpha=1}^n \{ \nabla_{\alpha} \delta(x_{\alpha} - x) + \delta(x_{\alpha} - x) \nabla_{\alpha} \}. \quad (2)$$

and the density operator is

$$\rho(x) = \sum_{\alpha} \delta(x_{\alpha} - x) \quad (3)$$

for n particle system. Therefore it seems to be cumbersome to follow the classical definition of the velocity.¹⁾

If one, however, writes the wave function in the form²⁾

$$\varphi(x) = R(x) \exp \frac{i}{\hbar} \Phi(x), \quad R, \Phi : \text{real}, \quad (4)$$

the classical definition gives

$$V(x) = \frac{1}{m} \frac{\partial \Phi(x)}{\partial x} = J(x) / \rho(x) \quad (5)$$

with

$$J(x) = \frac{\hbar}{2mi} \left\{ \varphi^*(x) \frac{\partial \varphi(x)}{\partial x} - \frac{\partial \varphi^*(x)}{\partial x} \varphi(x) \right\}. \quad (6)$$

Some modifications become necessary, if one deals with the quantized wave functions, and we must set*

$$\begin{aligned} \varphi^*(x) &= R(x) \Psi^*(x), \\ \varphi(x) &= \Psi(x) R(x), \end{aligned} \quad (7)$$

so that the current operator takes the form

$$J(x) = R(x) v(x) R(x), \quad (8)$$

where

$$v(x) = \frac{\hbar}{2mi} \left\{ \Psi^*(x) \frac{\partial \Psi(x)}{\partial x} - \frac{\partial \Psi^*(x)}{\partial x} \Psi(x) \right\}. \quad (9)$$

For boson assemblies,

$$\Psi^*(x) \Psi(x) = 1 - \varepsilon(x), \quad \Psi(x) \Psi^*(x) = 1, \quad (10)$$

where the operator $\varepsilon(x)$ is a projection operator which has the eigenvalue 1 if the density is absent at x and has the eigenvalue 0 if the density is present at x . From the commutation relations

$$[\rho(x'), \Psi^*(x)] = \Psi^*(x) \delta(x - x'), \quad (11)$$

$$[\rho(x'), \Psi(x)] = -\Psi(x) \delta(x - x'), \quad (12)$$

one finds the commutation relation between the density and the velocity of the form

$$[\rho(x'), v(x)] = \frac{i\hbar}{m} \{ 1 - \varepsilon(x) \} \frac{\partial}{\partial x} \delta(x - x'), \quad (13)$$

which means that $\rho(x')$ is commutative with the velocity at x wherever the density is absent.

The commutation relation between the velocity operators is expressed by

$$[v(x'), v(x)] = -\frac{\hbar}{mi} \left\{ \text{rot } v(x) + \frac{1}{2} \text{grad } \varepsilon(x) \times v(x) - \frac{1}{2} v(x) \times \text{grad } \varepsilon(x) \right\} \delta(x-x'). \quad (14)$$

The second terms of the right side are negligible except at the neighbourhoods of the points where the density vanishes. In my preceding paper³⁾, a velocity operator has been introduced in terms of the current operator by the formula:

$$J(x) = \frac{1}{2} \rho(x) V(x) + \frac{1}{2} V^*(x) \rho(x). \quad (15)$$

These operators are related to the velocity operator defined above as follows:

$$V(x) = R(x)^{-1} v(x) R(x), \\ V^*(x) = R(x) v(x) R(x)^{-1}, \quad (16)$$

if $R(x)$ is not zero at x .

One gets the stress tensor, $T(x)$, the energy $E(x)$ and the energy flow $h(x)$ in terms of the square root density $R(x)$, the velocity $v(x)$ and the projection operator $\varepsilon(x)$. Especially the hamiltonian is given in the form

$$H = \int E(x) dx = \int H(x) dx, \quad (17) \\ H(x) = \frac{m}{2} R(x) v(x)^2 R(x) + \frac{\hbar^2}{2m} \left| \frac{\partial R(x)}{\partial x} \right|^2 \\ + \frac{\hbar}{2i} \left\{ R(x) v(x) \frac{\partial R(x)}{\partial x} - \frac{\partial R(x)}{\partial x} v(x) R(x) \right\} \\ + \frac{1}{2} \int G(x, x') \{ \rho(x') - \hat{o}(x' - x) \} \\ \times \rho(x) dx', \quad (18)$$

where $G(x, x')$ is the interaction energy. This does not involve the projection operator explicitly. According to the familiar decomposition of the field quantity into the

gradient part and the rotational part, the velocity operator splits into the following two parts:

$$v(x) = v_0(x) + A(x), \quad (19)$$

$$v_0(x) = \frac{1}{m} \text{grad } \Phi(x), \\ A(x) = \text{rot } \dot{B}(x). \quad (20)$$

Here we consider a space region excluding the neighbourhoods of points at which the density vanishes. Taking the divergence of the commutation relation with respect to the coordinate x , one gets

$$\left[\rho(x'), \frac{1}{m} \Delta \Phi(x) \right] = -\frac{\hbar}{mi} \Delta \delta(x-x'). \quad (21)$$

Therefore the quantity $\Phi(x)$ may be regarded as canonically conjugate to the density up to an additive arbitrary function $\Gamma(x)$ subject to the Laplace equation:

$$\Delta \Gamma(x) = 0.$$

Such a quantity may, as usual, be eliminated by means of a suitable boundary condition.

The author thanks Professor Tomonaga for the valuable discussions and the interest in this problem.

* The author indebts much to Professor Tomonaga for this point.

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This note is written on occasion of receiving a manuscript from Professor D. Bohm, which has recently been published: *Phys. Rev.* **85** (1952), 166, 180.

where $H(X) = -J_\mu(X) A_\mu(X)$,

$$J_\mu(X) = j_\mu^{(p)}(X) + j_\mu^{(e)}(X), \quad (4)$$

an index d means the state with one proton, one electron and no real quantum, and n means the other states. In calculating $[]_d$ and $[]_n$ we define the vacuum state of longitudinal and scalar photons χ_0 after Utiyama et al. by

$$\chi_0 = H \left\{ \sum_n (-1)^n g_n \chi_{n,n}(\mathbf{k}) / (\sum_n g_n^2)^{1/2} \right\}, \quad (5)$$

$$\lim_{g_n \rightarrow 1} \left\{ \sum_n (2n+1) g_n^2 - \sum_n (2n+2) g_n g_{n+1} \right\} / \sum_n g_n^2 = 0, \quad (6)$$

where $\chi_{n,m}(\mathbf{k})$ represents a state where n longitudinal photons with the wave number vector \mathbf{k} and m scalar photons with the same \mathbf{k} are present. A straightforward calculation shows that the second term in (3) contains an absolutely divergent term of the form

$$\begin{aligned} & \frac{i}{16} \int d\sigma(X) \int^\sigma dX' \int^\sigma dX'' \int d\sigma(X''') \\ & \times ([J_d, J'_k] \cdot [J'_l, J''_l])_d (\partial_{kl} D(X' - X''')) \\ & - \frac{\partial^2}{\partial X'_k \partial X'_l} G(X' - X'') \\ & \times (\sum_n (2n+1) g_n^2 / \sum_n g_n^2) G^{(1)}(X - X'''), \end{aligned} \quad (7)$$

but this is cancelled out by the same term with opposite sign appearing in the third term of (3). Neglecting the Lamb shift type correction which contributes only small effects and also the self-energy terms, (3) can be written as follows:

$$\begin{aligned} V^4(A) &= V^2(a) + \text{Coulomb pot.} \\ &+ [H_0, S(e^2)] + V^4(a) + V^4(A, H) \\ &+ V^4_{\Lambda, \Pi}(a) + V^4_a([H_0, S(e^2)]) \\ &+ [H_0, S'(e^4)], \end{aligned} \quad (8)$$

where $V^4_a([H_0, S(e^2)])$ represents the transversal radiation correction to $[H_0, S(e^2)]$:

$$\begin{aligned} V^4_a([H_0, S(e^2)]) &= -\frac{1}{4} \int^\sigma dX'' \int^\sigma dX''' \\ &\times [[H_0, S(e^2)], J_\mu(X'')] a_\mu(X''') \\ &J_\nu(X''') a_\nu(X'''). \end{aligned} \quad (9)$$

Also we can verify that except time derivative terms $V^4(A, H)$ and $V^4_{\Lambda, \Pi}(a)$ satisfy the following identities:

$$\begin{aligned} & -\frac{i}{2} \left[I^{\alpha\beta}(a), \int^\epsilon dt' [H_0, S(e^2)] \right] \\ & = -V^4_{\Lambda, \Pi}(a), \\ & -\frac{i}{2} \left[\text{Coulomb pot.} + [H_0, S(e^2)], \right. \\ & \left. \int^\epsilon dt' [H_0, S(e^2)] \right] = -V^4(A, H). \end{aligned} \quad (10)$$

Then, if we apply the following canonical transformation to the state vector

$$\Psi \rightarrow \exp(-i \int^\epsilon dt' [H_0, S(e^2)]) \cdot \Psi, \quad (11)$$

there remains no main fourth order term due to $A(X)$ and $H(X)$ except $V^4_a([H_0, S(e^2)])$ and $[H_0, S'(e^4)]$. The former is the transversal radiation correction to $[H_0, S(e^2)]$ and the latter is the fourth order analogue to $[H_0, S(e^2)]$ and can be transformed into more higher order terms.

From the above considerations our anticipation seems to be convincing. It seems to us that the rôle played by Araki term ought to be interpreted not as the inapplicability of energy law in the concept of the potential, but as representing the effect of longitudinal and scalar photons. Also the definition of the vacuum state by Utiyama et al. suffers no serious difficulties up to e^4 .

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Vacuum Polarisation by Spin One Particles

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April 30, 1952

In view of the criticism by Umezawa and Kamefuchi¹⁾ of the method suggested to eliminate diverging integrals in the self-charge problem²⁾ it may be useful to indicate how we obtained these integrals in the study of vacuum polarisation by the virtual production of pairs of mesons with spin 1.

The Hamiltonian for the vector interaction between the electromagnetic field and spin 1 particles is³⁾

$$ic(\phi_{\mu\alpha}^* \chi_{\lambda\mu} - \chi_{\lambda\mu}^* \phi_{\mu\alpha}) A_{\lambda} + e^2(\dots\dots\dots),$$

where $\hbar=c=1$, m is the mass of the particle, ϕ_{μ} its wave function and

$$\chi_{\alpha\beta} = \frac{\partial \phi_{\beta}}{\partial x_{\alpha}} - \frac{\partial \phi_{\alpha}}{\partial x_{\beta}}.$$

The e^2 -term of the Hamiltonian does not influence the calculation of the induced current in the e^2 -approximation. The current-density vector is

$$ic(\chi_{\mu\alpha}^* \phi_{\alpha} - \phi_{\alpha}^* \chi_{\mu\alpha}) + e^2(\partial_{\mu\nu} - \partial_{\nu\mu} \partial_{\nu 4}) \times \{ -2A_{\nu} \phi_{\alpha}^* \phi_{\alpha} + A_{\alpha} (\phi_{\nu}^* \phi_{\alpha} + \phi_{\alpha}^* \phi_{\nu}) \}.$$

The second term will not yield a gauge invariant contribution. Following the procedure and notation of Schwinger⁴⁾ we see that the part of the induced current giving the self-charge is included in

$$1/2i \int_{-\infty}^{\infty} \epsilon(x-x') <[S_{\mu}(x), S_{\nu}(x')]>_0 \times A_{\nu}(x') d^4x',$$

where

$$S_{\mu} = ic(\chi_{\mu\nu}^* \phi_{\nu} - \phi_{\nu}^* \chi_{\mu\nu}), \quad \epsilon(x) = \frac{x_0}{|x_0|},$$

and the notation $<>_0$ denotes the expectation value when no real meson is present.

To evaluate the integral we note that

$$\begin{aligned} <\phi_{\alpha}(x) \phi_{\beta}^*(x')>_0 = <\phi_{\alpha}^*(x) \phi_{\beta}(x')>_0 \\ &= i \left(\partial_{\alpha\beta} - \frac{1}{m^2} \frac{\partial^2}{\partial x_{\alpha} \partial x_{\beta}} \right) \Delta^{(+)}(x-x'), \\ <\phi_{\alpha}(x) \phi_{\beta}(x')>_0 = <\phi_{\alpha}^*(x) \phi_{\beta}^*(x')>_0 = 0. \end{aligned}$$

It is convenient to transform to momentum representation, employing the following Schwinger representation of singular functions

$$\begin{aligned} \bar{\Delta}(x) &= -\frac{1}{2} \epsilon(x) \Delta(x) \\ &= \frac{1}{(2\pi)^4} P \int \frac{\exp(ik_{\alpha} x_{\alpha}) d^4k}{k_{\lambda}^2 + m^2}, \\ \Delta^{(1)}(x) &= \frac{1}{(2\pi)^3} \int \exp(ik_{\alpha} x_{\alpha}) \delta(k_{\lambda}^2 + m^2) d^4k, \\ \delta(q_{\lambda}^2 + m^2) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp ia(q_{\lambda}^2 + m^2) da, \\ P \frac{1}{q_{\lambda}^2 + m^2} &= -\frac{i}{2} \int_{-\infty}^{\infty} \exp ib(q_{\lambda}^2 + m^2) \frac{b}{|b|} db. \end{aligned}$$

The resulting calculations are lengthy. When principal values of the diverging integrals are taken in order to avoid the purely imaginary quadratically divergent contribution, we find that the divergent part in the self-charge term of the induced current is

$$-\frac{e}{2\pi} J_{\mu}(x) \lim_{z \rightarrow 0} \log \gamma m^2 z_0, \quad (1)$$

where $J_{\mu}(x)$ is the current producing the external field and

$$\alpha = \frac{e^2}{4\pi}, \quad \gamma = 1.78.$$

In these calculations the only point at which a gauge invariant diverging part of the induced current is omitted is where the principal values of diverging integrals are taken. The expression (1) differs from Feldman's result⁽⁵⁾ by a factor 2, which is irrelevant to the present discussion.

The diverging integrals which occur in the above calculations are

$$I_1 = \frac{1}{im^2} \int_{-\infty}^{\infty} \exp(im^2 z) / z^2 \sigma(z) dz,$$

$$I_2 = \int_{-\infty}^{\infty} \exp(im^2 z) / z \sigma(z) dz,$$

where

$$\sigma(z) = \frac{z}{|z|}.$$

We may remark that I_1 does not arise in these calculations from $\mathcal{A}^{(1)}(0)$, as Umezawa and Kamefuchi stated. The integrals I_1 and I_2 have singularities at $z=0$ and on integrating from $-\infty$ to $-z'_0$ and from z_0 to ∞ we find that

$$I_1 = 2 + I_2 + \frac{1}{im^2} \lim_{z_0 \rightarrow 0, z'_0 \rightarrow 0} \left(\frac{1}{z_0} - \frac{1}{z'_0} \right),$$

$$I_2 = - \lim_{z_0 \rightarrow 0, z'_0 \rightarrow 0} (\log \gamma m^2 z_0 - \log \gamma m^2 z'_0).$$

The integrals have no well-defined meaning and therefore no method of evaluating the self-charge term can decide whether or not it diverges quadratically. We chose to define the diverging integrals by their principal values, this choice being an *ad hoc* assumption to avoid the physical difficulty of having an imaginary contribution to the self-charge.

We may therefore conclude that the theory of vacuum polarisation by mesons with spin 1 contains ambiguities which cannot be obviated except by some new postulates. The postulate that principal

values of diverging integrals are to be taken in all cases has the physical basis that a purely imaginary term in the expression for the self-charge is thereby excluded. It also reduces the degree of divergence to a logarithmic one. This in turn may be compensated by self-charge contributions from the current induced in the vacuum by the virtual pair production of particles with spin 0 or 1/2.

I wish so thank Professor W. Heitler for stimulating comments.

- 1) H. Umezawa and S. Kamefuchi, Prog. Theor. Phys. **6** (1951), 543.
- 2) J. McConnell, Phys. Rev. **81** (1951), 275.
- 3) Cf. S. Kanesawa and S. Tomonaga, Prog. Theor. Phys. **3** (1948), 101.
- 4) J. Schwinger, Phys. Rev. **75** (1949), 651.
- 5) D. Feldman, Phys. Rev. **76** (1949), 1369.

Erratum

On the Nucleonic Components producing Large Cosmic-Ray Bursts under Thick Shield

[Prog. Theor. Phys. **7** (1952), 1]

T. Kameda and M. Wada

Fig. 8 and Fig. 10 in the recent paper should be altered to the figures shown here respectively. And the equations (2), (3) and (4') should be altered as follows;

$$N = 30 S^{1.16 \pm 0.03}, \quad 4 \leq S \leq 20, \quad (2)$$

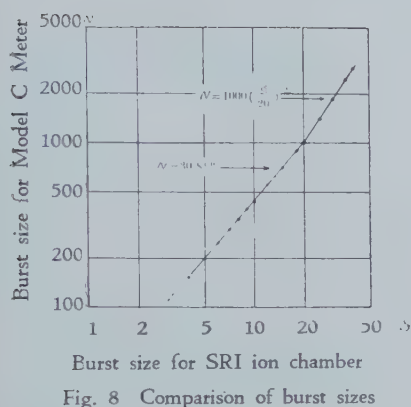
$$N = 1000 \left(\frac{S}{20} \right)^{1.48 \pm 0.05}, \quad S \geq 20,$$

$$N = 60 S_N^{1.17 \pm 0.04}, \quad 2 \leq S_N \leq 15, \quad (3)$$

$$N = 1300 \left(\frac{S_N}{15} \right)^{1.42 \pm 0.06}, \quad S_N \geq 15,$$

$$f = \frac{\alpha_2 S_N^{-1.17\gamma}}{\alpha_2' S_N^{-1.42\gamma}}, \quad 2 \leq S_N \leq 15, \quad (4')$$

$$f = \frac{\alpha_2 S_N^{-1.17\gamma}}{\alpha_2' S_N^{-1.42\gamma}}, \quad S_N \geq 15.$$



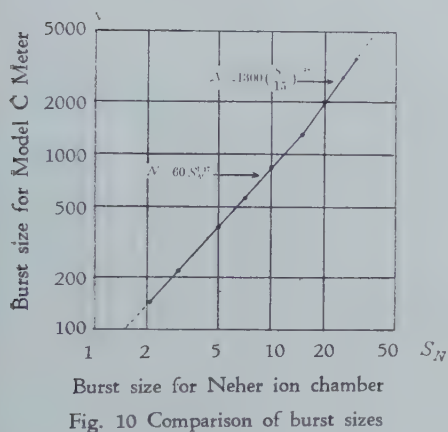
Owing to the relation (2), the conclusion that recombination and difference of the shape of chamber do not affect the size-frequency distribution of the bursts is not correct.**

* Size 1 corresponds to 1.91×10^6 ion-pairs.

** As seen from the differential form of (2) or (3)

$$\frac{dN}{dS} = x \frac{N}{S}$$

where x is the power factor of (2) or (3), these relations may be easily interpreted by considering recombination of ions. However, the experimental facts that x does not vary much with pressure and kinds of the gas make it uncertain to attribute the above relations to recombination.



Theory of the Antiferromagnetic Resonance Absorption in $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$

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T. Nagamiya has developed a theory of antiferromagnetic resonance absorption in the case that the applied static field strength is weaker than the geometrical mean of the exchange field and the anisotropy field. In this paper his theory is extended so as to apply also to the case that the former field is stronger than the latter geometrical mean, and is compared with the experimental results obtained for $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$ by Gorter *et al.*

§ 1. Introduction

By the experimental studies by Gorter *et al.*⁽¹⁾, there have been obtained two important results on the antiferromagnetism exhibited by $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$ single crystal. One is the appearance of the critical field. Under the smaller applied field than the critical value, the spins of the magnetic ions have an antiferromagnetic arrangement, pointing to the direction parallel or antiparallel to the preferred axis, i. e., x -axis, whereas, when the applied field strength exceeds this critical value, they turn to the perpendicular direction to the magnetic field. The writer discussed this phenomenon and derived the theoretical formula for this critical field in the earlier papers.^{2,3)}

The other interesting result is that on the antiferromagnetic resonance absorption. T. Nagamiya⁽⁴⁾ has developed a theory of antiferromagnetic resonance absorption introducing the anisotropy energy in the Van Vleck model in the case that the applied field is small compared with the square root of the product of the exchange field and the anisotropy field which gives a measure of the critical field strength. According to his theory, it is expected that below the Curie temperature, the resonance absorption cannot be observed in the ordinary antiferromagnetic materials such as MnO , Cr_2O_3 , etc. by the usual microwave with the wave-length of several centimeters. This expectation is consistent with experimental results for Cr_2O_3 by Trownson and others.⁽⁵⁾ In $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$ having a very low Curie temperature, the circumstances become a little different from those of the former materials having comparatively high Curie temperature. Since in this substance the critical field is comparatively weak, we must take into consideration the rotation of the spins in the calculations of the resonance frequency. The writer calculated the resonance frequencies when the external static field applied to the directions of three principal axes in the earlier paper (reference 3). In this paper we shall treat this problem in a little more details and compare the theoretical results with the experimental data recently reported by J. Ubbink.⁽⁶⁾

§ 2. The magnetic moment on each sublattice under the static field

When the static field H is applied in the xy -plane to the direction making an angle β with the x -axis, $+$ and $-$ magnetic moment on two sublattices deviate from their easy direction, i. e., x -axis to the direction perpendicular to this field by an angle $\psi - \beta$, ψ being an angle between the external field and this new direction of the magnetic moments.* And moreover the parallel and perpendicular components of each moment to this direction increase by $(1/2) \chi_{\parallel} H \cos \psi$ and $(1/2) \chi_{\perp} H \sin \psi$ respectively, χ_{\parallel} and χ_{\perp} being parallel and perpendicular susceptibilities in the Van Vleck theory. Therefore, denoting the x - and y -components of the magnetic moments on two sublattices by M_x^{\pm} and M_y^{\pm} , we obtain the following relations :

$$\begin{aligned} M_x^{+} &= \left(M_0 + \frac{1}{2} \chi_{\parallel} H \cos \psi \right) \cos(\psi - \beta) + \frac{1}{2} \chi_{\perp} H \sin \psi \sin(\psi - \beta), \\ M_y^{+} &= - \left(M_0 + \frac{1}{2} \chi_{\parallel} H \cos \psi \right) \sin(\psi - \beta) + \frac{1}{2} \chi_{\perp} H \sin \psi \cos(\psi - \beta), \\ M_x^{-} &= - \left(M_0 - \frac{1}{2} \chi_{\parallel} H \cos \psi \right) \cos(\psi - \beta) + \frac{1}{2} \chi_{\perp} H \sin \psi \sin(\psi - \beta), \\ M_y^{-} &= \left(M_0 - \frac{1}{2} \chi_{\parallel} H \cos \psi \right) \sin(\psi - \beta) + \frac{1}{2} \chi_{\perp} H \sin \psi \cos(\psi - \beta), \end{aligned} \quad (1)$$

where M_0 is the magnitude of the magnetic moment in the absence of the external field.

If we let the sum and the difference of the magnetic moments on two sublattices be M and M' , the x - and y -components of M and M' become

$$M_x = \chi_{\parallel} H \cos \psi \cos(\psi - \beta) + \chi_{\perp} H \sin \psi \sin(\psi - \beta), \quad (2)$$

$$M_y = -\chi_{\parallel} H \cos \psi \sin(\psi - \beta) + \chi_{\perp} H \sin \psi \cos(\psi - \beta),$$

$$M'_x = 2M_0 \cos(\psi - \beta), \quad (3)$$

$$M'_y = -2M_0 \sin(\psi - \beta).$$

As shown in the earlier paper (reference 2), an angle between the applied field and the direction of the magnetic moment on each sublattice ψ is determined by the following equations :

$$\tan 2\psi = \frac{\sin 2\beta}{\cos 2\beta - \lambda}, \quad (4)$$

$$\lambda = \frac{H^2}{H_C^2}, \quad H_C^2 = \frac{2AK_1}{1 - \chi_{\parallel}/\chi_{\perp}}, \quad (5)$$

where H_C is the critical field and A is $1/\chi_{\perp}$ and K_1 is the anisotropy constant in the xy -plane. AM_0 represents the exchange field and K_1/M_0 the anisotropy field.

* If we assume that the anisotropy constant in the xz -plane is larger than that in the xy -plane, we can consider that the magnetic moment is in the xy -plane in the case that the external field is applied in this plane.

Since AM_0 can be expressed by

$$AM_0 = \frac{|J|z}{N\mu_B^2} \cdot \frac{1}{2} N\mu_B = \frac{|J|z}{2} \frac{1}{\mu_B} = \frac{k\theta}{\mu_B}, \quad (6)$$

the exchange field AM_0 is estimated as 0.74×10^5 oersteds, using the experimental value of $5^\circ K$ for the curie temperature θ . On the other hand, using the experimental value of 7000 oersteds for H_C and assuming $\chi_b \sim 0$, Eq. (5) and just estimated value of AM_0 give the value of 3.3×10^2 oersteds for the anisotropy field K_1/M_0 . Thus in $\text{CuCl}_2\cdot 2\text{H}_2\text{O}$ the anisotropy field is small compared with the exchange field.

§ 3. The resonance condition

The magnetic anisotropy energy for the crystal of orthorhombic symmetry is expressed by the following form*:

$$F = \frac{1}{2} (K_1 \beta^{+2} + K_2 \gamma^{+2}) + \frac{1}{2} (K_1 \beta^{-2} + K_2 \gamma^{-2}), \quad (7)$$

where β^\pm and γ^\pm are respectively the direction cosines for the y - and z -axes of the magnetic moments on two sublattices, and we assume that $K_2 > K_1 > 0$. Then the torque acting on the two magnetic moments \mathbf{M}^+ and \mathbf{M}^- due to this anisotropy energy is given by

$$\left[-\frac{K_2 - K_1}{M_0^2} M_z^\pm M_y^\pm, \quad \frac{K_2}{M_0^2} M_x^\pm M_z^\pm, \quad -\frac{K_1}{M_0^2} M_x^\pm M_y^\pm \right]. \quad (8)$$

Taking into account this torque, Kittel's equation of motion for \mathbf{M}^\pm becomes

$$\begin{aligned} \frac{1}{\gamma} \frac{d\mathbf{M}^\pm}{dt} = & \mathbf{M}^\pm \times (\mathbf{H} - A\mathbf{M}^\mp) + \left[-\frac{(K_2 - K_1)}{M_0^2} M_z^\pm M_y^\pm, \right. \\ & \left. \frac{K_2}{M_0^2} M_x^\pm M_z^\pm, \quad -\frac{K_1}{M_0^2} M_x^\pm M_y^\pm \right], \end{aligned} \quad (9)$$

where $A\mathbf{M}^\pm$ is the exchange field and A is equal to $1/\chi_\perp$.

From Eq. (9), we obtain the equation of motion for the oscillation part $\delta\mathbf{M}^\pm$ of \mathbf{M}^\pm as

$$\begin{aligned} \frac{1}{\gamma} \frac{d\delta\mathbf{M}^\pm}{dt} = & \delta\mathbf{M}^\pm \times (\mathbf{H} - A\mathbf{M}^\mp) + \mathbf{M}^\pm (\delta\mathbf{H} - A\delta\mathbf{M}^\mp) \\ & + \left[-\frac{K_2 - K_1}{M_0^2} (\delta M_z^\pm M_y^\pm + M_z^\pm \delta M_y^\pm), \quad \frac{K_2}{M_0^2} (\delta M_x^\pm M_z^\pm + M_x^\pm \delta M_z^\pm), \right. \\ & \left. -\frac{K_1}{M_0^2} (\delta M_x^\pm M_y^\pm + M_x^\pm \delta M_y^\pm) \right]. \end{aligned} \quad (10)$$

Transforming the equation of motion (10) for $\delta\mathbf{M}^+$ and $\delta\mathbf{M}^-$ to that for $\delta\mathbf{M} = \delta\mathbf{M}^+ + \delta\mathbf{M}^-$ and $\delta\mathbf{M}' = \delta\mathbf{M}^+ - \delta\mathbf{M}^-$ to simplify the calculations, we obtain

* We should rather adopt as the anisotropy energy the following expression: $F = -K_1\beta^+\beta^- - K_2\gamma^+\gamma^-$. However, in our approximation used here, the same results are obtained.

$$\frac{1}{\gamma} \frac{d\delta \mathbf{M}}{dt} = \delta \mathbf{M} \times \mathbf{H} + \mathbf{M} \times \delta \mathbf{H} + \left[-\frac{K_2 - K_1}{2M_0^2} (\delta M_x M_y + M_x \delta M_y + \delta M'_x M'_y + M'_x \delta M'_y), \right. \\ \left. -\frac{K_2}{2M_0^2} (\delta M_x M_z + M_x \delta M_z + \delta M'_x M'_z + M'_x \delta M'_z), \right. \\ \left. -\frac{K_1}{2M_0^2} (\delta M_x M_y + M_x \delta M_y + \delta M'_x M'_y + M'_x \delta M'_y) \right], \quad (11a)$$

$$\frac{1}{\gamma} \frac{d\delta \mathbf{M}'}{dt} = \delta \mathbf{M}' \times (\mathbf{H} - \alpha \mathbf{M}) + \mathbf{M}' \times (\delta \mathbf{H} - \alpha \delta \mathbf{M}) + \left[-\frac{K_2 - K_1}{2M_0^2} (\delta M_x M'_y + M'_x \delta M_y + \delta M'_x M_y + M'_x \delta M_y), \right. \\ \left. -\frac{K_2}{2M_0^2} (\delta M_x M'_z + M'_x \delta M_z + \delta M'_x M_z + M'_x \delta M_z), \right. \\ \left. -\frac{K_1}{2M_0^2} (\delta M_x M'_y + M'_x \delta M_y + \delta M'_x M_y + M'_x \delta M_y) \right], \quad (11b)$$

Here we put $H_x = H \cos \beta$, $H_y = H \sin \beta$ and $H_z = 0$, and use the relations (2) and (3) for M_x , M_y , M'_x and M'_y , and $M_z = M'_z = 0$. And moreover we put $\delta H = 0$, and neglect the anisotropy fields K_1/M_0 and K_2/M_0 compared with the exchange field AM_0 , taking into consideration the estimation of these quantities made in the last paragraph of Section 2. Then we obtain the following equations:

$$\frac{i\omega}{\gamma} \delta M_x = -H \sin \beta \delta M_z + \frac{K_2 - K_1}{M_0} \sin(\psi - \beta) \delta M'_z, \\ \frac{i\omega}{\gamma} \delta M_y = H \cos \beta \delta M_z + \frac{K_2}{M_0} \cos(\psi - \beta) \delta M'_z, \\ \frac{i\omega}{\gamma} \delta M_z = H \sin \beta \delta M_x - H \cos \beta \delta M_y + \frac{K_1}{M_0} \sin(\psi - \beta) \delta M'_x - \frac{K_1}{M_0} \cos(\psi - \beta) \delta M'_y, \\ \frac{i\omega}{\gamma} \delta M'_x = \alpha H \cos \psi \sin(\psi - \beta) \delta M'_z + 2AM_0 \sin(\psi - \beta) \delta M_z, \\ \frac{i\omega}{\gamma} \delta M'_y = \alpha H \cos \psi \cos(\psi - \beta) \delta M'_z + 2AM_0 \cos(\psi - \beta) \delta M_z, \\ \frac{i\omega}{\gamma} \delta M'_z = -\alpha H \cos \psi \sin(\psi - \beta) \delta M'_x - \alpha H \cos \psi \cos(\psi - \beta) \delta M'_y \\ - 2AM_0 \sin(\psi - \beta) \delta M_x - 2AM_0 \cos(\psi - \beta) \delta M_y, \quad (12)$$

where α is equal to $1 - \chi_{\parallel}/\chi_{\perp}$ and $i\omega$ is used for d/dt .

The resonance frequency is then determined by the condition that the determinant constructed by the coefficients of δM_x , δM_y , ..., etc., should vanish. After some elementary calculations, this determinant can be written as

$$\left(\frac{\omega}{\gamma} \right)^4 - \left(\frac{\omega}{\gamma} \right)^2 [H^2 (\alpha^2 \cos^2 \psi + 1) + 2AK_2 - 2AK_1 \{ 2 \sin^2(\psi - \beta) - \cos^2(\psi - \beta) \}] \\ + \alpha^2 H^4 \cos^2 \psi - H^2 [2AK_1 \{ \cos \beta \sin \psi \sin(\psi - \beta) + \alpha \cos^2 \psi \cos 2(\psi - \beta) \}]$$

$$+ a \sin \beta \cos \phi \sin(\phi - \beta) \} + 2AK_2(a \cos^2 \phi - \sin^2 \phi) \} \\ + 2AK_1 \cos 2(\phi - \beta) \{ 2AK_2 - 2AK_1 \sin^2(\phi - \beta) \} = 0. \quad (13)$$

It is this equation that gives the resonance condition.

§ 4. The resonance condition in the case that the static field is applied to the directions of three principal axes

(1) *The case that the static field is applied to the direction of the x-axis*

In this case, since β is equal to zero, we obtain $\phi = 0$ for $H < H_C$ and $\phi = \pi/2$ for $H > H_C$ from the equations (4) and (5). Accordingly the equation (13) becomes

$$\left(\frac{\omega}{\gamma}\right)^4 - \left(\frac{\omega}{\gamma}\right)^2 \{ H^2(1 + a^2) + 2AK_2 + 2AK_1 \} + a^2 H^4 \\ - aH^2(2AK_1 + 2AK_2) + 2AK_1 \cdot 2AK_2 = 0, \quad \text{for } H < H_C, \quad (14)$$

and

$$\left(\frac{\omega}{\gamma}\right)^4 - \left(\frac{\omega}{\gamma}\right)^2 (H^2 + 2AK_2 - 4AK_1) + H^2(2AK_2 - 2AK_1) \\ - 2AK_1(2AK_2 - 2AK_1) = 0, \quad \text{for } H > H_C. \quad (15)$$

(14) and (15) furnish the following resonance frequencies:

$$\left(\frac{\omega}{\gamma}\right)^2 = \frac{1}{2} [H^2(1 + a^2) + 2AK_1 + 2AK_2 \\ + \sqrt{H^4(1 - a^2)^2 + 2H^2(1 + a^2)(2AK_1 + 2AK_2) + (2AK_2 - 2AK_1)^2}], \quad H < H_C, \quad (16a)$$

$$\left(\frac{\omega}{\gamma}\right)^2 = \frac{1}{2} [H^2(1 + a^2) + 2AK_1 + 2AK_2 \\ - \sqrt{H^4(1 - a^2)^2 + 2H^2(1 + a^2)(2AK_1 + 2AK_2) + (2AK_2 - 2AK_1)^2}], \quad H > H_C, \quad (16b)$$

and

$$\left(\frac{\omega}{\gamma}\right)^2 = 2A(K_2 - K_1), \quad H < H_C \quad (17a)$$

$$\left(\frac{\omega}{\gamma}\right)^2 = H^2 - 2AK_1, \quad H > H_C. \quad (17b)$$

(2) *The case that the static field is applied to the direction of the y-axis*

In this case, we can put $\beta = \pi/2$ and $\phi = \pi/2$ without any limitation for the magnitude of H from the equations (4) and (5). Then we have

$$\left(\frac{\omega}{\gamma}\right)^4 - \left(\frac{\omega}{\gamma}\right)^2 (H^2 + 2AK_2 + 2AK_1) + 2AK_2(H^2 + 2AK_1) = 0, \quad (18)$$

which gives the following resonance frequencies :

$$\left(\frac{\omega}{\gamma}\right)^2 = H^2 + 2AK_1, \quad (19a)$$

$$\left(\frac{\omega}{\gamma}\right)^2 = 2AK_2. \quad (19b)$$

(3) *The case that the static field is applied to the direction of the z-axis*

In this case we obtain the following resonance frequencies by the interchange of K_1 and K_2 :

$$\left(\frac{\omega}{\gamma}\right)^2 = H^2 + 2AK_2, \quad (20a)$$

$$\left(\frac{\omega}{\gamma}\right)^2 = 2AK_1. \quad (20b)$$

§ 5. The comparison of the theoretical resonance frequencies with the experimental data

In their electronic resonance experiments, Gorter *et al.*¹ have used the microwave with the frequency of 9400 Mc/sec.. The value of ω/γ corresponding to this frequency becomes

$$\left(\frac{\omega}{\gamma}\right) = \frac{\nu}{\gamma/2\pi} = 3.36 \times 10^3 \text{ (oersteds)}. \quad (21)$$

On the other hand, at the sufficiently low temperature $(2AK_1)^{1/2}$ and $(2AK_2)^{1/2}$ have the value of the order of 7000 oersteds, estimating from the experimental value for the critical field strength.

Accordingly the resonance magnetic fields possible for observation are only those corresponding to (16b) and (17b) for the case that the static field is applied to the direction of the x-axis. One of these resonance fields is smaller and the other is larger than the critical field. In the case that the static field is applied to the direction of the x-axis, Ubbink⁽⁶⁾ has observed two resonance peaks at the temperature range $1.2 \sim 1.4^\circ \text{ K}$ which correspond to 7300 oersteds and 5000 oersteds. At this temperature the critical field given by $(2AK_1/a)^{1/2}$ is about 7000 oersteds according to the experimental results obtained by Gorter *et al.*¹ Therefore, if we substitute $\omega/\gamma = 3.36 \times 10^3$, $2AK_1 = a \times 49 \times 10^6$ and $H = 7.3 \times 10^3$ into Eq. (17b), we obtain $a = 0.86$, this value being considered as a reasonable value. Furthermore, if we put $\omega/\gamma = 3.36 \times 10^3$, $2AK_1 = a \times 49 \times 10^6$, $a = 0.86$ and $2AK_2/2AK_1 \sim 3.3$ in (16b), we obtain the resonance field of the value of 5000 oersteds.

Thus two resonance peaks observed by Ubbink⁽⁶⁾ at the both sides of the critical field can be explained by the above developed theory.

Next we shall consider the case that the direction of the static field deviates from the x-axis to the y-axis in the xy -plane. For the branch corresponding to $H > H_c$, we can approximately put $\psi = \pi/2$ in (13) from Eqs. (4) and (5). Then Eq. (13) becomes

$$\left(\frac{\omega}{\gamma}\right)^4 - \left(\frac{\omega}{\gamma}\right)^2 \{H^2 + 2AK_2 - 2AK_1(2\cos^2\beta - \sin^2\beta)\} + H^2\{-2AK_1\cos^2\beta + 2AK_2\} + 2AK_1 \cdot 2AK_2(\sin^2\beta - \cos^2\beta) - (2AK_1)^2 \cos^2\beta(\sin^2\beta - \cos^2\beta) = 0, \quad (22)$$

which furnishes the following two resonance frequencies:

$$\left(\frac{\omega}{\gamma}\right)^2 = 2AK_2 - 2AK_1\cos^2\beta, \quad H > H_C, \quad (23a)$$

$$\left(\frac{\omega}{\gamma}\right)^2 = H^2 - 2AK_1\cos 2\beta, \quad H > H_C. \quad (23b)$$

(23b) coincides with (17b) when β tends to zero. Therefore, the resonance field corresponding to 7300 oersteds for $\beta=0$ decreases with increasing β as shows Curve A in Fig. 1.

It is, however, to be noted that (23b) approximately holds only so far as $H > H_C$ and when the resonance field calculated from (23b) becomes smaller than H_C , we cannot observe the resonance any longer.

For the branch corresponding to $H < H_C$, we can assume that the direction of the magnetic moment lies approximately in the x -direction which corresponds to the preferred axis as H is considerably smaller than H_C . Then we can consider the x - and y -components of H separately. The y -component $H \sin \beta$ furnishes the resonance frequencies corresponding to (19a) and (19b) which cannot be observed as mentioned above. The x -component $H \cos$

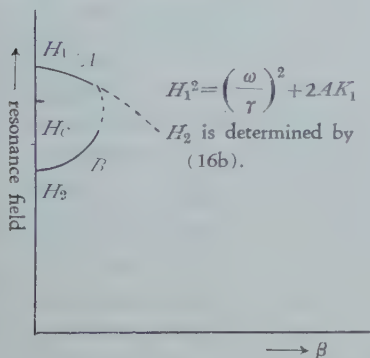


Fig. 1. The relation between the resonance field and the direction of the static field in the xy -plane, β being an angle between the static field and the x -axis.

β furnishes the observable resonance corresponding to (16b). Therefore the resonance field corresponding to 5000 oersteds for $\beta=0$ increases with increasing β as shows Curve B in Fig. 1.

The behavior of the resonance field with increasing β just considered above is in qualitative agreement with the results obtained by Ubbink.¹⁶ To obtain more precise knowledge, we must perform very complicated calculations using Eqs. (13), (4) and (5).

When the direction of the static field deviates from the x -axis to the z -axis in the xz -plane, we can consider again the x - and z -components of H separately since $2AK_2$ is about three times as large as $2AK_1$ and so we have not to take into consideration the rotation of the magnetic moment in the xy -plane. Then the z -component does not give any observable resonance. For the x -component we can expect two resonances corresponding to 5000 oersteds and 7300 oersteds for the case that the static field is applied to the x -axis. These two resonance fields are given by the following equations:

$$H = \frac{H_1}{\cos \beta}, \quad H = \frac{H_2}{\cos \beta}, \quad (24)$$

where β' is an angle between the static field and the x -axis and H_1 and H_2 are respectively the resonance fields corresponding to the case of $\beta'=0$. Thus two resonance fields which are equal to 5000 oersteds and 7300 oersteds when $\beta'=0$ increase with increasing β' . These tendencies are also qualitatively in agreement with the results observed by Ubbink.⁽⁶⁾

§ 6. The temperature dependence of the resonance field

According to the measurements by Ubbink, the resonance field belonging to the upper branch increases, whereas that belonging to the lower branch decreases with rising temperature. However, Eq. (17b) shows that the upper resonance field decreases with rising temperature since the anisotropy constant should decrease. This discrepancy between the theory and the experiments could be interpreted as follows.

According to the experiments, the critical field increases with increasing temperature and therefore the resonance field calculated on the basis of (17b) becomes smaller than the critical field above a certain temperature. If the resonance absorption occurs over a sufficiently broad range of the field, the resonance peak would appear at the critical field strength. Actually, the upper resonance occurs nearly at the critical field and have a lower peak and a larger breadth than those of the lower resonance in the experiments at 2.5°K.

The lower resonance field depends upon the temperature through u , K_1 and K_2 as shown by (16b). The experimental data for the temperature dependence of the critical field would enable to calculate the temperature variation of the lower resonance field.

From the above considerations it would be concluded that the experimental data on the electronic resonance absorption in $\text{CuCl}_2 \cdot 2\text{H}_2\text{O}$ can qualitatively be explained by the theory of antiferromagnetic resonance absorption which has been developed on the basis of the idea originally proposed by T. Nagamiya that the anisotropy energy plays an important rôle in the antiferromagnetism, and have had only the experimental proof that the resonance absorption disappears below the Curie temperature in ordinary antiferromagnetic materials.

In conclusion the writer would like to express his sincere thanks to Prof. T. Nagamiya for his illuminating discussions on this problem.

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Operator Calculus of Quantized Operator

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The notational ambiguities in Feynman's calculus are all remedied here by setting a more natural foundation of the ordered exponential operators, which will be called briefly "expansional" operators in this paper. The essential point to be stressed is that the pure exponential operator is merely a special case of the more wider class of operators, i.e., expansional operators, and the latter type of operators generally appears in quantum mechanics. A clear-cut distinction between these two types of operators is all-important. The so-called disentangling process is a device to decompose any one expansional operator into a product of some simple exponential operators. In the first place the general view-points concerning the expansional operators are presented, in which a rigorous organization of disentangling procedure is accomplished. Next are given some examples treating a generalized forced harmonic oscillator, where the transformation operator is completely disentangled and its representative yields automatically the classical action in its compact form, which process constitutes the original intention of this work. No accounts are given here concerning the quantized field theory.

§ 1. Introduction

The present work embodies a study of some aspects of operator calculus of quantized operator, in so far as it has concern with non-relativistic quantum theory. This study has been guided by the conviction that the relation of classical action to quantum mechanics, which was first suggested by Dirac¹⁾ and dominant in Feynman's "Space-time approach to non-relativistic quantum mechanics,"²⁾ can be traced somewhat more concretely. More precisely, the classical action is furnished quite automatically by the quantum mechanical transformation operator on taking its representative without solving well known Hamilton-Jacobi equation. The performance of this plan inevitably provokes an operator calculus of quantized operator, since for the evaluation of matrix element of any transformation operator the clear-cut positional separation of canonically conjugate operators involved is indispensable. This procedure, being termed "disentangling" according to Feynman,³⁾ has been organized almost on the same line of reasoning as with Feynman's and successfully utilized to the above-mentioned programme.⁴⁾ From the practical point of view we find no essential differences between his plan of this process and that presented here. But in my opinion, the former suffers from some inconvenient features in its logical side. What is wanted, and what I have striven after, is a logical well-ordering of main ideas concerning the operator calculus. The present study is entirely free from ambiguities in Feynman's notation, which might obscure the fundamental concepts of operator calculus and hamper the rigorous organization of the disentangling technique. The key to this solution is the following simple remark, that in general the transformation operator is not a pure exponential operator

defined in the same way as a numerical exponential function, but a more wider class of operators whose generating operator is not of so simple character that it will be impractical to use the same symbol for these two types of operators. The latter type is called here "expansional" operator and symbolized differently from the usual exponential operator in order to accentuate the situation that this is merely a special case of expansionals where the generating operator is commutable. As will be seen in § 2, any one expansional operator permits a decomposition into a product of any numbers of expansionals. Now for evaluating the matrix element of any expansional operator, it is at least essential that its generating operator is commutable. However, this alone does not suffice for our purpose, for there remains the above-mentioned difficulty concerning the mixed up occurrence of canonically conjugate operators. Accordingly the final result of our disentangling process appears as a decomposition of the complicated transformation operator into several exponential operators in such a way that the hopelessly entangled operators are systematically redistributed over all these exponentials in order to effectuate the direct evaluation of the matrix element.

The following two sections are devoted to the indication of above-mentioned circumstances characteristic to operator calculus and to the establishment of the disentangling technique. In the next section, we find an illustration for these general accounts treating a generalized forced harmonic oscillator, where the transformation operator is completely disentangled and its matrix element yields the classical action function in its well known form. Relating to this problem we must confess that the case in which the potential involves the space-coordinate to higher powers than the second or to inverse powers is far beyond the ability of our disentangling technique. Above all the omission of Coulomb potential is regrettable. In addition, two topics are appended, one is the operational treatment of the eigenfunction expansion of free oscillating kernel, which is intended to be an exemplification to the perspicuous symbolism due to Dirac.¹⁾ The other is non-relativistic motion of a particle in a constant uniform magnetic field, which may be interpreted as a superposition of two harmonic oscillators and easily managed on the basis of § 4.

As has already been stated, our interests are exclusively directed to the operator calculus itself and to its application to non-relativistic quantum mechanics. Respecting the former, no essentially new notions are introduced, but the business of this work is to render the process of disentangling explicit and—so far as may be efficient. Respecting the latter, the topics included are indeed worn-out problems and there are no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view, as Feynman said.

While no accounts are given here concerning the quantized field theory, the utility of the present work for this domain of study will soon be demonstrated by others.

§ 2. Expansional operator

We have an operator function $H(\lambda)$, which involves one real parameter λ and any numbers of constant operators having any prescribed commutation rules. The transformation operator generated by $H(\lambda)$ corresponding to the lapse of parameter from σ to τ is

an infinite product of the infinitesimal transformation operator $[I + d\lambda H(\lambda)]$ arranged in positional order from right to left corresponding to the succession of ordering parameter λ from σ to τ . When expanded in ascending powers of generating operator $H(\lambda)$, this is called "expansional" operator and symbolized by the following notation :

$$\text{Exp} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right] = E \int_{\sigma}^{\tau} d\lambda H(\lambda) = I + \sum_{n=1}^{\infty} \text{Exp}^{(n)} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right], \quad (1)$$

where of course

$$\text{Exp}^{(1)} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right] = \int_{\sigma}^{\tau} d\lambda H(\lambda), \quad (2)$$

$$\text{Exp}^{(n+1)} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right] = \int_{\sigma}^{\tau} d\mu H(\mu) \text{Exp}^{(n)} \left[\int_{\sigma}^{\mu} d\lambda H(\lambda) \right] \quad (n \geq 1). \quad (3)$$

Here the capital letter "E" is an indication to discriminate our expansional operator from the familiar exponential operator and the integral notation in "exponent" is of symbolical meaning standing for the above-mentioned positional ordering of infinitesimal transformation operators. It must be noted that the introduction of expansional operator is a direct consequence of the general situation: $[H(\lambda'), H(\lambda'')] \neq 0$ for $\lambda' \neq \lambda''$, which necessitates the conception of ordering. The special case where the generating operator commutes is especially simple, which will be referred to the next section. By the way, no restrictions are imposed on the integration limits σ and τ whether $\sigma < \tau$ or $\sigma > \tau$, and especially for $\sigma = \tau$ the expansional reduces to unit operator.

On differentiating eq. (1) by its upper limit τ , we have

$$\frac{\partial}{\partial \tau} \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right] = H(\tau) \cdot \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right], \quad (4)$$

and since the expression (3) may be rewritten as

$$\text{Exp}^{(n+1)} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right] = \int_{\sigma}^{\tau} d\mu \text{Exp}^{(n)} \left[\int_{\mu}^{\tau} d\lambda H(\lambda) \right] \cdot H(\mu), \quad (5)$$

the differentiation of eq. (1) by its lower limit σ gives also

$$\frac{\partial}{\partial \sigma} \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right] = -\text{Exp} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right] \cdot H(\sigma). \quad (6)$$

Now differentiate a combination of two successive expansionals $\text{Exp} \left[\int_{\mu}^{\tau} d\lambda H(\lambda) \right]$ and $\text{Exp} \left[\int_{\sigma}^{\mu} d\lambda H(\lambda) \right]$ by their conjunction μ , and it vanishes identically according to (4) and (6) wherever the conjunction may be (including the region outside the limits σ and τ):

$$\frac{\partial}{\partial \mu} \left\{ \text{Exp} \left[\int_{\mu}^{\tau} d\lambda H(\lambda) \right] \cdot \text{Exp} \left[\int_{\sigma}^{\mu} d\lambda H(\lambda) \right] \right\} = 0. \quad (7)$$

Hence the resultant of this product is independent of the specification of μ and the same as for taking $\mu = \sigma$ or $\mu = \tau$, i.e.,

$$\text{Exp} \left[\int_{\mu}^{\tau} d\lambda H(\lambda) \right] \cdot \text{Exp} \left[\int_{\sigma}^{\mu} d\lambda H(\lambda) \right] = \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right]. \quad (8)$$

The special case of this expression for $\sigma = \tau$ affords an inverse expansional:

$$\left\{ \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right] \right\}^{-1} = \text{Exp} \left[\int_{\tau}^{\sigma} d\lambda H(\lambda) \right], \quad (9)$$

that is, we have only to interchange the upper and lower limits in the integral in order to get an inverse expansional operator. Hitherto we have confirmed that for the symbolical integration in expansional the additive law for the upper and lower limits holds as usual.

The next stage of our consideration is concerned with "integrand". Splitting the generating operator into two parts: $H(\lambda) = F(\lambda) + G(\lambda)$, we assume correspondingly that the expansional $\text{Exp} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right]$ can be decomposed into two parts in the following manner:

$$\text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \{F(\lambda) + G(\lambda)\} \right] = \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda F(\lambda) \right] \cdot \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \bar{G}(\lambda) \right] \quad (10)$$

$$= \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \bar{F}(\lambda) \right] \cdot \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda G(\lambda) \right]. \quad (11)$$

In order to determine two unknown operators \bar{F} and \bar{G} , first differentiate eq. (10) by its upper limit τ :

$$G(\tau) \cdot \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \{F(\lambda) + G(\lambda)\} \right] = \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda F(\lambda) \right] \cdot \bar{G}(\tau) \cdot \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \bar{G}(\lambda) \right]. \quad (12)$$

Converting this equation and rewriting λ for τ , the operator \bar{G} reads

$$\bar{G}(\lambda) = \text{Exp} \left[\int_{\lambda}^{\sigma} d\mu F(\mu) \right] \cdot G(\lambda) \cdot \text{Exp} \left[\int_{\sigma}^{\lambda} d\mu F(\mu) \right]. \quad (13)$$

By similar procedures we get the expression for \bar{F} :

$$\bar{F}(\lambda) = \text{Exp} \left[\int_{\lambda}^{\tau} d\mu G(\mu) \right] \cdot F(\lambda) \cdot \text{Exp} \left[\int_{\tau}^{\lambda} d\mu G(\mu) \right]. \quad (14)$$

Accordingly establish the decomposition rules:

$$\begin{aligned} & \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \{F(\lambda) + G(\lambda)\} \right] \\ &= \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda F(\lambda) \right] \cdot \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \text{Exp} \left[\int_{\lambda}^{\sigma} d\mu F(\mu) \right] \cdot G(\lambda) \cdot \text{Exp} \left[\int_{\sigma}^{\lambda} d\mu F(\mu) \right] \right] \end{aligned} \quad (15)$$

$$= \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \text{Exp} \left[\int_{\lambda}^{\tau} d\mu G(\mu) \right] \cdot F(\lambda) \cdot \text{Exp} \left[\int_{\tau}^{\lambda} d\mu G(\mu) \right] \right] \cdot \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda G(\lambda) \right], \quad (16)$$

where the roles of F and G are always interchangeable. The meaning of these formulas may be stated as follows. First the interpretation of eq. (1) is — as has already been mentioned — that the infinite product of infinitesimal transformation operators is converted into the form of discrete plotting of generating operators $H(\lambda)$ on a continuous sheet composed of infinite numbers of unit operators. Now this sheet of unit operators is

replaced by that constructed from infinite numbers of the infinitesimal transformation operator $[I + d\lambda F(\lambda)]$ generated by $F(\lambda)$, on which is discretely plotted another generating operator $G(\lambda)$. This point of view corresponds to eq. (15) and interchanged roles of F and G are found in eq. (16).

Formal rewritings of eqs. (15) and (16) immediately give the composition rules for two expansionals in the following form:

$$\begin{aligned} & \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda F(\lambda) \right] \cdot \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda G(\lambda) \right] \\ &= \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \left\{ F(\lambda) + \text{Exp} \left[\int_{\sigma}^{\lambda} d\mu F(\mu) \right] \cdot G(\lambda) \cdot \text{Exp} \left[\int_{\lambda}^{\sigma} d\mu F(\mu) \right] \right\} \right] \end{aligned} \quad (17)$$

$$= \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \left\{ G(\lambda) + \text{Exp} \left[\int_{\sigma}^{\lambda} d\mu G(\mu) \right] \cdot F(\lambda) \cdot \text{Exp} \left[\int_{\lambda}^{\sigma} d\mu G(\mu) \right] \right\} \right]. \quad (18)$$

The direct apprehension of their meaning is not so easy a task. Furthermore, it may sound rather strange that one single expression permits two equivalents. But the latter question can be settled as follows, that is, interchanging σ and τ — at the same time F and G in eq. (18), we have

$$\begin{aligned} & \text{Exp} \left[\int_{\tau}^{\sigma} d\lambda G(\lambda) \right] \cdot \text{Exp} \left[\int_{\tau}^{\sigma} d\lambda F(\lambda) \right] \\ &= \text{Exp} \left[\int_{\tau}^{\sigma} d\lambda \left\{ F(\lambda) + \text{Exp} \left[\int_{\sigma}^{\lambda} d\mu F(\mu) \right] \cdot G(\lambda) \cdot \text{Exp} \left[\int_{\lambda}^{\sigma} d\mu F(\mu) \right] \right\} \right], \end{aligned} \quad (19)$$

which is the very inverse of eq. (17) and we find no essential difference between these two expressions. Simply iterating above composition or decomposition rules, we can easily attain the general composition and decomposition rules for any numbers of expansionals. Here we are to mention only two important cases of triple composition law which will be used in the later examples:

$$\begin{aligned} & \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda F(\lambda) \right] \cdot \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda G(\lambda) \right] \cdot \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right] \\ &= \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \left\{ F(\lambda) + \text{Exp} \left[\int_{\sigma}^{\lambda} d\mu F(\mu) \right] G(\lambda) \text{Exp} \left[\int_{\lambda}^{\sigma} d\mu F(\mu) \right] \right. \right. \\ & \quad \left. \left. + \text{Exp} \left[\int_{\sigma}^{\lambda} d\mu F(\mu) \right] \text{Exp} \left[\int_{\sigma}^{\lambda} d\mu G(\mu) \right] H(\lambda) \text{Exp} \left[\int_{\lambda}^{\sigma} d\mu G(\mu) \right] \text{Exp} \left[\int_{\lambda}^{\sigma} d\mu F(\mu) \right] \right\} \right] \end{aligned} \quad (20)$$

$$\begin{aligned} &= \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \left\{ H(\lambda) + \text{Exp} \left[\int_{\sigma}^{\lambda} d\mu H(\mu) \right] G(\lambda) \text{Exp} \left[\int_{\lambda}^{\sigma} d\mu H(\mu) \right] \right. \right. \\ & \quad \left. \left. + \text{Exp} \left[\int_{\sigma}^{\lambda} d\mu H(\mu) \right] \text{Exp} \left[\int_{\sigma}^{\lambda} d\mu G(\mu) \right] F(\lambda) \text{Exp} \left[\int_{\lambda}^{\sigma} d\mu G(\mu) \right] \text{Exp} \left[\int_{\lambda}^{\sigma} d\mu H(\mu) \right] \right\} \right]. \end{aligned} \quad (21)$$

Concluding the abstract accounts of expansional operators, it may be valuable to accentuate some of their characteristic features. The integration in the "exponent" of expansional operator is of symbolical character enforcing the ordering by parameter. Hence

there is no reason in general to perform the integration as it stands, excepting only one special case which will be discussed in the next section. Moreover the positions of integration limits are of importance admitting no interchange, for they regulate the positional ordering of infinitesimal transformation operators. The duplication of the integration symbol is meaningless: $\text{Exp} \left[\int d\lambda \{F(\lambda) + G(\lambda)\} \right] \neq \text{Exp} \left[\int d\lambda F(\lambda) + \int d\lambda G(\lambda) \right]$. But the followings are permissible, first to make a linear transformation in the ordering parameter, secondly to put out a numerical constant factor in the generating operator before the integration symbol: $\text{Exp} \left[\int d\lambda c H(\lambda) \right] = \text{Exp} \left[c \int d\lambda H(\lambda) \right]$.

§ 3. Exponential operator

In the last section, we have assumed a general circumstance; $[H(\lambda'), H(\lambda'')] \neq 0$ for $\lambda' \neq \lambda''$, which has enforced the conception of ordering and consequently the introduction of expansional operators. But there arise no such complications for commuting generating operators, for these are not to be distinguished from numerical functions and completely dissolves the reason for their special treatments. Now the concept of ordering is irrelevant, the abandonment of which converts the expansional operators into usual exponential operators. Accordingly, as soon as the commutativity of the generating operator is established, the characteristic capital letter "E" must be turned into its small letter and the integration in exponent must be performed according to its usual definition, that is, $\text{Exp} \left[\int d\lambda H(\lambda) \right] \rightarrow \exp \left[\int d\lambda H(\lambda) \right]$. Of course each term of their power expansions is to show the following correspondence: $\text{Exp}^{(n)} \left[\int d\lambda H(\lambda) \right] \rightarrow (1/n!) \left\{ \int d\lambda H(\lambda) \right\}^n$. For example the inverse operator is [see eq. (9)]

$$\left\{ \exp \left[\int_{\sigma}^{\tau} d\lambda H(\lambda) \right] \right\}^{-1} = \exp \left[\int_{\tau}^{\sigma} d\lambda H(\lambda) \right] = \exp \left[- \int_{\sigma}^{\tau} d\lambda H(\lambda) \right], \quad (22)$$

as is well known. Furthermore there occurs a quite different situation concerning the parameter derivative of an exponential operator compared with that for an expansional operator [see eqs. (4) and (6)], where the generating operator stands only on the right or only on the left of the expansional operator to advocate the conception of ordering. Here we find no such restrictions, since the generating operator does commute with the exponential operator itself.

Particularly the most familiar type of exponential operators is that for which the generating operator has lost its functional dependency on parameter and become one single constant operator; for example assuming $\sigma=0$ and $\tau=1$,

$$\text{Exp} \left[\int_0^1 d\lambda H \right] = \exp \left[\int_0^1 d\lambda H \right] = \exp [H]. \quad (23)$$

Now the decomposition rule reads rewriting eq. (15)

$$\exp [F + G] = \exp \left[\int_0^1 d\lambda (F + G) \right]$$

$$\begin{aligned}
&= \exp \left[\int_0^1 d\lambda F \right] \cdot \text{Exp} \left[\int_0^1 d\lambda \exp \left[\int_\lambda^0 d\mu F \right] \cdot G \cdot \exp \left[\int_0^\lambda d\mu F \right] \right] \\
&= \exp [F] \cdot \text{Exp} \left[\int_0^1 d\lambda \{ e^{-\lambda F} G e^{\lambda F} \} \right].
\end{aligned} \tag{24}$$

Similarly we have from eq. (16)

$$\exp [F + G] = \text{Exp} \left[\int_0^1 d\lambda \{ e^{(1-\lambda)G} F e^{-(1-\lambda)G} \} \right] \cdot \exp [G], \tag{25}$$

and for the composition rules (17) and (18)

$$\exp [F] \cdot \exp [G] = \text{Exp} \left[\int_0^1 d\lambda \{ F + e^{\lambda F} G e^{-\lambda F} \} \right] \tag{26}$$

$$= \text{Exp} \left[\int_1^0 d\lambda \{ G + e^{-(1-\lambda)G} F e^{(1-\lambda)G} \} \right]. \tag{27}$$

We now see that the composition or decomposition processes involving only pure exponential operators unavoidably induce a more wider class of operators, which is not to be included in the narrow category of exponentials. There is no hope whatever to exclude these operators from our calculus, while a temporary interpretation of an exponential operator as an ordered operator is often in danger of confusion and ambiguity. A strict discrimination between these two different types of operators is all-important, that is why we have taken the special trouble to introduce an operator termed "expansional."

The very equations (24) and (25) correspond to the starting point of Feynman's operator calculus, where the order of operation is regulated by an index attached to the operator. Now such a plan of attack is purposefully avoided and the ordering subscript to the operator is replaced by that composition or decomposition processes of expansionals, which are equivalent to successive applications of canonical transformations in essence. Of course the position of operator written on the paper holds its importance as usual, and from this point an efficient and rigorous organization of our operator calculus does start. It must be noted that the pure exponential operator taken as the basis of operator calculus is rather misleading and its proper status is to be estimated from a more general point of view.

Concluding this section an important formula is appended which will be frequently used without reference in the following sections, that is,

$$\exp [-A] B \exp [A] = \sum_{n=0}^{\infty} B_n, \tag{28}$$

where A and B are arbitrary operators and

$$B_0 = B, \quad B_n = [B_{n-1}, A]. \tag{29}$$

§ 4. Generalized forced harmonic oscillator

Here we will work out a generalized forced harmonic oscillator as an example of the

computation rules given in preceding sections. The time dependent Hamiltonian is

$$H(t) = \frac{\omega}{2} [a(t)p^2/(m\omega) + m\omega\beta(t)q^2] = i\hbar\omega [a(t)P + \beta(t)Q], \quad (30)$$

where the capital letters P and Q denote the anti-hermitian dimensionless operators

$$P = p^2/(2m\omega i\hbar), \quad Q = (m\omega/2i\hbar)q^2. \quad (31)$$

According to the fundamental commutation rule: $[q, p] = i\hbar$, we will introduce a third anti-hermitian dimensionless operator

$$R = [Q, P] = (pq + qp)/(2i\hbar) = pq/(i\hbar) + \frac{1}{2}, \quad (32)$$

whose commutators with P and Q are

$$[Q, R] = 2Q, \quad [P, R] = -2P. \quad (33)$$

From eqs. (32) and (33) we have for any number λ ,

$$e^{-\lambda R} P e^{\lambda R} = e^{-2\lambda} P, \quad e^{-\lambda R} Q e^{\lambda R} = e^{2\lambda} Q, \quad (34)$$

$$e^{-\lambda P} R e^{\lambda P} = R + 2\lambda P, \quad e^{-\lambda Q} R e^{\lambda Q} = R - 2\lambda Q, \quad (35)$$

$$e^{-\lambda Q} P e^{\lambda Q} = P - \lambda R + \lambda^2 Q, \quad e^{-\lambda P} Q e^{\lambda P} = Q + \lambda R + \lambda^2 P. \quad (36)$$

Making a substitution $\lambda = \omega t$, the transformation operator reads

$$T(\tau, \sigma) = \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda H(\lambda)/(i\hbar) \right] = \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \{a(\lambda)P + \beta(\lambda)Q\} \right]. \quad (37)$$

As is easily seen from eqs. (32) and (33), three operators P , Q and R make up a closed system under the operation to take commutators. Hence we may safely expect that the transformation operator (37) can be decomposed into three simple exponential operators, each of which contains only one of these operators, that is,

$$T(\tau, \sigma) = \exp[a(\tau)P] \exp[b(\tau)R] \exp[c(\tau)Q] \quad (38)$$

$$= \exp[f(\tau)Q] \exp[g(\tau)R] \exp[h(\tau)P]. \quad (39)$$

Six unknown functions $a(\tau) \dots h(\tau)$ are functionals of the given functions $a(\lambda)$ and $\beta(\lambda)$ and satisfy the common initial condition to vanish for $\tau = \sigma$. The triple composition rule (20) unifies the latter expression (39) into one single expansional operator in the following manner:

$$\begin{aligned} & \exp[f(\tau)Q] \exp[g(\tau)R] \exp[h(\tau)P] \\ &= \exp \left[\int_{\sigma}^{\tau} d\lambda f'(\lambda)Q \right] \exp \left[\int_{\sigma}^{\tau} d\lambda g'(\lambda)R \right] \exp \left[\int_{\sigma}^{\tau} d\lambda h'(\lambda)P \right] \end{aligned}$$

$$\begin{aligned}
&= \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \{ f'(\lambda) Q + g'(\lambda) e^{f(\lambda)Q} R e^{-f(\lambda)Q} \right. \\
&\quad \left. + h'(\lambda) e^{f(\lambda)Q} e^{g(\lambda)R} P e^{-g(\lambda)R} e^{-f(\lambda)Q} \} \right] \\
&= \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \{ f'(\lambda) Q + g'(\lambda) [R + 2f(\lambda)Q] + h'(\lambda) e^{2g(\lambda)} [P + f(\lambda)R + f^2(\lambda)Q] \} \right], \quad (40)
\end{aligned}$$

where the use is made of eqs. (34), (35) and (36). Comparing (40) with (37), we get the differential equations for $f(\lambda)$, $g(\lambda)$ and $h(\lambda)$:

$$h'(\lambda) \exp[2g(\lambda)] = \alpha(\lambda), \quad (41)$$

$$g'(\lambda) + \alpha(\lambda)f(\lambda) = 0, \quad (41')$$

$$f'(\lambda) - \alpha(\lambda)f^2(\lambda) = \beta(\lambda). \quad (41'')$$

Quite similarly for $a(\lambda)$, $b(\lambda)$ and $c(\lambda)$,

$$c'(\lambda) \exp[-2b(\lambda)] = \beta(\lambda), \quad (42)$$

$$b'(\lambda) - \beta(\lambda)a(\lambda) = 0, \quad (42')$$

$$a'(\lambda) - \beta(\lambda)a^2(\lambda) = \alpha(\lambda). \quad (42'')$$

Under the following substitution in eq. (41''):

$$f(\lambda) = -y'(\lambda)/u(\lambda)y(\lambda), \quad (43)$$

a second order differential equation for $y(\lambda)$ follows:

$$y'' - (u'/u)y' + \alpha\beta y = 0. \quad (44)$$

A particular solution of eq. (44) satisfying the initial conditions: $y'(\sigma) = 1$ and $y'(\sigma) = 0$, is

$$y(\lambda) = \{y_2'(\sigma)y_1(\lambda) - y_1'(\sigma)y_2(\lambda)\}/u(\sigma), \quad (45)$$

where $y_1(\lambda)$ and $y_2(\lambda)$ are particular solutions of eq. (44), and the latter is constructed from $y_1(\lambda)$ as

$$y_2(\lambda) = y_1(\lambda) \int_{\sigma}^{\lambda} d\mu u(\mu)/y_1^2(\mu). \quad (46)$$

Now the integration of the equations (41) and (41') is straightforward:

$$g(\lambda) = \log[y(\lambda)], \quad h(\lambda) = \int_{\sigma}^{\lambda} d\mu u(\mu)/y^2(\mu). \quad (47)$$

According to eq. (46), another function $z(\lambda)$ defined as

$$z(\lambda) = y(\lambda)h(\lambda) \quad (48)$$

is also a particular solution of (44) under the initial conditions: $z(\sigma) = 0$ and $z'(\sigma) = u(\sigma)$, since $z'(\lambda) = y'(\lambda)h(\lambda) + u(\lambda)/y'(\lambda)$, which is equivalent to

$$yz' - y'z = u. \quad (49)$$

The consideration of second system (42), (42') and (42'') is an easy task, since the following substitution in terms of $z(\lambda)$ is sufficient to fulfil the equation (42''):

$$a(\lambda) = u(\lambda)z(\lambda)/z'(\lambda). \quad (50)$$

The solutions of the remaining equations (42) and (42') are self-evident:

$$b(\lambda) = \log[u(\lambda)/z'(\lambda)], \quad c(\lambda) = -y'(\lambda)/z'(\lambda), \quad (51)$$

and for the proof of the latter expression the identity (49) is required. Therefore the final result of the disentangling process reads

$$T(\tau, \sigma) = \exp[uzP/z'] \exp[\log(u/z')R] \exp[-y'Q/z'] \quad (52)$$

$$= \exp[-y'Q/uy] \exp[(\log y)R] \exp[zP/y], \quad (53)$$

where the argument of these functions y, y', z and z' is of course τ .

The next step of our calculation is to take the matrix element of this transformation operator, for which purpose we will give a few preliminaries concerning the matrix element of an operator $\exp[\lambda pq/(i\hbar)]$ for any number λ . The equation to be established is

$$\hat{\xi}(\lambda) = \langle p' | \exp[\lambda pq/(i\hbar)] | q' \rangle = (2\pi\hbar)^{-1/2} \exp[e^\lambda p' q'/(i\hbar)]. \quad (54)$$

First the differential equation for $\hat{\xi}(\lambda)$ is

$$\begin{aligned} i\hbar \partial \hat{\xi}(\lambda) / \partial \lambda &= \langle p' | pq \exp[\lambda pq/(i\hbar)] | q' \rangle = p' \langle p' | \exp[\lambda pq/(i\hbar)] q | q' \rangle \\ &= p' q' \langle p' | \exp\{\lambda(pq + |q, p\rangle)/(i\hbar)\} | q' \rangle = p' q' e^\lambda \hat{\xi}(\lambda), \end{aligned} \quad (55)$$

where the trivial identity: $q(pq)^n = (qp)^n q$ must be noted. Now from the initial condition for $\hat{\xi}(\lambda)$

$$\hat{\xi}(0) = \langle p' | q' \rangle = (2\pi\hbar)^{-1/2} \exp[p' q'/(i\hbar)], \quad (56)$$

the validity of (54) is evident. A generalization of this formula is

$$\langle p' | \exp[\lambda pq^{n+1}/(i\hbar)] | q' \rangle = (2\pi\hbar)^{-1/2} \exp[p' q' (1 - n\lambda q'^n)^{-1/n}/(i\hbar)], \quad (57)$$

and the proof of this formula may be similarly given. According to eqs. (32) and (54), we have

$$\langle p' | \exp(\lambda R) | q' \rangle = (2\pi\hbar)^{-1/2} \exp(\lambda/2) \exp[e^\lambda p' q'/(i\hbar)], \quad (58)$$

and its complex conjugate

$$\langle q' | \exp(\lambda R) | p' \rangle = (2\pi\hbar)^{-1/2} \exp(-\lambda/2) \exp[e^{-\lambda} p' q'/(i\hbar)]. \quad (59)$$

On these grounds we can at once evaluate the matrix element of the transformation operator $T(\tau, \sigma)$ as follows:

$$\langle p'' | T(\tau, \sigma) | q' \rangle = (u/2\pi\hbar z')^{1/2} \exp[iS(p'', q')/\hbar], \quad (60)$$

$$S(p'', q') = -(u/2z') \{zp''^{1/2}/(m\omega) + 2p''q' - (m\omega y'/u)q'^2\}, \quad (60')$$

$$\langle q'' | T(\tau, \sigma) | p' \rangle = (2\pi\hbar y)^{-1/2} \exp[iS(q'', p')/\hbar], \quad (61)$$

$$S(q'', p') = (1/2y) \{(m\omega y'/u)q''^2 + 2q''p' - (z/m\omega)p'^2\}. \quad (61')$$

When an external force is acting, a term $-\omega \gamma(\lambda)q$ linear in q must be added to the Hamiltonian (30), where $\gamma(\lambda)$ is an arbitrary function in λ and ω is inserted for convenience. The disentangling process for this case is easily effected on the following considerations. Noting the commutators of q with large P and R : $[q, P] = p/(m\omega)$ and $[q, R] = q$, we have

$$e^{-\lambda P} q e^{\lambda P} = q + \lambda p/(m\omega), \quad e^{-\lambda R} q e^{\lambda R} = e^{\lambda} q, \quad (62)$$

and accordingly from eq. (53)

$$\begin{aligned} T(\sigma, \lambda) q T(\lambda, \sigma) &= \exp[-zP/y] \exp[-(\log y)R] q \exp[(\log y)R] \exp[zP/y] \\ &= \exp(\log y)[q + \{z(\lambda)/m\omega y(\lambda)\}p] = y(\lambda)q + z(\lambda)p/(m\omega). \end{aligned} \quad (63)$$

Now the disentangling of the forced transformation operator $T^F(\tau, \sigma)$ proceeds in the following way:

$$T^F(\tau, \sigma) = \text{Exp} \left[\int_{\sigma}^{\tau} d\lambda \{a(\lambda)P + \beta(\lambda)Q - \gamma(\lambda)q/(i\hbar)\} \right] = T(\tau, \sigma) \cdot F(\tau, \sigma), \quad (64)$$

$$\begin{aligned} F(\tau, \sigma) &= \text{Exp} \left[(i/\hbar) \int_{\sigma}^{\tau} d\lambda \gamma(\lambda) T(\sigma, \lambda) q T(\lambda, \sigma) \right] \\ &= \text{Exp} \left[(i/\hbar) \int_{\sigma}^{\tau} d\lambda \gamma(\lambda) \{y(\lambda)q + z(\lambda)p/(m\omega)\} \right] \\ &= \exp \left[(ip/\hbar m\omega) \int_{\sigma}^{\tau} d\lambda \gamma(\lambda) z(\lambda) \right] \cdot \text{Exp} \left[(i/\hbar) \int_{\sigma}^{\tau} d\lambda \gamma(\lambda) y(\lambda) \right. \\ &\quad \times \exp \left[-(ip/\hbar m\omega) \int_{\sigma}^{\lambda} d\mu \gamma(\mu) z(\mu) \right] q \exp \left[(ip/\hbar m\omega) \int_{\sigma}^{\lambda} d\mu \gamma(\mu) z(\mu) \right] \Big] \\ &= \exp \left[(ip/\hbar m\omega) \int_{\sigma}^{\tau} d\lambda \gamma(\lambda) z(\lambda) \right] \cdot \exp \left[(i/\hbar) \int_{\sigma}^{\tau} d\lambda \gamma(\lambda) y(\lambda) \right] \\ &\quad \times \exp \left[-(i/\hbar m\omega) \int_{\sigma}^{\tau} d\lambda \gamma(\lambda) y(\lambda) \int_{\sigma}^{\lambda} d\mu \gamma(\mu) z(\mu) \right]. \end{aligned} \quad (65)$$

Once performing the well known Gaussian integral, we get the matrix element for this operator, using the identity (49):

$$\begin{aligned} \langle q'' | T^F(\tau, \sigma) | q' \rangle &= \int \langle q'' | T(\tau, \sigma) | p' \rangle dp' \langle p' | F(\tau, \sigma) | q' \rangle \\ &= \{m\omega/2\pi\hbar iz(\tau)\}^{1/2} \exp[iS^F(q'', q')/\hbar], \end{aligned} \quad (66)$$

where the classical action function is

$$S^F(q'', q') = \{1/z(\tau)\} [(m\omega/2) \{z'(\tau) q''^2/a(\tau) + y(\tau) q'^2 - 2q''q'\} \\ + q' \int_{\sigma}^{\tau} d\lambda \gamma(\lambda) \{z(\tau) y(\lambda) - y(\tau) z(\lambda)\} + q'' \int_{\sigma}^{\tau} d\lambda \gamma(\lambda) z(\lambda) \\ - (1/m\omega) \int_{\sigma}^{\tau} d\lambda \gamma(\lambda) \{z(\tau) y(\lambda) - y(\tau) z(\lambda)\} \int_{\sigma}^{\lambda} d\mu \gamma(\mu) z(\mu)] . \quad (66')$$

The usual harmonic oscillator corresponds to the specification: $a(\lambda) = \beta(\lambda) = 1$, for which we have according to (45) and (48), $y(\lambda) = \cos(\lambda - \sigma)$ and $z(\lambda) = \sin(\lambda - \sigma)$. The substitution of these formulas into (66) and (66') affords well known kernel and action function for the forced harmonic oscillator. Another specification: $a(\lambda) = 1$ and $\beta(\lambda) = 0$, describes the non-relativistic motion of a particle in arbitrarily varying external field, the result of which is all the same well known. A slightly complicated example is

$$a(\lambda) = 1, \quad \beta(\lambda) = 2(n+1) \operatorname{sech}^2 \lambda (1 - n \operatorname{cosech}^2 \lambda). \quad (67)$$

According to eq. (44), one particular solution is $y_1(\lambda) = (\tanh \lambda)^{n+1}$ and another is evaluated by simple quadrature from (49):

$$y_2(\lambda) = (\tanh \lambda)^{n+1} \int_{\sigma}^{\lambda} d\mu (\coth \mu)^{2(n+1)}. \quad (68)$$

Then $y(\lambda)$ can be constructed by (45) and for $z(\lambda)$ the following prescription will be more convenient than (48):

$$z(\lambda) = y_2(\sigma) y_1(\lambda) - y_1(\sigma) y_2(\lambda). \quad (69)$$

Detailed discussion of this example is omitted here.

In conclusion we will briefly refer to the practical side of our disentangling technique. As is easily seen in above considerations, the most fundamental computation rule in this procedure is the transformation (28), in which generally appears an infinite number of multiple commutators and the commutator plays a principal role. Consequently the key for this process to be successful in a finite manner seems to be that the series expansion (28) should terminate in finite terms, in other words, different operators appearing in this should form together a closed system respecting the commutator operation. Judging from this criterion, the only favorable case in non-relativistic quantum mechanics is our generalized forced harmonic oscillator. Other examples, in which the potential involves the space-coordinate to higher powers than the second or to inverse powers are all excluded. Above all, the omission of Coulomb potential is quite regrettable, while we see a subtle clue of regularity.

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Appendix 1. Eigenfunction expansion of free oscillating kernel

In this appendix an operational treatment of the eigenfunction expansion of free oscillating kernel is intended, which otherwise necessitates cumbersome computations in

terms of the Hermite polynomials. In order to treat this problem operationally, or to free the calculations completely from representatives the bra-ket symbolism due to Dirac¹⁾ is indispensable, accordingly a few preliminaries of which are presented in the first place. Now we will introduce the conception of standard ket, which is to be grasped as the total summation of eigenkets for any one particular operator over all its possible eigenvalues. Respecting the coordinate operator q , we have a standard ket

$$|s(q)\rangle = \int_{-\infty}^{\infty} dq' |q'\rangle, \quad (\text{A.1})$$

where of course $q|q'\rangle = q'|q'\rangle$. The characteristic property of this ket is as is well known $\langle q'|s(q)\rangle = 1$ for any eigenvalue q' . Correspondingly for the momentum operator p , we have

$$|s(p)\rangle = \int_{-\infty}^{\infty} dp' |p'\rangle \quad \text{and} \quad \langle p'|s(p)\rangle = 1, \quad (\text{A.2})$$

for any eigenvalue p' . With the help of the transformation function (56), any eigenket of q may be transformed in the following way:

$$\begin{aligned} |q'\rangle &= \int |p'\rangle dp' \langle p'|q'\rangle = (2\pi\hbar)^{-1/2} \int \exp [ip'q'/(i\hbar)] |p'\rangle dp' \\ &= (2\pi\hbar)^{-1/2} \exp [ip'q/(i\hbar)] |s(p)\rangle. \end{aligned} \quad (\text{A.3})$$

Especially for $q'=0$, the eigenket of q is identical with the standard ket in p -space, that is, $|0(q)\rangle = (2\pi\hbar)^{-1/2} |s(p)\rangle$, where $q|0(q)\rangle = 0$. Hence for any q' we have $|q'\rangle = \exp [ip'q/(i\hbar)] |0(q)\rangle$. The same is also valid for p : $|p'\rangle = (2\pi\hbar)^{-1/2} \exp (ip'q/\hbar) |s(q)\rangle$, $|0(p)\rangle = (2\pi\hbar)^{-1/2} |s(q)\rangle$ and $|p'\rangle = \exp (ip'q/\hbar) |0(p)\rangle$. A connection between these two standard kets is easily established once performing the Gaussian integral:

$$\exp(P/i\mu) |s(p)\rangle = (\mu m \omega)^{1/2} \exp(\mu Q/i) |s(q)\rangle, \quad (\text{A.4})$$

where μ is a number. By the way, since $P|s(q)\rangle = (2\pi\hbar)^{-1/2} P|0(p)\rangle = 0$, we have identically

$$\exp(P/i\mu) |s(q)\rangle = |s(q)\rangle. \quad (\text{A.5})$$

The key to the present problem is the recognition of the following simple transformation, which may be easily established on the basis of eqs. (35) and (36):

$$\frac{1}{\hbar\omega} H = i(P+Q) = \exp(iQ) \exp(-iP/2) R \exp(iP/2) \exp(-iQ). \quad (\text{A.6})$$

Then for the free oscillating transformation operator an operator $\exp(-i\lambda R)$ plays a principal role, which may be transformed according to eq. (58):

$$\begin{aligned} \exp(-i\lambda R) &= \iint |p'\rangle dp' \langle p'| \exp(-i\lambda R) |q'\rangle dq' \langle q'| \\ &= (2\pi\hbar)^{-1/2} \exp(-i\lambda/2) \iint |p'\rangle dp' \exp[e^{-i\lambda} p'q'/(i\hbar)] dq' \langle q'| \\ &= (2\pi\hbar)^{-1/2} \sum_{n=0}^{\infty} (1/n!) \exp\left\{-i\lambda\left(n + \frac{1}{2}\right)\right\} \cdot p^n |s(p)\rangle \langle s(q)| (q/i\hbar)^n. \end{aligned} \quad (\text{A.7})$$

Therefore the transformation operator reads

$$\begin{aligned} T(\lambda, 0) &= \exp[\lambda(P+Q)] = \exp(iQ) \exp(-iP/2) \exp(-i\lambda R) \exp(iP/2) \exp(-iQ) \\ &= \sum_{n=0}^{\infty} \exp\left\{-i\lambda\left(n + \frac{1}{2}\right)\right\} \cdot T_n. \end{aligned} \quad (\text{A} \cdot 8)$$

On the basis of eqs. (A·4) and (A·5), this linear operator T_n can be transformed as follows :

$$\begin{aligned} T_n &= (2\pi\hbar)^{-1/2} (1/n!) \exp(iQ) p^n \exp(-iP/2) |s(p)\rangle \\ &\quad \times \langle s(q) | (q/i\hbar)^n \exp(iP/2) \exp(-iQ) \\ &= (m\omega/\pi\hbar)^{1/2} (1/n!) \exp(iQ) p^n \exp(-iQ) \exp(-iQ) |s(q)\rangle \\ &\quad \times \langle s(q) | \exp(-iQ) \exp(iQ) \exp(-iP/2) (q/i\hbar)^n \exp(iP/2) \exp(-iQ) \\ &= \{(2\pi m\omega)^{-1/2} e^{iQ} p e^{-iQ}\}^n (1/n!)^{1/2} (m\omega/\pi\hbar)^{1/4} e^{-iQ} |s(q)\rangle \\ &\quad \times \langle s(q) | e^{-iQ} (m\omega/\pi\hbar)^{1/4} (1/n!)^{1/2} \{(2\hbar m\omega)^{1/2} e^{iQ} e^{-iP/2} (q/i\hbar) e^{iP/2} e^{-iQ}\}^n. \end{aligned} \quad (\text{A} \cdot 9)$$

According to Dirac¹⁾ the normalized eigenket of the n -th quantum state takes the following form :

$$|n\rangle = (1/n!)^{1/2} (m\omega/\pi\hbar)^{1/4} \eta^n \exp(-iQ) |s(q)\rangle, \quad (\text{A} \cdot 10)$$

where the dimensionless auxiliary operator η is

$$\eta = (2\hbar m\omega)^{-1/2} (p + im\omega q) = (2\hbar m\omega)^{-1/2} \exp(iQ) p \exp(-iQ), \quad (\text{A} \cdot 11)$$

of which the hermitian conjugate is

$$\eta^\dagger = (2\hbar m\omega)^{1/2} \exp(iQ) \exp(-iP/2) (q/i\hbar) \exp(iP/2) \exp(-iQ). \quad (\text{A} \cdot 12)$$

Now the above linear operator T_n reads simply : $T_n = |n\rangle\langle n|$, and the Schrödinger representative of the n -th eigenket is

$$\begin{aligned} \langle q' | n \rangle &= (1/n!)^{1/2} (m\omega/\pi\hbar)^{1/4} (2\hbar m\omega)^{-n/2} \langle q' | e^{iQ} p^n e^{-iQ} | s(q) \rangle \\ &= i^n (1/n!)^{1/2} (m\omega/\pi\hbar)^{1/4} (\hbar/2m\omega)^{n/2} e^{-iQ'} \{e^{2iQ'} (-\partial/\partial q')^n e^{-2iQ'}\} \\ &= i^n (1/n!)^{1/2} (m\omega/\pi\hbar)^{1/4} \exp(-x^2/4) \{\exp(x^2/2) (-\partial/\partial x)^n \exp(-x^2/2)\} \\ &= i^n (2m\omega/\hbar)^{1/4} \phi_n(x), \end{aligned} \quad (\text{A} \cdot 13)$$

where $x = (2m\omega/\hbar)^{1/2} q'$ and $\phi_n(x)$ is a normalized Hermite function

$$\phi_n(x) = (n!)^{-1/2} (2\pi)^{-1/4} \exp(-x^2/4) H_n(x), \quad (\text{A} \cdot 14)$$

and $H_n(x)$ is a Hermite polynomial, i. e., $H_n(x) = \exp(x^2/2) (-\partial/\partial x)^n \exp(-x^2/2)$. Accordingly the final result of our calculation is

$$T(\lambda, 0) = \sum_{n=0}^{\infty} \exp\left\{-i\lambda\left(n + \frac{1}{2}\right)\right\} |n\rangle\langle n|, \quad (\text{A} \cdot 15)$$

and the colinear expansion of free oscillating kernel is as is well known

$$K(q'', q') = \langle q'' | T(\lambda, 0) | q' \rangle = (2m\omega/\hbar)^{1/2} \times \sum_{n=0}^{\infty} \exp \left\{ -i\omega t \left(n + \frac{1}{2} \right) \right\} \cdot \psi_n(\{2m\omega/\hbar\}^{1/2} q'') \psi_n^*(\{2m\omega/\hbar\}^{1/2} q'). \quad (\text{A} \cdot 16)$$

Appendix 2. Non-relativistic particle in constant uniform magnetic field

This problem is essentially three dimensional, and we have three pairs of operators, (p_1, q_1) , (p_2, q_2) and (p_3, q_3) . When the magnetic field is directed along the q_3 -axis, the third pair is irrelevant only describing a free motion along this axis, which will be neglected throughout this appendix. Now the Hamiltonian reads

$$H = (\Pi_1^2 + \Pi_2^2)/2m, \quad (\text{A} \cdot 17)$$

where $\Pi_1 = p_1 - m\omega q_2$, $\Pi_2 = p_2 + m\omega q_1$ ($m\omega = eH/2c$). In view of the commutator $[\Pi_2, \Pi_1] = i\hbar \cdot 2m\omega$, we may expect that there exists a canonical transformation, which converts the pair (Π_2, Π_1) into the pair (q_1, p_1) . For this purpose, first introduce two anti-hermitian operators

$$P = p_1 p_2 / (m\omega i\hbar), \quad Q = m\omega q_1 q_2 / (i\hbar), \quad (\text{A} \cdot 18)$$

whose commutators with small letters are

$$\left. \begin{aligned} [q_1, P] &= p_2 / (m\omega), & [q_2, P] &= p_1 / (m\omega), \\ [Q, p_1] &= m\omega q_2, & [Q, p_2] &= m\omega q_1. \end{aligned} \right\} \quad (\text{A} \cdot 19)$$

Next we will introduce a third operator

$$R = [Q, P] = (q_1 p_1 + p_2 q_2) / (i\hbar) = I + (p_1 q_1 + p_2 q_2) / (i\hbar), \quad (\text{A} \cdot 20)$$

whose properties are $[Q, R] = 2Q$ and $[P, R] = -2P$. Consequently these three operators P , Q and R are algebraically identical with those introduced in § 4. Consider a transformation operator $U(\mu) = \exp[\mu(P + Q)]$ for any number μ , which can be disentangled according to eqs. (52) and (53) in the following manner:

$$\begin{aligned} U(\mu) &= \exp[(\tan \mu) Q] \exp[(\log \cos \mu) R] \exp[(\tan \mu) P] \\ &= \exp[(\tan \mu) P] \exp[(\log \sec \mu) R] \exp[(\tan \mu) Q]. \end{aligned} \quad (\text{A} \cdot 21)$$

Therefore the specification $\mu = \pi/4$ gives

$$\left. \begin{aligned} \Pi_1 &= \sqrt{2} \cdot U^+(\pi/4) p] U(\pi/4), \\ \Pi_2 &= \sqrt{2} m\omega \cdot U^+(\pi/4) q_1 U(\pi/4), \end{aligned} \right\} \quad (\text{A} \cdot 22)$$

and the Hamiltonian reduces to

$$H = 2 \cdot U^+(\pi/4) \cdot (\omega/2) [p_1^2 / (m\omega) + m\omega q_1^2] \cdot U(\pi/4), \quad (\text{A} \cdot 23)$$

which is essentially that of harmonic oscillator excepting a numerical factor 2. Now the

transformation operator reads simply $U^+(\pi/4)T(2\lambda, 0)U(\pi/4)$ and its matrix element is easily evaluated on the basis of § 4. According to eq. (58) we have

$$\begin{aligned} \langle p_1' p_2' | U(\pi/4) | q_1' q_2' \rangle &= \sqrt{2} (2\pi\hbar)^{-1} \\ &\times \exp \left[\{ p_1' p_2' / (m\omega) + m\omega q_1' q_2' + \sqrt{2} (p_1' q_1' + p_2' q_2') \} / (i\hbar) \right], \end{aligned} \quad (\text{A} \cdot 24)$$

and from eqs. (60) or (61), once performing the Gaussian integral

$$\begin{aligned} \langle p_1'' | T(2\lambda, 0) | p_1' \rangle &= \{ 2\pi\hbar m\omega i \sin(2\lambda) \}^{-1/2} \\ &\times \exp \left[i \{ (\cot 2\lambda) (p_1''^2 + p_1'^2) - 2(\operatorname{cosec} 2\lambda) p_1'' p_1' \} / (2\hbar m\omega) \right]. \end{aligned} \quad (\text{A} \cdot 25)$$

The final result of our calculation is

$$\begin{aligned} &\langle q_1'' q_2'' | U^+(\pi/4) T(2\lambda, 0) U(\pi/4) | q_1' q_2' \rangle \\ &= \iiint d p_2' \langle q_1'' q_2'' | U^+(\pi/4) | p_1'' p_2' \rangle d p_1'' \langle p_1'' | T(2\lambda, 0) | p_1' \rangle d p_1' \langle p_1' p_2' | U(\pi/4) | q_1' q_2' \rangle \\ &= \{ m\omega / (2\pi\hbar i \sin \lambda) \} \exp \left[i S(q_1'' q_2''; q_1' q_2') / \hbar \right], \end{aligned} \quad (\text{A} \cdot 26)$$

where the classical action is

$$\begin{aligned} S(q_1'' q_2''; q_1' q_2') / (m\omega) &= \frac{1}{2} \cot \lambda \{ (q_1'' - q_1')^2 + (q_2'' - q_2')^2 \} \\ &+ (q_1'' q_2' - q_2'' q_1'). \end{aligned} \quad (\text{A} \cdot 27)$$

References

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Operator Calculus in Quantized Field Theory

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Following the suggestion of Feynman, we analyze the operators corresponding to the electron-positron field by means of the operator calculus. Thus we can deduce operationally Feynman's theory from the second quantized theory. With the aid of the external spinors, which are considered as the analogues of the Schwinger's prescribed source, we obtain formally a closed form of the S-matrix. Using this result we can derive the diffusion equation for the Dirac field very simply. The relations to the Schwinger's new theory of Green function are also discussed. Finally the formula for multiple boson or fermion pair-production are presented as illustrations for applications.

§ 1. Introduction and summary

Recently Schwinger¹⁾ and many others have made attempts to obtain the closed forms of the S-matrix and the Green's functions of quantized fields, avoiding the use of the perturbation procedures. We have here discussed a closed form of the S-matrix formally, at least in the case of the external electromagnetic field, with the aid of the operator calculus. The calculus of quantized operator has been, originally and extensively, used by Feynman²⁾, but we adopt here its developed scheme given by Fujiwara³⁾.

Using the operator calculus, in this paper two matters are intended; one is the proof of the equivalence of Feynman's theory and the usual quantized field theory of Tomonaga-Schwinger, and the other is the above mentioned closed form of the S-matrix. The first problem has already been discussed by many authors, Dyson proved this equivalence in the frame of perturbation expansion and Feynman's proof was attained by means of his operator calculus. But Feynman did not take up the quantized Dirac field, and used the results of his own positron theory, and carried out the disentangling of the radiation field only. We now here disentangle the quantized Dirac field operators and provide the direct proof of the above equivalence operationally. Our aim is to deduce consistently as many results as possible from the definition of the transformation operator U and the commutation relations between the field variables. It means that without explicit applications of physical considerations except basic prescriptions set up above, we resort only to the means of the mathematical procedures of "disentangling"—avoiding the expansion with coupling constant—: for example, for one-body kernel of which we know only its definition: ${}_0\langle P(S\bar{\psi}(1)\psi(2)) \rangle_0 \in (1, 2)$, we seek the integral equation which it satisfies.

The program was suggested by Feynman²⁾, but contrarily to his conjecture the analysis of this problem does not throw any new light on the problem of disentanglement of the form $\int_0^\infty \exp[i(\gamma p + m)W]dW$, because, although in this form anticommuting quantity γ

appears in the *linear* form and causes essential difficulty owing to its *anticommutativity*, in our problem the anticommuting quantities ψ and $\bar{\psi}$ appear always in the *bilinear* form and do not cause any trouble owing to its *anticommutativity*. A prominent point of the concept of Feynman's ordered operators is that we can disentangle the different fields separately step by step, i.e. during the process of disentangling of one field (in our case ψ , and in Feynman's²⁾ case A_μ) we can treat the other field (in our case A_μ , and in Feynman's²⁾ case ψ) as if it were a c-number, but we must remember that the latter field operates in time order, not in positional order.*)

In a brief outline the procedure is the following. In § 2 we have disentangled the transformation operator U with respect to the Dirac field and illustrate our mathematical method in some details. Even though it is complicated a little in the integral equation method treated there, there seems some possibility of handling the bound state problems. In the Appendix we give a sketch of the differential equation method, which is simpler than that of the integral equation. In § 3 we have taken the expectation values of U -operator and have discussed the relation between the S-matrix and the kernels, and also the integral equations which are satisfied by them.

In § 4 we introduce external spinors ϕ and $\bar{\phi}$ which are the analogues of the Schwinger's prescribed source η and $\bar{\eta}$, and intend to describe the real spinor particles in the initial and final states, which are present on both edges of the electron lines in Dyson-diagram. Using this new techniques we can construct the S-matrix in a closed form formally, and prove the equivalence of our procedure and the usual one. These forms have recently been obtained by M. Neuman¹⁾ independent to us from the point of view of the Fredholm theory of integral equations. In § 5 are presented the simple derivation of diffusion equations for the Dirac field which were introduced by Fukuda and Tomonaga⁷⁾ after very elaborate calculations. We derive this equation operationally as was done by Feynman for the case of the radiation field. Of course the diffusion equation can be obtained directly, if we use the functional derivatives⁶⁾ or the Wick's product symbol⁷⁾ formally, but this does not mean, I think, any important advance to solve the problem. The relations to the Schwinger's new theory of Green function¹⁾ have been also discussed. § 6 are devoted to some illustration of our operator calculus and derivations of the formula for multiple production of bosons and fermions; the boson case was already treated by Glauber⁸⁾, and of course his results agree with ours.

§ 2. Structure of the transformation operator $U(\sigma_b, \sigma_a)$

Concerning the systems treated, we may restrict ourselves with Quantum Electrodynamics without loss of generality. Starting with the interaction representation, we take the interaction Hamiltonian as**)

*) Concerning the principles and techniques of our operator calculus the reader are requested to refer to the Fujiwara's work³⁾. But we have not gained any new results, since we do not use any new principles and techniques, and it is only a mathematical re-expression of old materials.

**) In this paper we use Schwinger's notations⁹⁾ mainly and take $\hbar=c=1$.

$$H = -i\bar{\psi}A\psi, \quad A \equiv e\gamma_\mu A_\mu, \quad (2.1)$$

where ψ and $\bar{\psi}$ are the quantized Dirac spinors, and A_μ is the quantized electromagnetic field, but we can treat the latter as if it were a c-number in the meaning of Feynman's ordered operators. We decompose the spinor fields into positive and negative frequency parts. The commutation relations are

$$\left. \begin{aligned} \{\psi^+(x), \bar{\psi}^-(x')\} &= -iS^{(+)}(x-x'), \\ \{\psi^-(x), \bar{\psi}^+(x')\} &= -iS^{(-)}(x-x'), \\ \text{Other combinations anticommute.} \end{aligned} \right\} \quad (2.2)$$

As it is well known, the transformation operator $U(\sigma_b, \sigma_a)$ is defined as

$$U(\sigma_b, \sigma_a) = \text{Exp} \left\{ -i \int_{\sigma_a}^{\sigma_b} d\tau H(x) \right\}, \quad (2.3)$$

where Exp means Fujiwara's expansion operator. We disentangle the U -operator so as to bring the annihilation operators ψ^+ and $\bar{\psi}^+$ on the right side and the creation operators ψ^- and $\bar{\psi}^-$ on the left. Feynman²⁾ and many others disentangle the operators A_μ^+ and A_μ^- first, but contrarily to their procedures we would like to disentangle the spinor fields first.

We follow Fujiwara's method of disentangling of Exp-operators and put

$$P_1 = -\bar{\psi}^+ A \psi^+, \quad R_1 = -(\bar{\psi}^- A \psi^+ + \bar{\psi}^+ A \psi^-), \quad Q_1 = -\bar{\psi}^- A \psi^-. \quad (2.4)$$

In the following the terms, in which ψ 's are contained in the form $\bar{\psi}^+ \psi^+$, $\bar{\psi}^- \psi^-$, $\bar{\psi}^- \psi^+$ or $\bar{\psi}^+ \psi^-$, are referred as P -, Q -, R -part respectively and we use the same capital letters to represent them. Following the above program we put^{*)}

$$\left. \begin{aligned} U(\sigma_b, \sigma_a) &= \text{Exp} \left\{ \int d\tau (P_1 + R_1 + Q_1) \right\} \\ &= \text{exp} \left\{ \int d\tau Q \right\} \text{Exp} \left\{ \int d\tau R \right\} \text{exp} \left\{ \int d\tau P \right\} \\ &= \text{Exp} \left[\int d\tau \left\{ Q + e^{\int_{\sigma_a}^{\sigma_b} d\tau' Q'} Re^{-\int_{\sigma_a}^{\sigma_b} d\tau' Q'} + e^{\int_{\sigma_a}^{\sigma_b} d\tau' Q'} (E^{\int_{\sigma_a}^{\sigma_b} d\tau' R'} PE^{\int_{\sigma_a}^{\sigma_b} d\tau' R'}) e^{-\int_{\sigma_a}^{\sigma_b} d\tau' Q'} \right\} \right], \end{aligned} \right\} \quad (2.5)$$

where

$$P = \bar{\psi}^+ f \psi^+, \quad R = \bar{\psi}^+ g \psi^- + \bar{\psi}^- g \psi^+, \quad Q = \bar{\psi}^- f \psi^-. \quad (2.6)$$

And now we must determine the functional dependency of f and g on A . With the aid of the commutation relations (2.2) we get

$$\left. \begin{aligned} [Q, P] &= -i(\bar{\psi}^- f S^{(-)} f' \psi^{+'} - \bar{\psi}^{+'} f' S^{(+)} f \psi^-) \dots R\text{-part} \\ [Q, R] &= -i(\bar{\psi}^- f S^{(-)} g' \psi^{-'} - \bar{\psi}^{-'} g' S^{(+)} f \psi^-) \dots Q\text{-part} \\ [P, R] &= -i(\bar{\psi}^+ f S^{(+)} g' \psi^{+'} - \bar{\psi}^{+'} g' S^{(-)} f \psi^+) \dots P\text{-part} \end{aligned} \right\} \quad (2.7) **$$

*) For the sake of simplicity, in this section we abbreviate $\int_{\sigma_a}^{\sigma_b} d\tau$ as \int and $\int_{\sigma_a}^{\sigma_b} d\tau' R' = \int_{\sigma_a}^{\sigma_b} d\tau' R(x')$,

$\int_{\sigma} d\tau' R' = \int_{\sigma}^{\sigma_a} d\tau' R(x')$. σ means the space-like surface passing the point x , and since $[Q, Q'] = [P, P'] = 0$, we can write $E^{\int d\tau Q} = e^{\int d\tau Q}$, $E^{\int d\tau Q} = e^{\int d\tau Q}$ but for R -part we must use the Exp.

**) In full writing this means

$[Q(x), P(x')] = -i(\bar{\psi}^-(x)f(x)S^{(-)}(x-x')f(x')\psi^+(x') - \bar{\psi}^+(x')f(x')S^{(+)}(x'-x)f(x)\psi^-(x))$, etc.

and

$$\left. \begin{aligned} e^{\int_{\sigma_a}^{\sigma} d\tau' Q'} R e^{-\int_{\sigma_a}^{\sigma} d\tau' Q'} &= R - i \int d\tau' (\bar{\psi}^{-'} f' S^{(-)} \psi^{-'} - \bar{\psi}^{-'} S^{(+)} f' \psi^{-'}) \\ e^{\int_{\sigma_a}^{\sigma} d\tau' Q'} P e^{-\int_{\sigma_a}^{\sigma} d\tau' Q'} &= P - i \int d\tau' (\bar{\psi}^{-'} f' S^{(-)} f \psi^{+} - \bar{\psi}^{-'} f S^{(+)} f' \psi^{-'}) \\ &\quad - (-i)^2 \int d\tau' \int d\tau'' (\bar{\psi}^{-''} f'' S^{(-)} f S^{(+)} f' \psi^{-'} + \bar{\psi}^{-'} f' S^{(-)} f S^{(+)} f'' \psi^{-''}) \end{aligned} \right\} \quad (2.8)$$

From (2.8) we see that $e^{\int_{\sigma_a}^{\sigma} d\tau' Q'} R e^{-\int_{\sigma_a}^{\sigma} d\tau' Q'}$ has no P -part. $E^{\int_{\sigma_a}^{\sigma} d\tau' R'} P E^{\int_{\sigma_a}^{\sigma} d\tau' R'}$ contains P -part only, and yet, even if we now multiply $e^{\int_{\sigma_a}^{\sigma} d\tau' Q'}$, $e^{-\int_{\sigma_a}^{\sigma} d\tau' Q'}$ from both sides respectively, it does not yield any new P -part. And so identifying the P -parts in (2.5), we get

$$E^{\int_{\sigma_a}^{\sigma} d\tau' R'} P E^{\int_{\sigma_a}^{\sigma} d\tau' R'} = P_1 \quad (2.9)$$

Similarly identifying the R -parts in (2.5), and using (2.9) we have

$$R + R \text{ part of } (e^{\int_{\sigma_a}^{\sigma} d\tau' Q'} P_1 e^{-\int_{\sigma_a}^{\sigma} d\tau' Q'}) = R_1.$$

And now when we apply (2.8) in this equation we get,

$$R - R_1 = \int_{\sigma_a}^{\sigma} d\tau' [P_1, Q']. \quad (2.10R)$$

In the same way the Q -part satisfies

$$Q = Q_1 + \int_{\sigma_a}^{\sigma} d\tau' [R_1, Q'] + \frac{1}{2} \int_{\sigma_a}^{\sigma} d\tau' \int_{\sigma_a}^{\sigma} d\tau'' [\tau'' [Q', [Q'', P_1]]]. \quad (2.10Q)$$

(2.10R) and (2.10Q) are the equations which must be solved.

It is rather an easy task to solve these equations in power series of A and to prove the general validity of them with mathematical induction.

But we avoid the perturbation procedure and use the integral equation method to solve (2.10). In the following, we use familiar Feynman's K_+ function, instead of $S^{(+)}$ or $S^{(-)}$.*) First we take up (2.10Q). Obviously we can put.

$$\int_{\sigma_a}^{\sigma} d\tau' Q' = - \int_{\sigma_a}^{\sigma} d\tau' \bar{\psi}^{-'} A' \psi^{-'} + \int_{\sigma_a}^{\sigma} d\tau' \int_{\sigma_a}^{\sigma} d\tau'' \bar{\psi}^{-''} A' K_+^A(\sigma, \sigma_a) A'' \psi^{-''}, ** \quad (2.11)$$

and we seek the integral equation which $K_+^A(\sigma, \sigma_a)$ satisfies. From (2.11) we have

$$Q = -\bar{\psi}^{-} A \psi^{-} + \int_{\sigma_a}^{\sigma} d\tau' (\bar{\psi}^{-} A K_+^A(\sigma, \sigma_a) A' \psi^{-'} + \bar{\psi}^{-'} A' K_+^A(\sigma, \sigma_a) A \psi^{-})$$

*) As is well known, Feynman's K_+ function is defined as

$$K^-(x) = \begin{cases} -iS^{(+)}(x) & \text{for } x_0 > 0 \\ +iS^{(-)}(x) & \text{for } x_0 < 0, \end{cases}$$

then we see that $S^{(+)}$ and $S^{(-)}$ functions appearing in the above and in the following equations can be substituted by K_+ function.

**) In full writing this means $\int_{\sigma_a}^{\sigma} d\tau' \int_{\sigma_a}^{\sigma} d\tau'' \bar{\psi}^{-}(x') A(x') K_+^A(x', x''; \sigma, \sigma_a) A(x'') \psi^{-}(x'')$.

$$+ \int_{\sigma_a}^{\sigma} d\tau' \int_{\sigma_a}^{\sigma} d\tau'' \bar{\psi}^{-'} A' \frac{\partial K_+^A(\sigma, \sigma_a)}{\partial \sigma(x)} A'' \psi^{-''}, \quad (2.12)$$

and in the same way we express the other terms of (2.10Q), and inserting them into (2.10Q) we readily obtain

$$\begin{aligned} & \int_{\sigma_a}^{\sigma} d\tau' \bar{\psi}^{-} A \left\{ K_+^A(\sigma, \sigma_a) - K_+ + \int_{\sigma_a}^{\sigma} d\tau'' K_+ A'' K_+^A(\sigma, \sigma_a) \right\} A' \psi^{-'} \\ & + \int_{\sigma_a}^{\sigma} d\tau' \bar{\psi}^{-'} A' \left\{ K_+^A(\sigma, \sigma_a) - K_+ + \int_{\sigma_a}^{\sigma} d\tau'' K_+^A(\sigma, \sigma_a) A'' K_+ \right\} A \psi^{-} \\ & + \int_{\sigma_a}^{\sigma} d\tau' \int_{\sigma_a}^{\sigma} d\tau'' \bar{\psi}^{-'} A' \left\{ \frac{\partial K_+^A(\sigma, \sigma_a)}{\partial \sigma(x)} \right. \\ & \left. + (K_+ - \int_{\sigma_a}^{\sigma} d\tau''' K_+^A(\sigma, \sigma_a) A''' K_+) A (K_+ - \int_{\sigma_a}^{\sigma} d\tau^{IV} K_+ A^{IV} K_+^A(\sigma, \sigma_a)) \right\} A'' \psi^{-''} \\ & = 0. \end{aligned} \quad (2.13)$$

To solve this equation we get the inspection that $K_+^A(\sigma, \sigma_a)$ should satisfy

$$K_+^A(\sigma, \sigma_a) - K_+ + \int_{\sigma_a}^{\sigma} d\tau' K_+ A' K_+^A(\sigma, \sigma_a) = 0. \quad (2.14)$$

Then the first and the second terms of (2.13) become zero.*) Now we must show that K_+^A function defined in this manner makes the third term of (2.13) zero. By the operation $\frac{\partial}{\partial \sigma(x)}$ on (2.14) we have

$$\frac{\partial K_+^A(\sigma, \sigma_a)}{\partial \sigma(x)} + K_+ A K_+^A(\sigma, \sigma_a) + \int_{\sigma_a}^{\sigma} d\tau' K_+ A' \frac{\partial K_+^A(\sigma, \sigma_a)}{\partial \sigma(x)} = 0.$$

If the expression obtained from (2.14) for K_+ is used for K_+ in this equation, and if we put

$$\frac{\partial K_+^A(\sigma, \sigma_a)}{\partial \sigma(x)} + K_+^A(\sigma, \sigma_a) A K_+^A(\sigma, \sigma_a) \equiv \mathfrak{R}(\sigma(x), \sigma_a),$$

then we can easily see that \mathfrak{R} satisfies

$$\mathfrak{R}(\sigma(x), \sigma_a) + \int_{\sigma_a}^{\sigma} d\tau' K_+ A' \mathfrak{R}(\sigma(x), \sigma_a) = 0. \quad (2.15)$$

On the other hand we see from (2.14) that $\{\dots\dots\}$ part in the third term of (2.13)

*) The condition :

$$K_+^A(\sigma, \sigma_a) - K_+ + \int_{\sigma_a}^{\sigma} d\tau' K_+^A(\sigma, \sigma_a) A' K_+ = 0,$$

that is required for the vanishment of the second term of (2.13) is automatically satisfied by (2.14), when (2.14) has unique solution and the Liouville-Neuman series converges. In the other cases we must impose this condition in addition to (2.14).

in question is just $\mathfrak{R}(\sigma(x), \sigma_a)$. Now, (2.15) is just the homogeneous equation belonging to the integral equation (2.14), and according to the general theory of integral equation¹⁰⁾ (2.15) has no solution except zero, when the coupling constant e has a general value. And thus in this general case (2.14) has a unique solution. Thus we see that (2.14) is necessary and sufficient in this case. If e has a special value, (2.15) has solutions except zero, and in this special case there is a possibility of the existence of the solution of (2.14), but this solution has ambiguous in some extent and does not satisfy $\mathfrak{R}=0$, in general. And even though (2.14) has no solution, there is also a possibility, that the sum of three terms of (2.13) vanishes with an extremely special A . Corresponding to the problems here treated (mainly the scattering problem in which e have a general value), we do not discuss this hard question here and will restrict our discussion to the case, when (2.14) has unique solution. We arrive at the result

$$\left. \begin{aligned} \int_{\sigma_a}^{\sigma_b} d\tau Q &= - \int_{\sigma_a}^{\sigma_b} d\tau \bar{\psi}^- A \left(\psi^- - \int_{\sigma_a}^{\sigma_b} d\tau' K_+^A(\sigma_b, \sigma_a) A' \psi'^- \right) \\ &= - \int_{\sigma_a}^{\sigma_b} d\tau (\bar{\psi}^- - \int_{\sigma_a}^{\sigma_b} d\tau' \bar{\psi}'^- A' K_+^A(\sigma_b, \sigma_a)) A \psi^-. \end{aligned} \right\} \quad (2.16)$$

To determine the R -part we insert (2.16) into (2.10R) and readily obtain

$$R = -\bar{\psi}^+ A \left\{ \psi^- - \int_{\sigma_a}^{\sigma} d\tau' K_+^A(\sigma, \sigma_a) A' \psi'^- \right\} - \left\{ \bar{\psi}^- - \int_{\sigma_a}^{\sigma} d\tau' \bar{\psi}'^- A' K_+^A(\sigma, \sigma_a) \right\} A \psi^+. \quad (2.17)$$

If we, remembering $(\gamma_\mu \partial_\mu + m)K_+ = -i$, apply $(\gamma_\mu \partial_\mu + m)$ to (2.14), we obtain^{*)}

$$\left. \begin{aligned} \{\gamma_\mu (\partial_\mu - ieA_\mu) + m\} K_+^A(\sigma_b, \sigma_a) &= -i \quad \text{for } \sigma_a \leq \sigma \leq \sigma_b \\ \{\gamma_\mu \partial_\mu + m\} K_+^A(\sigma_b, \sigma_a) &= -i \quad \text{for } \sigma_a > \sigma, \text{ or } \sigma > \sigma_b \end{aligned} \right\} \quad (2.18)$$

Now we see that our $K_+^A(\infty, -\infty)$ is identical with the well known Feynman's one-body kernel with interaction, but it does not contain the vacuum polarization effects. Our $K_+^A(\sigma_b, \sigma_a)$ is the solution of the differential equation (2.18) with the boundary condition $K_+^A(\sigma_a, \sigma_a) = K_+$.

Now if we put

$$\psi^-(\sigma_b, \sigma_a) \equiv \psi^- - \int_{\sigma_a}^{\sigma_b} d\tau' K_+^A(\sigma, \sigma_a) A' \psi'^-, \quad (2.19)$$

we see that ψ^- satisfies the following Yang-Feldman type equation

$$\psi^-(\sigma_b, \sigma_a) = \psi^- - \int_{\sigma_a}^{\sigma_b} d\tau' K_+ A' \psi'^-, \quad (2.20)$$

and

$$\{\gamma_\mu (\partial_\mu - ieA_\mu) + m\} \psi^-(\sigma_b, \sigma_a) = 0, \quad (\sigma_a \leq \sigma \leq \sigma_b). \quad (2.21)$$

Similarly we also define $\bar{\psi}^-$.

Since we restrict our discussion to the case when (2.15) has no solution, we can

*) In Feynman's notations our A becomes iA .

write (2.14), (2.19) symbolically as

$$K_+^A(\sigma_b, \sigma_a) = [1 + \int_{\sigma_a}^{\sigma_b} d\tau' K_+ A']^{-1} K_+ = K_+ [1 + \int_{\sigma_a}^{\sigma_b} d\tau' A' K_+]^{-1}. \quad (2.14')$$

$$\phi^-(\sigma_b, \sigma_a) = [1 + \int_{\sigma_a}^{\sigma_b} d\tau' K_+ A']^{-1} \phi^-, \quad \bar{\phi}^-(\sigma_b, \sigma_a) = \bar{\phi}^- [1 + \int_{\sigma_a}^{\sigma_b} d\tau' A' K_+]^{-1}. \quad (2.19')$$

Making use of (2.12) we obtain

$$Q = -\phi^-(\sigma, \sigma_a) A \phi^-(\sigma, \sigma_a). \quad (2.22)$$

If, in the outset, we take the charge symmetrized Hamiltonian^{*)}

$$H_s = -\frac{i}{2} [\bar{\psi}, A\psi], \quad (2.1')$$

we can easily see $Q_s = Q$, and get the following results for R -part.

$$\begin{aligned} R_s &= -\frac{1}{2} (\bar{\psi}^+ A \psi^- - \psi^- A^T \bar{\psi}^+) + \bar{\psi}^+ A \int_{\sigma_a}^{\sigma} d\tau' K_+^A(\sigma, \sigma_a) A' \psi^- \\ &\quad - \frac{1}{2} (\bar{\psi}^- A \psi^+ - \psi^+ A^T \bar{\psi}^-) + \int_{\sigma_a}^{\sigma} d\tau' \bar{\psi}^- A' K_+^A(\sigma, \sigma_a) A \psi^+ \\ &= \phi^-(\sigma, \sigma_a) A^T \bar{\psi}^+ - \bar{\phi}^-(\sigma, \sigma_a) A \psi^+ - \int_{\sigma_a}^{\sigma} d\tau' S \rho(K_+ A K_+^A(\sigma, \sigma_a) A') - \frac{e}{2} S \rho(S^{(1)}(0) \gamma_\mu) A_\mu, \end{aligned}$$

and the last term vanishes as was shown by Schwinger^{v)} from the symmetry property of $S^{(1)}$.

At the last step we must determine P -part from (2.9). But as we know that P and Q -part must be symmetrical^{**) ,} so we get on account of (2.22)

$$\begin{aligned} P &= E \int_{\sigma_a}^{\sigma} d\tau' R' (-\bar{\psi}^+ A \psi^+) E \int_{\sigma_a}^{\sigma} d\tau' R' = -\bar{\psi}^+(\sigma, \sigma_a) A \psi^+(\sigma, \sigma_a), \\ \therefore \psi^+ E \int_{\sigma_a}^{\sigma} d\tau' R' &= E \int_{\sigma_a}^{\sigma} d\tau' R' \psi^+(\sigma, \sigma_a). \end{aligned} \quad (2.23)$$

And also from the symmetry of Q -and P -part we get

$$E \int_{\sigma_a}^{\sigma_b} d\tau' R' \phi^- = \phi^-(\sigma_b, \sigma) E \int_{\sigma_a}^{\sigma_b} d\tau' R' \quad (2.24)$$

These equations are useful in the next section.

And thus we attain to the demanded result, that is, the complete disentangling of the U -operator with respect to ϕ and $\bar{\phi}$.

$$U(\sigma_b, \sigma_a) = e^{-\int_{\sigma_a}^{\sigma_b} d\tau \bar{\phi}^- A D(\sigma_b, \sigma_a) \phi^-} E^{+\int_{\sigma_a}^{\sigma_b} d\tau \phi^- D(\sigma, \sigma_a) A^T \bar{\phi}^+}$$

*) We designate the charge symmetrized quantities with subscript s .

**) Or repeat the above analysis by interchanging the role of Q -and P -part. (See also the first equation of (A 2) in the Appendix).

$$\begin{aligned} & \times E^{-\int_{\sigma_a}^{\sigma_b} d\tau} \bar{\psi}^- D(\sigma, \sigma_a) A \psi^+ e^{-\int_{\sigma_a}^{\sigma_b} d\tau} \bar{\psi}^+ A D(\sigma_b, \sigma_a) \psi^+ \\ & \times e^{-\int_{\sigma_a}^{\sigma_b} d\tau \int_{\sigma_a}^{\sigma} d\tau' S p(K_+ A D(\sigma, \sigma_a) K_+ A')} \end{aligned} \quad (2.25)$$

where

$$D(\sigma, \sigma_a) = [1 + \int_{\sigma_a}^{\sigma} d\tau' K_+ A']^{-1} \quad (2.26)$$

Of course in these equations ψ and $\bar{\psi}$ operate in positional order, but A operate in time-order.

We can treat for example the meson-photon system in β -formalism in an completely analogous manner, and obtain the same form as (2.25) save for a sign change in the last term, which arises from the different statistics associated with the integral spin field. In this case, of course, K_+ means the propagation function of mesons in β -formalism. As we will show in the next section, the last term in (2.25) just represents the vacuum-vacuum amplitude C_v . Thus we obtain the Feynman's results, that is, $C_v = e^{iL}$ for bosons and $C_v = e^{-iL}$ for fermions, without any physical consideration.

§ 3. Relations between the U -operator and the kernels

i) *Vacuum-vacuum amplitude.* If we take up the vacuum expectation value of the spinor field (2.25), and therein make use of the defining properties of the spinor vacuum :

$$\psi^+|_s = \bar{\psi}^+|_s = 0 \quad \text{and} \quad {}_s\langle \psi^- = {}_s\langle \bar{\psi} = 0, \quad (3.1)$$

we obtain

$$\langle U(\sigma_b, \sigma_a) \rangle_s \equiv C_v(\sigma_b, \sigma_a) = \exp \left\{ - \int_{\sigma_a}^{\sigma_b} d\tau \int_{\sigma_a}^{\sigma} d\tau' S p(K_+ A D(\sigma, \sigma_a) K_+ A') \right\}, \quad (3.2)$$

since in (2.25) all the spinor operators now vanish. As we can easily see

$$\begin{aligned} & \int_{\sigma_a}^{\sigma_b} d\tau \int_{\sigma_a}^{\sigma} d\tau_1 \int_{\sigma_a}^{\sigma} d\tau_2 \cdots \int_{\sigma_a}^{\sigma} d\tau_{n-1} S p(K_+ A K_+ A_1 K_+ A_2 \cdots K_+ A_{n-1}) \\ & = \frac{1}{n!} \int_{\sigma_a}^{\sigma_b} d\tau \int_{\sigma_a}^{\sigma_b} d\tau_1 \cdots \int_{\sigma_a}^{\sigma_b} d\tau_{n-1} S p(K_+ A K_+ A_1 \cdots K_+ A_{n-1}), \end{aligned}$$

if we expand (3.2) with respect to A , we get

$$C_v(\sigma_b, \sigma_a) = \exp \left\{ \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n!} \int_{\sigma_a}^{\sigma_b} d\tau_1 \cdots \int_{\sigma_a}^{\sigma_b} d\tau_n S p(K_+ A_1 K_+ A_2 \cdots K_+ A_n) \right\}, \quad (3.3)$$

and we recognize that it agrees completely with the Feynman's result⁽¹⁾, when we make $\sigma_a \rightarrow -\infty$, $\sigma_b \rightarrow +\infty$ (cf. the footnote of page 454.) And we can write symbolically,

$$C_v(\sigma_b, \sigma_a) = \exp \left\{ S p \log \left(1 + \int_{\sigma_a}^{\sigma_b} d\tau' K_+ A' \right) \right\} \quad (3.4)$$

since $\int_{\sigma_a}^{\sigma_b} d\tau' S\dot{p}(K_+ A') = 0$. This expression has already been obtained by M. Neuman¹⁾.

ii) *One-body kernel and two-body kernel.* To simplify the notations we write (2.25) as

$$U(\sigma_1, \sigma_2) = e^{Q(\sigma_1, \sigma_2)} E^{R(\sigma_1, \sigma_2)} e^{P(\sigma_1, \sigma_2)} C_v(\sigma_1, \sigma_2), \quad (2.25')$$

and then, making use of the equations of previous section we obtain*)

$$U(\sigma_1, \sigma_2) \psi(2) = e^{Q(\sigma_1, \sigma_2)} E^{R(\sigma_1, \sigma_2)} (\psi^-(2) - \int_{\sigma_2}^{\sigma_1} d\tau_{-} K_+(2, 3) l(3) \psi^+(3; \sigma_1, \sigma_2)) e^{P(\sigma_1, \sigma_2)} C_v(\sigma_1, \sigma_2) + U(\sigma_1, \sigma_2) \psi^+(2),$$

or from (2.24)

$$\begin{aligned} &= e^{Q(\sigma_1, \sigma_2)} \psi(2; \sigma_1, \sigma_2) E^{R(\sigma_1, \sigma_2)} e^{P(\sigma_1, \sigma_2)} C_v(\sigma_1, \sigma_2) + U(\sigma_1, \sigma_2) \psi^+(2; \sigma_1, \sigma_2) \\ &= \psi^-(2; \sigma_1, \sigma_2) U(\sigma_1, \sigma_2) + U(\sigma_1, \sigma_2) \psi^+(2; \sigma_1, \sigma_2). \end{aligned} \quad (3.5)$$

Similarly

$$\psi(2) U(\sigma_2, \sigma_a) = \psi^-(2; \sigma_2, \sigma_a) U(\sigma_2, \sigma_a) + U(\sigma_2, \sigma_a) \psi^+(2; \sigma_2, \sigma_a).$$

In these equations we see that Q -, P -, R -parts represent pair-creation, pair-annihilation, and radiative correction respectively, or in Feynman's word $P(Q)$ -part represents the potential repelling fermions to the past (future) direction.

(Fig. 1.)

From (3.5) we obtain

$$\begin{aligned} {}_s \langle \bar{\psi}(1) U(\sigma_1, \sigma_2) \psi(2) \rangle_s &= \{ \bar{\psi}^+(1), \psi^-(2; \sigma_1, \sigma_2) \}_s \langle U(\sigma_1, \sigma_2) \rangle_s \\ &= -C_v(\sigma_1, \sigma_2) K_+^A(2, 1; \sigma_1, \sigma_2). \end{aligned}$$

More generally

$$U(\sigma_1, \sigma_2) \psi(2) U(\sigma_2, \sigma_a) \rangle_s = U(\sigma_1, \sigma_2) \psi^-(2; \sigma_2, \sigma_a) U(\sigma_2, \sigma_a) \rangle_s,$$

and we obtain from (2.14')

$$K_+^A(\sigma_b, \sigma_a) = [(K_+^A(\sigma_b, \sigma_1))^{-1} + (K_+^A(\sigma_1, \sigma_a))^{-1} - K_+^{-1}]^{-1}. \quad (3.6)$$

The arguments (σ_b, σ_1) etc. in this equation can be considered as showing a domain wherein coupling exists. After some manipulation we obtain

$$\begin{aligned} U(\sigma_1, \sigma_2) \psi(2) U(\sigma_2, \sigma_a) \rangle_s &= [1 + \int_{\sigma_2}^{\sigma_1} d\tau_{-} K_+ A]^{-1} [1 + \int_{\sigma_a}^{\sigma_2} d\tau_{-} K_+ A]^{-1} \left[1 - \left\{ + \int_{\sigma_2}^{\sigma_1} d\tau_{-} K_+ A \right\}^{-1} \right. \\ &\quad \times \left. \left(- \int_{\sigma_2}^{\sigma_1} d\tau_{-} K_+ A \right) \left\{ 1 + \int_{\sigma_a}^{\sigma_2} d\tau_{-} K_+ A \right\}^{-1} \left(- \int_{\sigma_a}^{\sigma_2} d\tau_{-} K_+ A \right) \right]^{-1} \psi^-(2) U(\sigma_1, \sigma_2) \rangle_s \\ &= \left[1 + \int_{\sigma_a}^{\sigma_1} d\tau_{-} K_+ A \right]^{-1} \psi^-(2) U(\sigma_1, \sigma_a) \rangle_s \end{aligned}$$

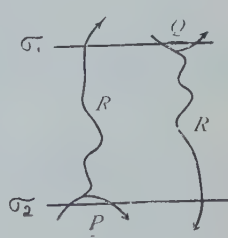


Fig. 1

*) σ_1, σ_2 mean the space-like surfaces passing x_1, x_2 respectively.

and finally

$${}_s\langle U(\sigma_b, \sigma_1)\bar{\psi}(1)U(\sigma_1, \sigma_2)\psi(2)U(\sigma_2, \sigma_a)\rangle_s \in (2, 1) = +C_v(\sigma_b, \sigma_a)K_+^A(2, 1; \sigma_b, \sigma_a). \quad (3.7)$$

Similarly we obtain

$${}_s\langle U(\sigma_b, \sigma_2)\psi(2)U(\sigma_2, \sigma_1)\bar{\psi}(1)U(\sigma_1, \sigma_a)\rangle_s \in (2, 1) = +C_v(\sigma_b, \sigma_a)K_+^A(2, 1; \sigma_b, \sigma_a). \quad (3.7')$$

If A represents the external field, C_v represents only the normalization constant, and in this case if we define the one-body kernel as^{*)}

$$K_+^A(1, 2) = {}_s\langle P(U(\infty, -\infty)\bar{\psi}(2)\psi(1))\rangle_s \in (1, 2) / {}_s\langle U(\infty, -\infty)\rangle_s \quad (3.8)$$

we can derive the Feynman's equation

$$K_+^A(1, 2) = K_+(1, 2) - \int_{-\infty}^{\infty} d\tau_{11} K_+(1, 3)A(3)K_+^A(3, 2),$$

from the second quantized theory, only by the use of the commutation relations and mathematical manipulation. And thus we can prove operationally the equivalence of Feynman's theory and the second quantized theory.

In an analogous manner for two-body kernel we obtain

$$\begin{aligned} {}_s\langle P(U(\infty, -\infty)\bar{\psi}(1)\psi(2)\bar{\psi}(3)\psi(4))\rangle_s \in (1, 2, 3, 4) \\ = C_v(K_+^A(2, 1)K_+^A(4, 3) - K_+^A(2, 3)K_+^A(4, 1)), \end{aligned} \quad (3.9)$$

but the actual derivation of these equations needs some clumsy calculations. We shall give in § 5 more simple and compact derivation with the aid of the external spinors.

§ 4. A closed form of the S-matrix

Instead of (2.1') we take the interaction Hamiltonian as

$$H = -\frac{i}{2} [\bar{\psi} + \bar{\phi}, A(\psi + \phi)], \quad (4.1)$$

ϕ and $\bar{\phi}$ are the external spinors, which are analogues of the Schwinger's¹⁾ η and $\bar{\eta}$, and represent real fermions in the initial and final states. ϕ and $\bar{\phi}$ are to describe the separate real fermions, and so they all anticommute with each other as well as ψ and $\bar{\psi}$.^{**) (3.9)}

The calculations proceed in the same line as given in the previous section, (The details are also given in Appendix, which are simpler than those of § 2) and we obtain^{***)}

$$\begin{aligned} S = U(\infty, -\infty) = \exp \{ -(\bar{\psi}^- AD\psi^- + \bar{\psi}^- AD\phi + \bar{\phi} AD\psi^-) \} \exp \{ +\phi^- D_\sigma A^T \bar{\psi}^+ \} \\ \times \exp \{ -\bar{\psi}^- AD_\sigma \phi^+ \} \exp \{ -(\bar{\psi}^+ AD\phi^+ + \bar{\psi}^+ AD\phi + \bar{\phi} AD\phi^+) \} \\ \times \exp \{ S\rho \log(1 + \int_{-\infty}^{\infty} d\tau' K_+ A') \} \exp \{ -\bar{\phi} AD\phi \} \end{aligned} \quad (4.2)$$

*) We abbreviate $K_+^A(1, 2; \infty, -\infty)$ as $K_+^A(1, 2)$ and $C_v(\infty, -\infty)$ as C_v .

**) Incidentally our $\phi, \bar{\phi}$ correspond to Dyson's free operators.

***) In this section we consider S-matrix only instead of U-matrix to avoid the unnecessarily complicated notation.

where

$$\left. \begin{aligned} \bar{\psi}^- AD\psi^- &= \int_{-\infty}^{\infty} d\tau \bar{\psi}^- A \left[1 + \int_{-\infty}^{\infty} d\tau' K_+ A' \right]^{-1} \psi^- \\ \bar{\psi}^- AD_0 \psi^+ &= \int_{-\infty}^{\infty} d\tau \bar{\psi}^- A \left[1 + \int_{-\infty}^{\infty} d\tau' K_+ A' \right]^{-1} \psi^+, \end{aligned} \right\} \text{etc.} \quad (4.3)$$

Taking the vacuum expectation values of the virtual spinors ψ and $\bar{\psi}$ in this equation (4.2) we obtain

$$\left. \begin{aligned} \langle S \rangle_s &= C_r \cdot C_r \\ C_r &= \exp \{ S \bar{\psi} \log(1 + K_+ A) \} \\ C_r &= \exp \{ -\bar{\psi} A [1 + K_+ A]^{-1} \psi \} \end{aligned} \right\} \quad (4.4)$$

Here we omit the integral sign $\int_{-\infty}^{\infty} d\tau$ to simplify the notations.

C_r and C_r describe the closed loops and the real transitions respectively. Actually in our theory 0-0 expectation value of the S-matrix with respect to the virtual spinor fields is the only quantity of physical interest. If we expand C_r in power of $\bar{\psi} \psi$, it is no other than the expansion of the S-matrix in the number of real fermions, and it is almost evident that the terms containing $(\bar{\psi} \psi)^n$ represent the n - n expectation value of the usual S-matrix. (cf. also next section.)

And thus we obtain a closed form of the S matrix in the situation, when the arbitrary numbers of real fermions are present, or at least in such a situation as the case of external electromagnetic field. And in this case C_r and C_r are disconnected, and C_r is the only normalization constant.

If A_μ represent the quantized field, as we have emphasized it above, the expression (4.4) is the operator equation of A_μ , not in the positional order but in the time order. And now we use Nambu's⁽⁶⁾ procedure. We review it here. Let $F(A)$ be any functional of operator A_μ , not in the well order, (i.e. A_μ^- on the left and A_μ^+ on the right) but in the time order. Remembering the commutation relations: $[A_\mu^+(x), A_\nu^-(x')] = i\delta_{\mu\nu} D^{(+)}(x-x')$ and $[A_\nu^-(x), A_\mu^+(x')] = i\delta_{\mu\nu} D^{(-)}(x-x')$, when we bring $F(A)$ in the well order, and denote the well ordered $F(A)$ by $\tilde{F}(\tilde{A})$, we obtain

$$\begin{aligned} \tilde{A}_\mu(x) &= A_\mu^+(x) + i \int_{-\infty}^{\infty} d\tau' D^{(+)}(x-x') \frac{\delta}{\delta A_\mu^-(x')} + A_\mu^-(x) - i \int_{-\infty}^{\infty} d\tau' D^{(-)}(x-x') \frac{\delta}{\delta A_\mu^+(x')} \\ &\approx A_\mu(x) + \frac{1}{2} \int_{-\infty}^{\infty} d\tau' D_F(x-x') \frac{\delta}{\delta A_\mu(x')} \quad *) \end{aligned} \quad (4.5)$$

where

$$D_F(x) = i\epsilon(x)[D(x) - i\epsilon(x)D^{(1)}(x)] = \begin{cases} 2iD^{(+)}(x) & \text{if } x_0 > 0, \\ -2iD^{(-)}(x) & \text{if } x_0 < 0, \end{cases}$$

*) means that the equality holds as far as A_μ^+ and A_μ^- appear in the combination $A_\mu^+ + A_\mu^-$.

and also $\tilde{A}_\mu(x)$ can be written

$$\tilde{A}_\mu(x) = V^{-1} A_\mu(x) V$$

$$V = \exp \left[-\frac{1}{4} \int d\tau' \int \frac{\partial}{\partial A_\mu(x)} D_F(x-x') \frac{\partial}{\partial A_\mu(x')} \right] = \exp \left[-\frac{1}{4} \frac{\partial}{\partial A_\mu} D_F \frac{\partial}{\partial A_\mu} \right]. \quad (4.6)$$

Thus we can write (4.4) as

$$\begin{aligned} {}_s\langle S \rangle_s &= \exp \{ S \rho \log(1 + K_+ \tilde{A}) \} \exp \{ -\tilde{\phi} \tilde{A} [1 + K_+ \tilde{A}]^{-1} \phi \} \\ \text{or} \\ &= \exp \left\{ \frac{1}{4} \frac{\partial}{\partial A_\mu} D_F \frac{\partial}{\partial A_\mu} \right\} \exp \{ S \rho \log(1 + K_+ A) \} \exp \left\{ -\phi A [1 + K_+ A]^{-1} \phi \right\}, \end{aligned} \quad (4.7)$$

In the latter equation we can treat A_μ as a true c-number, but this does not make any important advance above (4.4) in practice.

Thus, if A_μ 's in C_n and A_μ 's in C_r are contracted, (i.e. $\frac{\partial}{\partial A_\mu}$ in C_n operate on A_μ 's in C_r in (4.7) and vice versa) it represents the effect of vacuum polarization, and if A_μ 's in the same $\bar{\phi} A D \phi$ of C_r are contracted, it represents the self interaction of real fermions, and if A_μ 's in the two $\bar{\phi} A D \phi$ of C_r are contracted, it represents the mutual interaction of real fermions.

To prove that our procedure is equivalent to the usual one^{*)}, we take up the case of quantized radiation field. In the usual theory, as we have seen above, if we replace the operators $A_\mu(x)$ by $A_\mu(x) + \frac{1}{2} \int d\tau' D_F(x-x') \frac{\partial}{\partial A_\mu(x')}$, we can treat the operators A_μ as if it were a c-number, and carrying out the differentiation $\frac{\partial}{\partial A_\mu}$ the remaining A_μ represent the real photons. On the other hand we use in our treatment separate variables to describe the real and the virtual particles. The variables A_μ (or ϕ and $\bar{\phi}$ above used), which refer to the real particles or to the given external potentials, commute (or anticommute) with each other as well as A_μ^i ^{**)} which refer to the virtual particles, when A_μ obey the Bose statistics (or Fermi statistics). Remembering that we take always the vacuum expectation value of the S-matrix with respect to the virtual particles, our prescriptions are summarized as follows: Replace A_μ by $A_\mu^i + A_\mu^e$, where A_μ^e mean the external (c-number) potential and take the vacuum expectation value of the S-matrix with respect to the virtual photon A_μ^i .

Since A_μ^i can be treated as A_μ in the usual theory, we can treat $A_\mu^i + A_\mu^e$ as if it were a c-number after replacing them by $A_\mu^i + A_\mu^e + \frac{1}{2} \int D_F \frac{\partial}{\partial A_\mu^i}$; and if we follow our prescription, we must put all the remaining A_μ^i zero after differentiation $\frac{\partial}{\partial A_\mu^i}$. Now since

*) The author is indebted to Mr. H. Tanaka for pointing out this proof.

**) The superindex e and i in A_μ^e and A_μ^i are the abbreviations of external and internal.

$A_\mu^i + A_\mu^e$ always appear in the S-matrix in this combination, $A_\mu^i + A_\mu^e + \frac{1}{2} \int D_F \frac{\delta}{\delta A_\mu^i}$ is equivalent to $A_\mu^i + A_\mu^e + \frac{1}{2} \int D_F \frac{\delta}{\delta A_\mu^i}$, and so, according to our prescription that we take the vacuum expectation value with respect to A_μ^i , we can put $A_\mu^i = 0$ from the beginning. (Because there is no differentiation $\frac{\delta}{\delta A_\mu^i}$.) After we have treated A_μ^i and A_μ^e in this manner A_μ^e in our theory play the role of A_μ in the usual theory, and thus we can prove their equivalence.

§ 5. Diffusion equations, and the relations of our theory to the Schwinger's new theory of Green function

In the previous section we have obtained a closed form of the S-matrix. Using it, we can derive the so-called diffusion equation for the Dirac field very simply, though it was introduced by Fukuda and Tomonaga⁵⁾ after very elaborate calculations. The diffusion equation for the radiation field was introduced by Feynman¹²⁾ and also was derived very simply by the same author by means of the operator calculus²⁾. We here take the operational method.

After the so-called Fukuda-Tomonaga transformation: (it is only a scale-change of the field variables.)

$$e \rightarrow b\sqrt{a}, \quad A_\mu \rightarrow \frac{1}{\sqrt{a}} A_\mu, \quad \psi \rightarrow \frac{1}{\sqrt{b}} \psi, \quad \phi \rightarrow \frac{1}{\sqrt{b}} \phi \text{ etc.}$$

the fundamental equation of our theory (4.6) now becomes*)

$$S = C_r C_r = \exp \{ S \hat{p} \log(1 + bK_+ A) \} \exp \{ -\bar{\phi} A [1 + bK_+ A]^{-1} \phi \} \quad (5.1)$$

Now

$$K_+ a \frac{\partial^2}{\partial \phi \partial \phi} \langle S \rangle_s = S \hat{p} (K_+ A [1 + bK_+ A]^{-1})_s \langle S \rangle_s + (\bar{\phi} A [1 + bK_+ A]^{-1} K_+ A [1 + bK_+ A]^{-1} \phi)_s \langle S \rangle_s \quad (5.2)$$

In this equation the first term represents the construction of one closed loop, which has been produced by the operation $\frac{\partial^2}{\partial \phi \partial \phi}$ on the same real electron line,**) (Fig. 2a.) The second term of the equation (5.2) represents the construction of one real electron line, which has been produced by the operation $\frac{\partial^2}{\partial \phi \partial \phi}$ on the separate real electron lines (Fig. 2b.)¹³⁾

*) We put $A = \gamma_\mu A_\mu$.

**) Real electron line means the electron line which has $\bar{\phi}$ and ϕ on both edges of it.

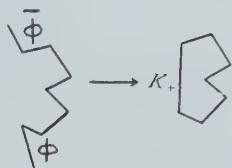


Fig. 2a.

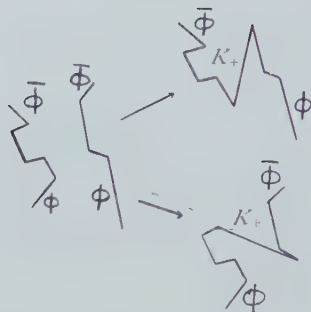


Fig. 2b.

On the other hand

$$\left. \begin{aligned} \frac{dC_v}{db} &= S\hat{p}([1 + bK_+ \mathbf{A}]^{-1} K_+ \mathbf{A}) C_v \\ \frac{dC_r}{db} &= (\bar{\phi} \mathbf{A} [1 + bK_+ \mathbf{A}]^{-1} K_+ \mathbf{A} [1 + bK_+ \mathbf{A}]^{-1} \phi) C_r \end{aligned} \right\} \quad (5.3)$$

Collecting (5.2) and (5.3) we obtain

$$\frac{d_s \langle S \rangle_s}{db} = K_{+\alpha\beta} \frac{\partial_s^2 \langle S \rangle_s}{\partial \phi_\beta \partial \phi_\alpha}$$

or

$$\frac{d_s \langle S \rangle_s}{db} = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' K_{+\alpha\beta}(x, x') \frac{\partial_s^2 \langle S \rangle_s}{\partial \phi_\beta(x') \partial \phi_\alpha(x)} \quad (5.4)$$

This is just the same diffusion equation of Fukuda-Tomonaga.

If we put

$$\frac{i\dot{c}}{2} [\bar{\psi}, \gamma_\mu \psi] = j_\mu, \quad i\dot{c} \bar{\psi} \mathbf{A} = \eta, \quad i\dot{c} \mathbf{A} \psi = \gamma, \quad i\dot{c} \gamma_\mu \psi = j_\mu, \quad (5.5)$$

our interaction Hamiltonian (4.1) becomes

$$H = -(j_\mu A_\mu + \bar{\eta} \psi + \bar{\psi} \eta + J_\mu A_\mu) \quad (5.6)$$

η , $\bar{\eta}$ and J_μ are just the Schwinger's prescribed source, and our theory agrees completely with that of Schwinger¹¹. In this notation (4.6) becomes

$${}_s \langle S \rangle_s = \exp \{ S\hat{p} \log(1 + eK_+ \mathbf{A}) \} e^{-\bar{\eta} K_+^A \eta} e^{iJ_\mu A_\mu} \quad (5.7)$$

or

$$= C_v e^{-\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \bar{\eta}(x) K_+^A(x, x') \eta(x')} e^{i \int_{-\infty}^{\infty} d\tau J_\mu(x) A_\mu(x)}$$

It is interesting to compare this equation with

$${}_p \langle S \rangle_p = e^{-\frac{1}{4} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' j_\mu(x) D_F(x, x') j_\mu(x')} e^{i \int_{-\infty}^{\infty} d\tau j_\mu(x) B_\mu(x)} \quad (5.8)$$

The bracket ${}_p\langle \rangle_p$ denotes here vacuum expectation values with respect to the virtual photon field, and B_μ represent the external potential. (γ_μ also represent real photons in our treatment.) Obviously

$$\begin{aligned} \frac{\partial_s \langle S \rangle_s}{\partial \bar{\eta}(x')} &= {}_s\langle \frac{\partial S}{\partial \bar{\eta}(x')} \rangle_s = -{}_s\langle P(S\bar{\psi}(x')) \rangle_s \\ \frac{\partial_s^2 \langle S \rangle_s}{\partial \bar{\eta}(x) \partial \bar{\eta}(x')} &= +{}_s\langle P(S\psi(x)\bar{\psi}(x')) \rangle_s \epsilon(x, x'), \end{aligned} \quad (5.9)$$

and on account of (5.7) we obtain

$$\left[\frac{\partial_s^2 \langle S \rangle_s}{\partial \bar{\eta}(x) \partial \bar{\eta}(x')} \right]_{\eta=\bar{\eta}=0} = -C_v K_+^A(x, x') e^{i \int d\tau J_\mu A_\mu}$$

and thus

$${}_s\langle P(S\psi(x)\bar{\psi}(x')) \rangle_s \epsilon(x, x') \Big|_{\eta=\bar{\eta}=0} = -C_v K_+^A(x, x') e^{i \int d\tau J_\mu A_\mu} \quad (5.10)$$

This is just (3.7), when the external source J_μ exists. From (2.19) and (5.10), if $\eta=\bar{\eta}=0$, $J_\mu \neq 0$, and if we put

$${}_s\langle P(S\psi(x)\bar{\psi}(x')) \rangle_s \epsilon(x, x') = K(x, x'),$$

we obtain

$$\{ \gamma_\mu (\partial_\mu - i e A_\mu) + m \} K(x, x') = i {}_s\langle S \rangle_s \delta(x, x').$$

where

$${}_s\langle S \rangle_s = \exp \left\{ S \bar{p} \log (1 + K_+ A) \right\} e^{i \int d\tau J_\mu A_\mu}$$

This is the operator equation of A_μ with time order, and so we obtain^{*)}

$$\left\{ \gamma_\mu (\partial_\mu - e \frac{\partial}{\partial J_\mu}) + m \right\} {}_p\langle K(x, x') \rangle_p = i {}_0\langle S \rangle_0 \delta(x, x')$$

or

$$\left\{ \gamma_\mu (\partial_\mu - e \frac{\partial}{\partial J_\mu}) + m \right\} \frac{{}_p\langle K(x, x') \rangle_p}{{}_0\langle S \rangle_0} = i \delta(x, x') + e \gamma_\mu \frac{{}_p\langle K(x, x') \rangle_p}{{}_0\langle S \rangle_0} \frac{\frac{\partial}{{\partial J_\mu}} {}_0\langle S \rangle_0}{{}_0\langle S \rangle_0}$$

The factor ${}_0\langle S \rangle_0$ in the denominator is needed for the normalization, i.e. to make the vacuum-vacuum (here vacuum means $\rangle_s \rangle_p$ i.e. \rangle_0) amplitude to be unit. This means also that we can neglect the disconnected Dyson-diagram. And finally if we put

$$G_+(x, x') = {}_0\langle P(S\psi(x)\bar{\psi}(x')) \rangle_0 \epsilon(x, x') / {}_0\langle S \rangle_0,$$

we get the following equation.

*) We abbreviate $\langle \langle \rangle \rangle_p$ as $\langle \rangle_p$.

$$\left\{ \gamma_\mu \left(\partial_\mu - i e \frac{\langle P(A_\mu(x) S) \rangle_0}{\langle S \rangle_0} - e \frac{\delta}{\delta j_\mu} \right) + m \right\} G_+(x, x') = i \delta(x, x') \quad (5.11)$$

This is just the equation of Green function given by Schwinger¹⁾, and contrally to Feynman's one, $G_+(x, x')$ contains the effects of vacuum polarization.

In the same way for the two-body kernel, we have, for a typical order of the times $(\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \sigma_4)$,

$$\begin{aligned} & \left. {}_s \langle P(S_{\bar{\eta}}(1) \psi(2) \bar{\psi}(3) \psi(4)) \rangle \right|_{\eta=\bar{\eta}=0} = \left. \frac{\partial^4 \langle S \rangle_s}{\partial \eta(1) \partial \bar{\eta}(2) \partial \eta(3) \partial \bar{\eta}(4)} \right|_{\eta=\bar{\eta}=0} \\ &= \frac{\partial^2}{\partial \eta(1) \partial \bar{\eta}(2)} \left[(K_+^A(4, 3) - \int d\tau K_+^A(4, x) \eta(x) \int d\tau' \bar{\eta}(x') K_+(x', 3)) {}_s \langle S \rangle_s \right]_{\eta=\bar{\eta}=0} \\ &= C_v(K_+^A(4, 3) K_+^A(2, 1) - K_+^A(4, 1) K_+^A(2, 3)). \end{aligned} \quad (5.12)$$

This is just (3.9), and thus we can derive the results of § 3 in an compact way.

§ 6. Applications—Multiple production of bosons and fermions

As an illustration of our operational calculus, we give here the formula for multiple production of bosons and fermions. Although this formula has already been obtained in boson case by Glauber³⁾, we reproduce his results by means of our operational calculus. In his calculation some special considerations were needed for vacuum fluctuation. In our method, since we rearrange the quantized operators in well order, we can separate the vacuum fluctuation from the beginning and extract the real emission amplitude directly and attain the final result more easily.

As it has been shown by Feynman²⁾

$$S = E^i \{ d\tau j_\mu A_\mu = e^i \{ d\tau j_\mu A_\mu^- e^i \{ d\tau j_\mu A_\mu^+ S_{00} \} \} \quad (6.1)$$

where $S_{00} = e^{-\frac{1}{4} \int \int d\tau' d\tau' j_\mu D_F j_\mu'}$ and now $\Psi(\infty) = S\Psi(-\infty)$. Thus if $\Psi(-\infty) = \rangle_p$ and $\Psi(\infty) = \Psi_n$, where Ψ_n is the n -photon state, we obtain

$$\Psi_n = \frac{1}{n!} (i \int d\tau j_\mu A_\mu^-) {}_p \langle S_{00} \rangle$$

$$\therefore \omega_n = (\Psi_n, \Psi_n) = \frac{1}{(n!)^2} |S_{00}|^2 {}_p \langle (\int d\tau j_\mu A_\mu^+)^n (\int d\tau j_\mu A_\mu^-)^n \rangle_p$$

We may, however, employ the easily established theorem

$$g(B)f(A) = \sum_{n=0} \frac{([B, A])^n}{n!} \frac{\partial^n f(A)}{\partial A^n} = \frac{\partial^n g(B)}{\partial B^n}, \quad (6.2)$$

for operators A and B that commute with their commutator $[B, A]$. Thus

$$\begin{aligned}\omega_n &= \frac{1}{(n!)^2} |S_{00}|^2 n! \left(\int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 [A_{\mu}^+(1), A_{\nu}^-(2)] j_{\mu}(1) j_{\nu}(2) \right)^n \\ &= \frac{1}{n!} |S_{00}|^2 \left(\frac{1}{2} \iint d\tau_1 d\tau_2 j_{\mu}(1) D^{(1)}(1, 2) j_{\mu}(2) \right)^n \equiv \frac{1}{n!} |S_{00}|^2 W^n\end{aligned}$$

where $W = \frac{1}{2} \iint d\tau d\tau' j_{\mu} D^{(1)} j_{\mu}'$. On the other hand we easily see: $|S_{00}|^2 = e^{-W}$.

Thus we obtain the Poisson distribution for ω_n ,

$$\omega_n = \frac{W^n}{n!} e^{-W} \quad (6.3)$$

Similarly in the fermion case, the probability ω_n for the production of n -pairs from spinor vacuum is given by

$$\omega_n = \frac{1}{(n!)^2} |C_v|^2 {}_s \langle P^n Q^n \rangle_s = \frac{1}{n!} W_{2n} |C_v|^2 \quad (6.4)$$

where $Q = -\bar{\psi}^+ A D \psi^-$, $P = -\bar{\psi}^+ A D \psi^+$ in (4.3), and W_{2n} are given by

$$W_2 = G_2, \quad W_4 = G_4 + G_2^2, \quad W_6 = 2G_6 + 3G_4 G_2 + G_2^3, \dots \quad (6.5)$$

where

$$G_{2i} = - \int_{-\infty}^{\infty} d\tau_1 \cdots \int_{-\infty}^{\infty} d\tau_{2i} S p(A(1) S_-^A(1, 2) A(2) S_+^A(2, 3) A(3) \cdots A(2i) S_+^A(2i, 1)), \quad (6.6)$$

and

$$\begin{aligned}S_-^A(1, 2) &= S_-(1, 2) - \int_{-\infty}^{\infty} d\tau_3 K_+(1, 3) A(3) S_-^A(3, 2), \\ S_+^A(1, 2) &= S_+(1, 2) - \int_{-\infty}^{\infty} d\tau_3 K_+(1, 3) A(3) S_+^A(3, 2).\end{aligned}$$

I have performed the calculations up to W_{18} ; and though our method is more efficient than the usual one, but as the expression is so cumbersome and moreover it has no practical interest, we shall omit them here.

In the above arguments the systems in quantum electrodynamics are mainly treated, but perhaps there will be no difficulties in handling with the case of nucleon-meson systems, especially with direct coupling cases.

In this paper, we scarcely consider photon interaction, and the applications of our theory to the related problems are in progress, especially the renormalization problems will be treated in a future investigation.

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valuable discussions.

Appendix ; Derivation of (4.2)

In § 2 we have shown an integral equation method, but here we show a sketch of the differential equation method. Interaction Hamiltonian is (4.1) and we now put

$$\begin{aligned}
 U(\sigma_b, 0) &= \text{Exp} \left\{ - \int_0^{\sigma_b} d\tau (\bar{\psi} A \psi + \bar{\psi} A \phi + \dot{\bar{\psi}} A \psi + \dot{\bar{\psi}} A \phi) \right\} \\
 &\equiv e^{Q(\sigma_b)} E^{R(\sigma_b)} e^{P(\sigma_b)} e^{S(\sigma_b)} \\
 &= \text{exp} \int_0^{\sigma_b} d\tau \left\{ \frac{\partial Q}{\partial \sigma(x)} + e^{Q(\sigma)} \frac{\partial R}{\partial \sigma(x)} e^{-Q(\sigma)} \right. \\
 &\quad \left. + e^{Q(\sigma)} \left(E^{R(\sigma)} \frac{\partial P}{\partial \sigma(x)} E^{R(\sigma)} \right) e^{-Q(\sigma)} + \frac{\partial S}{\partial \sigma(x)} \right\} \quad (\text{A.1}) \\
 &= e^{(\bar{\psi}^- f(\sigma_b) \psi^- + \bar{\psi}^- F(\sigma_b) + \bar{F}(\sigma_b) \psi^-)} e^{(\bar{\psi}^+ g(\sigma_b) \psi^- + \bar{\psi}^- g(\sigma_b) \psi^+)} \\
 &\quad \times e^{(\bar{\psi}^+ f(\sigma_b) \psi^+ + \bar{\psi}^+ F(\sigma_b) + \bar{F}(\sigma_b) \psi^+)} e^{\bar{\psi} S(\sigma_b) \phi}
 \end{aligned}$$

Following similar arguments as in § 2 we obtain the relations corresponding to (2.9) and (2.10),

$$\begin{aligned}
 e^{-g(\sigma)K_+} \frac{\partial f(\sigma)}{\partial \sigma(x)} e^{-K_+g(\sigma)} &= -A, \\
 \frac{\partial g(\sigma)}{\partial \sigma(x)} + f(\sigma) K_+ A &= \frac{\partial g(\sigma)}{\partial \sigma(x)} + A K_+ f(\sigma) = -A, \\
 \frac{\partial f(\sigma)}{\partial \sigma(x)} - (f(\sigma) K_+ \frac{\partial g(\sigma)}{\partial \sigma(x)} + \frac{\partial g(\sigma)}{\partial \sigma(x)} K_+ f(\sigma)) - f(\sigma) K_+ A K_+ f(\sigma) &= -A, \\
 e^{-g(\sigma)K_+} \frac{\partial F(\sigma)}{\partial \sigma(x)} - A K_+ F(\sigma) &= -A \phi, \quad \text{and its adjoint,} \\
 \frac{\partial F(\sigma)}{\partial \sigma(x)} - \frac{\partial g(\sigma)}{\partial \sigma(x)} K_+ F(\sigma) - f(\sigma) K_+ (e^{-g(\sigma)K_+} \frac{\partial F(\sigma)}{\partial \sigma(x)} - A K_+ F(\sigma)) &= -A \phi, \\
 \text{and its adjoint,} \\
 -(\bar{F}(\sigma) K_+ e^{-g(\sigma)K_+} \frac{\partial F(\sigma)}{\partial \sigma(x)} + \frac{\partial \bar{F}(\sigma)}{\partial \sigma(x)} e^{-K_+g(\sigma)} K_+ F(\sigma)) \\
 - \bar{F}(\sigma) K_+ A K_+ F(\sigma) + \phi \frac{\partial S(\sigma)}{\partial \sigma(x)} \phi &= -\phi A \phi
 \end{aligned} \quad (\text{A.2})$$

This can be reduced to

$$\frac{\partial f(\sigma)}{\partial \sigma(x)} + (f(\sigma) K_+ + 1) A (1 + K_+ f(\sigma)) = 0.$$

As the solution of this equation with the boundary condition $f(0)=0$, we get

$$f(\sigma) = - \int_0^{\sigma} A [1 + \int_0^{\sigma} K_+ A]^{-1}, \quad (\text{A.3})$$

and from this we obtain

$$g(\sigma) = -K_+^{-1} \log(1 + \int_0^\sigma K_+ A). \quad (\text{A.4})$$

Using (A.2), (A.3), (A.4) we obtain

$$\frac{\delta F(\sigma)}{\delta \sigma(x)} + [1 + \int_0^\sigma K_+ A]^{-1} K_+ A F(\sigma) + [1 + \int_0^\sigma K_+ A]^{-1} A \phi = 0,$$

and the solution of this linear equation with the initial condition $F(0) = 0$ can be readily found

$$F(\sigma) = - \int_0^\sigma A [1 + \int_0^\sigma K_+ A]^{-1} \phi. \quad (\text{A.5})$$

Substituting (A.3), (A.4) and (A.5) into (A.1) we obtain the previous result (4.2).

Note added in proof: To fill up the vacant space some remarks and formulas may be inserted here.

a) Taking the expectation value of the S -matrix given by (4.4) with respect to the photon field operator A_μ we would be able to obtain a true closed form of the S -matrix, though it is a very hard task. Not to say about the general S -matrix itself, even about the one-body kernel G_+ , or about the $K_+ A$ which neglects the vacuum polarization effect (ϕ from G_+), it is hard to obtain ${}_p \langle K_+ A \rangle_p$ in a closed form. Recently we have made a formal evaluation of ${}_p \langle K_+ A \rangle_p$ with the aid of Feynman's proper-time formulation (Y. Katayama and K. Yamazaki, "Convergence problem in Quantum Electrodynamics", Prog. Theor. Phys., in press, and also independently S. Hori private communication, makes a similar consideration) in connection with the Dyson's argument on the convergence of the perturbation series. (Dyson, Phys. Rev. 85, (1952), 631).

According to our results similar to Dyson's conclusion, it seems probable that the Dyson (perturbation) series does not converge, and indeed it is only an asymptotic expansion of the true solution. Yet, this fact may be expected from the beginning for the non-linearity of the interaction term. The origin of this sort of divergence comes from the fact that in the very higher order processes the rates of increase of the number of Dyson diagrams overcome the smallness of coupling. Hence, if the expectation value of the S -matrix or G_+ with respect to A_μ field has any closed solution, setting aside the problem for a time whether it is possible or not to perform the renormalization program without perturbation procedures, it seems to me that its expansion will be allowed only in a form of inverse power expansion of coupling. Therefore to obtain a closed form of such an expression as ${}_p \langle {}_s \langle S \rangle_s \rangle_p$ from our ${}_s \langle S \rangle_s$, the operator calculus of Feynman-Fujiwara's form can never be used, since it necessarily stands on assumption that there exists the solution which gives a convergent power (not inverse power) series expansion, and indeed it is also nothing but a perturbation procedure. It will need some very new mathematical tools.

β) We shall give several useful formulas for applications without proof.

$$a) \quad (\phi + q)^n = n! \sum_{j=0}^{(n/2)} \frac{D^j}{j! 2^j} \sum_{i=0}^{n-2j} \frac{1}{(n-2j-i)! i!} q^{n-2j-i} \phi^i,$$

for operators ϕ and q that commute with their commutator $[\phi, q] = D$, and $(n/2)$ represents $n/2$ or $(n-1)/2$ for n respectively when it is even or odd.

b) In the following a and b are c-numbers.

i) Quadratic case (Bosons).

$$\begin{aligned} & \exp\{(\hat{p}+q)a(\hat{p}+q)+b(\hat{p}+q)\} \\ &= \exp(qK\bar{a}q+bK\bar{q}) \text{Exp}\left\{2\int_0^1 d\mu \bar{q}K(\mu)a\hat{p}\right\} \exp(\hat{p}K\bar{a}\hat{p}+bK\bar{q}) \exp\left(-\frac{1}{2}\log K\right) \exp\left(\frac{1}{2}bK\bar{D}b\right), \\ & \langle q' | \exp\{(\hat{p}+q)a(\hat{p}+q)+b(\hat{p}+q)\} | f' \rangle \\ &= \exp K\left\{(\hat{p}'+q')a(\hat{p}'+q') + b(\hat{p}'+q') + \frac{1}{2}bK\bar{D}b\right\} \exp\left(-\frac{1}{2}\log K\right) \langle q' | f' \rangle, \end{aligned}$$

where $[\hat{p}, q] = D$, $K(\mu) = (1 - 2aD\mu)^{-1}$ and $K = K(1)$; and Dirac's bra-ket symbols are used.

ii) Bilinear case (Bosons).

$$\begin{aligned} & \exp\{(\bar{\hat{p}}+\bar{q})a(\hat{p}+q)+\bar{b}(\hat{p}+q)+(\bar{\hat{p}}+\bar{q})b\} \\ &= \exp(\bar{q}K\bar{a}q+\bar{b}K\bar{q}+\bar{q}K\bar{b}) \text{Exp}\left\{\int_0^1 d\mu (\bar{q}K(\mu)a\hat{p}+qK(\mu)a\bar{\hat{p}})\right\} \\ & \quad \times \exp(\bar{\hat{p}}K\bar{a}\hat{p}+\bar{b}K\bar{q}+\bar{\hat{p}}K\bar{b}) \exp(-\log K) \exp(\bar{b}K\bar{a}D\bar{b}), \end{aligned}$$

where $[\bar{\hat{p}}, q] = [\hat{p}, \bar{q}] = D$, other commutators 0; and $K(\mu) = (1 - aD\mu)^{-1}$, $K = K(1)$.

iii) Bilinear case (Fermions).

Identical equation to ii) holds for this case, except that $+qK(\mu)a\bar{\hat{p}}$ and $-\log K$ now read $-qK(\mu)a\bar{\hat{p}}$ and $+\log K$ respectively; and $-\{\bar{\hat{p}}, q\} = \{\hat{p}, \bar{q}\} = D$, other anticommutators 0; and now \bar{b} and b anticommute with each other as well with \hat{p} , q , $\bar{\hat{p}}$ and \bar{q} .

$$c) \text{Exp}\left\{i\int_{\sigma_2}^{\sigma_1} d\tau j_{\mu}(x)A_{\mu}(\quad)\right\} = \text{exp}\left\{i\int_{\sigma_2}^{\sigma_1} d\tau j_{\mu}(x)A_{\mu}(x)\right\} \text{exp}\left\{\frac{i}{2}\int_{\sigma_2}^{\sigma_1} d\tau \int_{\sigma_2}^{\sigma_1} d\tau' j_{\mu}(x)\bar{D}(x, x')j_{\mu}(x')\right\}$$

when j_{μ} can be treated as a c-number.

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Notes on the Decay of the Neutron, I

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The beta-decay of the neutron is theoretically discussed and compared with the experimental results of Robson. As was first pointed out by Beretetsky and Pomeranchuk, the form of the beta-spectrum derived on the pseudoscalar interaction is different from the so-called allowed spectrum. The final decision about the type of interaction is not, however, offered from the beta-spectrum alone. Attempts are therefore made to obtain some theoretical predictions which might be useful for the decision of the interaction type when compared with some future experiments. They are the angular correlation of electrons and protons and the momentum spectrum of protons. The calculation shows that in principle it is possible, though not easy, to carry through such experiments.

The beta-spectrum obtained by Robson is not reliable in the low energy region, but an explanation is given how to account for the shape of the spectrum in this region, if the observed spectrum is correct there.

§ 1. Introduction

Experimental confirmation that a neutron decays radioactively into a proton, a beta-particle and a neutral particle, usually assumed to be a neutrino, was reported by Robson¹⁾. He identified the beta-particle with the electron resulting from the neutron decay by observing its coincidence with the proton also produced by the decay. By this method the end point of the beta-spectrum was found to be 782 Kev. with a probable error of ± 13 Kev. The half-life of the neutron was found to be 12.8 minutes with an error of ± 2.5 minutes. He also found that above 300 Kev. the energy distribution of the beta-particles was consistent with what was to be expected for an allowed transition.

The purpose of the present paper is to give a theoretical consideration about the neutron decay process aiming at the determination of the type of interaction between nucleon and electron-neutrino fields. The types for the interaction including the derivative of wave function are certainly in disagreement with the accurate measurements²⁾. Even when no derivatives are included in the interaction, the form for the interaction is not

uniquely determined by the condition of relativistic invariance. As is well known, five independent relativistic invariant expressions can be chosen for the interaction Hamiltonian density functions, which are usually called the scalar, vector, tensor, pseudovector (or axial vector) and pseudoscalar interaction (and for short denoted respectively by S ., V ., T ., A . and P . or 1, 2, 3, 4 and 5)³⁾.

It is known that the form of the beta-spectrum in the first approximation is the same for these five invariants in the nuclear beta-decay processes.* So it seems at first sight that no information about the type of interaction would be given by the form of the spectrum. But as was first pointed out by Beretetsky and Pomeranchuk^{4)**}, the P . interaction gives distinctly different spectrum in case of neutron decay. Because, I_z appearing in the P . interaction is equal to $(\vec{\sigma} \cdot \vec{p})/2M$ in the non-relativistic approximation, where $\vec{\sigma}$, \vec{p} and M are the spin operator, momentum and mass of nucleon, respectively. Thus it is expected that in the P . interaction the nucleon recoil will play an important role⁵⁾, in contrast with the fact that the large components in the other four interactions are not related with this effect. Such being the case, the result will be that the nuclear matrix element for the P . interaction can not take the constant value in the neutron decay, while in the usual nuclear beta-decay it could be regarded as constant. In the recent theory of nuclear beta-decay, this P . interaction is called a relativistic term⁶⁾ and neglected in the allowed transition because its magnitude is only 1/10 as large as the other interactions. But this statement is based on the assumption the coupling constants for all five interactions are of the same order of magnitude, and it may happen that the P . interaction is equally important in beta decay processes. We shall therefore study in § 2 the results for the neutron decay for each of five interactions including the P . interaction too. The calculation is substantially the same with the above cited Russian authors', but we shall repeat it for the sake of completeness. In the preceding papers, it was made sure that the reactive corrections by meson⁷⁾ and photon⁸⁾ play no essential role in the beta decay processes; in particular, they do not effectively give rise to mixing of interactions of different types, so we first deal with five pure interactions separately. The effects of mixing them will be treated in the last section.

In § 3 and § 4 of this paper we shall also study about the momentum spectrum of protons as well as the angular correlations of the directions in which protons and electrons are emitted. These discussions would be extremely interesting from a theoretical point of view, because the various types of interaction lead to different predictions in these phenomena and there seems to be a possibility, though not easy at present, that future experiments would decide the actual types of interaction by observing the proton spectrum and the correlation.

* For short, the term of "nuclear beta-decay" means the beta-decay processes of complex nuclei in this paper.

** (Note added in proof.) Recently this fact has also been emphasized by O. Kofoed-Hanson, Phil. Mag. 42 (1951), 1411.

§ 2. Electron energy spectrum

We shall write H_1, \dots, H_5 for the Hamiltonian density functions as follows;

$$H_\nu = G_\nu (\bar{\psi} O_\nu^\dagger \tau_{PN} \psi) (\bar{\varphi} o_\nu^L \varphi), \quad (2.1)$$

where G_ν ($\nu=1, 2, 3, 4, 5$) are the coupling constants for the five coupling types, and O_ν^\dagger and o_ν^L are the Dirac operators given in the following table:

Type of interaction	$S.$	$V.$	$T.$	$A.$	$P.$
G_ν	G_1	G_2	G_3	G_4	G_5
O_ν^\dagger	1	$i\Gamma_\mu$	$\sum_{\mu\rho}$	$i\Gamma_5\Gamma_\mu$	$i\Gamma_5$
o_ν^L	1	$i\gamma_\mu$	$\sigma_{\mu\rho}$	$i\gamma_5\gamma_\mu$	$i\gamma_5$

(2.2)

$\sum_{\mu\rho}$ being $(\Gamma_\mu I'_\rho - I'_\rho \Gamma_\mu)/2i$. Further $\bar{\psi} (= \psi^* I'_4)$, ψ , $\bar{\varphi}$ and φ are the wave functions of final proton, initial neutron, electron and antineutrino, respectively. τ_{PN} is a transition operator transforming a neutron into a proton. Throughout this paper we employ the natural unit; $\hbar=c=1$.

If the influence of the electromagnetic interaction between the electron and the proton resulting from the neutron decay is neglected, four particles can be represented by plane waves. Hereafter, this approximation will be used in this paper because it does not give rise to any serious errors, as will be shown at the end of this section. By the ordinary perturbation theory, the total transition probability for the decay of the neutron at rest is given by the expression:

$$P = \frac{G^2}{2\pi^3} \int_0^{V\varepsilon^2 - m^2} d\rho \int_{-1}^1 d(\cos \theta) \frac{\rho^2 q^2 \cdot S\rho_{\nu\nu}}{64 M_N \rho_0 q (F_0 + q + \rho \cos \theta)}. \quad (2.3)$$

In (2.3) the following notations are used:

M_P and M_N : masses of the nucleons. (The lower suffixes P and N are used in order to distinguish between proton and neutron, whenever the mass-difference of these two particles ($M_N - M_P = \varepsilon_0$) can not be neglected.)

F_0 and F : energy and momentum of the proton.

ρ_0 , ρ and m : energy, momentum and mass of the electron. ($\rho_0^2 = \rho^2 + m^2$.)

q : energy of the neutrino. (It is assumed that the rest mass of the neutrino is equal to zero.⁽⁹⁾)

ε : maximum energy which is determined by experiments for the electron resulting from the decay. (The difference ($\varepsilon_0 - \varepsilon$) is nearly equal to the kinetic energy of proton and in our approximation this difference can be neglected.)

θ : angle between the directions of emission of electron and neutrino.

$S\rho_{\nu\nu}$'s are given in Table (I) for each five interactions. The conservations of energy and momentum ($M_N = F_0 + \rho_0 + q$, and $0 = \vec{F} + \vec{\rho} + \vec{q}$) are satisfied for four particles in this case. By using these conservation laws, q can be eliminated by expressing it as a function of ρ_0 and θ .

If the whole expression (2.3) is expanded into power series of (m/M) and only the first term is to be taken into account,* approximate $S\hat{p}_{\nu\nu}$'s given by the expressions in the second lines of Table I are to be used in (2.3). Then the integration over θ is easily performed. Those approximate expressions except for the P interaction are evidently the same ones which were calculated by Hamilton⁽¹⁰⁾ in order to get the angular correlation of the electron and the neutrino. In this approximation, the energy of neutrino (q) is nearly equal to $(\varepsilon - p_0)$.

Thus, we obtain

$$P = (G_v^2 a_v^2 / 2\pi^3) \int_0^{V\varepsilon^2 - m^2} dp N_{\nu\nu}(p), \quad (2.4)$$

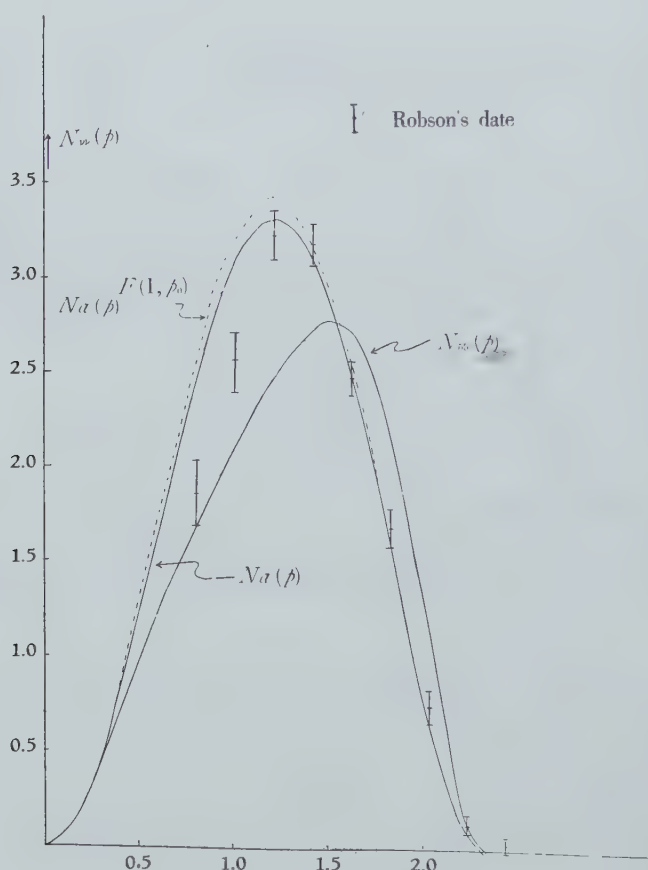


Fig. 1. Momentum spectra of emitted electrons.

The expressions of $N_\alpha(p)$, $N_\beta(p)$ and $N_\alpha(p)F(1, p_0)$ are given by (2.6), (2.7) and (2.14), respectively.

* However, remaining terms of this expansion can contribute to highly forbidden transitions, though this fact has hitherto been overlooked.

where a 's are given by the following table ;

a_1	a_2	a_3	a_4	a_5	(2.5)
1	1	$\sqrt{12}$	$\sqrt{3}$	$(m/2M)$	

Here, in cases of the first four types a 's are the so-called nuclear matrix elements themselves, but in the case of the fifth type a_5 is defined by the equation :

$$\int \bar{\psi} \gamma_5 \psi d\tau \simeq a_5 \int \bar{\psi} \left(\frac{\vec{\sigma} \cdot \vec{F}}{m} \right) \psi d\tau.$$

The shape of the beta-spectrum for the neutron decay can be calculated according to the following $N_{\nu\nu}(p)dp$:

- I) In cases of H_1 , H_2 , H_3 and H_4 interactions, we get $N_{11}=N_{22}=N_{33}=N_{44}=N_a$, and

$$N_a(p) \doteq p^2(\epsilon - p_0)^2. \quad (2.6)$$

This $N_a(p)$ gives the well known allowed spectrum and is shown in Fig. (1) (momentum spectrum) and Fig. (5) (energy spectrum), using Robson's value ($\epsilon=2.53m$). The maximum point of the spectrum corresponds to $p=1.22m$.*

- II) In the case of H_5 interaction,⁴⁾ we obtain

$$N_{55}(p) \doteq \{ p^2 + (\epsilon - p_0)^2 - (2p^2(\epsilon - p_0)/3p_0) \} p^2(\epsilon - p_0)^2/m^2. \quad (2.7)$$

$N_{55}(p)$ given by this formula is evidently different from $N_a(p)$. This difference is the result of the disturbance due to the recoil of the proton. The plots for $N_{55}(p)$ are shown in Figs. (1) and (5).

Fig. 1 shows that $N_{55}(p)$ obviously disagrees with the experimental spectrum obtained by Robson. But it will be interesting to know the magnitude of G_5 , which is necessary to give the observed neutron life-time, disregarding the disagreement is the spectrum. If we define that

$$m^5 f_\nu(\eta) = \int_0^{\sqrt{\epsilon^2 - m^2}} dp N_{\nu\nu}(p), \quad (2.8)$$

the following expressions are given :

for $N_a(p)=N_{11}(p)$, (2.6),^{11),12)}

$$f_a(\eta) = \frac{1}{30} \eta^5 - \frac{1}{12} \eta^3 - \frac{1}{4} \eta + \frac{1}{4} \sqrt{\eta^2 + 1} \log(\eta + \sqrt{\eta^2 + 1}), \quad (2.9)$$

for $N_{55}(p)$, (2.7),¹³⁾

* The maximum point of the energy spectrum corresponds to the value obtained from the following expressions :

$$p_0 = (\epsilon + \sqrt{\epsilon^2 + 8m^2})/4.$$

$$f_5(\eta) = \frac{1}{70} \eta^7 - \frac{1}{20} \eta^5 - \frac{1}{3} \eta^3 - \frac{1}{4} \eta + \frac{1}{4} (\eta^2 + 1)^{3/2} \log(\eta + \sqrt{\eta^2 + 1}), \quad (2 \cdot 10)$$

where $m^2 \eta^0 = \epsilon^2 - m^2$.

Table 1. Exact expressions of $S_{\nu\nu}$ and their first approximations in the power of (m/M) . Symbols as (fq) mean the four dimensional inner product. $((fq) = \not{p}q \cos \theta - \not{p}_0 q)$.

$$\begin{aligned} S_{p_{11}} &= -16 [F_0 + M_P] M_N (pq) \\ &\doteq 32 M_N M_P q \not{p}_0 [1 - (\not{p}/\not{p}_0) \cos \theta] \\ &\doteq 32 M_N M_P [(\epsilon^2 - m^2 - F^2)/2] \\ S_{p_{22}} &= -32 M_N [q(F\not{p}) + \not{p}_0(Fq) - M_P(pq)] \\ &\doteq 32 M_N M_P q \not{p}_0 [1 + (\not{p}/\not{p}_0) \cos \theta] \\ &\doteq 32 M_N M_P [2(\epsilon - \not{p}_0) \not{p}_0 - (\epsilon^2 - m^2 - F^2)/2] \\ S_{p_{33}} &= 128 M_N [F_0(pq) - 2\not{p}_0(F\not{p}) - 2q(Fq)] \\ &\doteq 12 \cdot 32 M_N M_P q \not{p}_0 [1 + (\not{p}/3\not{p}_0) \cos \theta] \\ &\doteq 4 \cdot 32 M_N M_P [4(\epsilon - \not{p}_0) \not{p}_0 - (\epsilon^2 - m^2 - F^2)/2] \\ S_{p_{44}} &= -32 M_N [q(F\not{p}) + \not{p}_0(Fq) + M_P(pq)] \\ &\doteq 3 \cdot 32 M_N M_P q \not{p}_0 [1 - (\not{p}/3\not{p}_0) \cos \theta] \\ &\doteq 32 M_N M_P [2\not{p}_0(\epsilon - \not{p}_0) + (\epsilon^2 - m^2 - F^2)/2] \\ S_{p_{55}} &= -16 M_N (F_0 - M_P) (pq) \\ &\doteq 16 M_N M_P \not{p}_0 q [(\not{p}^2 + (\epsilon - \not{p}_0)^2 + 2\not{p}(\epsilon - \not{p}_0) \cos \theta)/2 M_P^2] \\ &\quad \times [1 - (\not{p}/\not{p}_0) \cos \theta] \\ &\doteq 32 M_N M_P (m/2M)^2 (F^2/m^2) [(\epsilon^2 - m^2 - F^2)/2] \end{aligned}$$

The magnitudes of the coupling constants (G_ν 's) are now calculated from the life-time of neutron by the following expression* :

$$G_\nu^2 a_\nu^2 = \frac{1}{f_\nu T} \cdot 9.25_6 \times 10^{-22} \left[\frac{1}{m^2} \right]. \quad (2 \cdot 11)$$

T is the neutron half-life in minutes. If we use Robson's experimental values ($T=12.8$ and $\epsilon=2.53m$), we find $f_\alpha(2.32)=1.63_1^{(1)}$ and $f_\beta(2.32)=3.45_9$ and so

* $\left[\frac{1}{m^2} \right] = (mc^2) \left(\frac{\hbar}{mc} \right)^3 = 4.70_4 \times 10^{-38} [\text{erg} \cdot \text{cm}^3]$.

$$G_k u_k = 6.6_3 \times 10^{-12} \left[\frac{1}{m^2} \right] = 3.1_3 \times 10^{-49} [\text{erg. cm}^3], \quad (2 \cdot 12)$$

and

($k=1, 2, 3$ and 4),

$$G_5 u_5 = 4.5_6 \times 10^{-12} \left[\frac{1}{m^2} \right],$$

$$G_5 = 7.8_3 \times 10^{-46} [\text{erg. cm}^3]. \quad (2 \cdot 13)$$

Thus we find that G_5 should be about 1000 times as large as G_1 , G_2 , G_3 and G_4 in order to account for the observed life-time. This is to be expected, because, as stated above, the P . interaction give rise to the beta-decay only through the recoil of the nucleon.¹⁴⁾

The formula (2.4) gives the shape of the beta-spectrum without taking into account the influence of the electromagnetic interaction between electron and proton; this is a reasonable approximation for neutron decay. But to obtain the correct shape, the influence of the electromagnetic interaction must, of course, be considered. Since the relativistic two body problem consisting of electron and proton is not yet solved, the only possible method is to introduce this effect for the allowed transition by neglecting the recoil of proton and taking into account the distortion of the electron wave function by the Coulomb field of the proton at rest. This means that the formula,

$$P = \frac{G_k^2}{2\pi^3} \left| \int \bar{\psi} O_k^H \tau_{PN} \psi d\tau \right|^2 \int_0^{\sqrt{\varepsilon^2 - m^2}} d\rho N_\alpha(\rho) F(Z, \rho_0) \quad (2 \cdot 14)$$

($k=1, 2, 3$ and 4)

is used insted of (2.4). A well known factor $F(Z, \rho_0)$ has been given by the equation (20) of Konopinski's paper¹⁵⁾. Thereupon, a difficult problem arises. It is that $F(Z, \rho_0)$ tends to infinity when R included in it approaches to zero. In the usual nuclear beta-decay, this difficulty has been avoided by assuming R to be equal to the nuclear radius, but in our neutron decay, this assumption can not be used owing to the unknown factor of "proton radius" and the large recoil of proton. However, even when R is taken as the meson Compton wave length or one-thousandth of that length, the numerical value of $R^{\sqrt{1-\alpha^2}-1}$ is nearly equal to unity, because $\sqrt{1-\alpha^2}-1$ can be regarded as zero for the neutron decay. Thus by using the approximate formula (2.14) and putting $R^{\sqrt{1-\alpha^2}-1} \approx 1$, the change of the momentum spectrum for the allowed one can be obtained. The result is shown by the dotted line in Fig. 1. As is seen from this figure, the effect of Coulomb force is really of minor importance; therefore we can use, hereafter, the plane wave approximation for all particles concerned without introducing any appreciable errors.

§ 3. Angular correlation of electron and proton, neglecting the electromagnetic interaction between them

Unfortunately, in the case of the so-called allowed transition, the shape of the beta-spectrum is the same for all types of interactions except for the P . interaction. That is to say, a clue about the choice between the first four types is not offered by the momentum

spectrum of electrons. Attempts will, therefore, be made in this section to compare the theoretical predictions on the various interactions with future experiments on the angular correlation of electron and proton. The calculation will be simplified by disregarding the electromagnetic influence acting between electron and proton. As has been stated in § 2, this effects is small enough, so that this simplification is permissible in our treatment.

The transition probability for the emission of an electron with an angle between φ and $\varphi + d\varphi$, φ denoting the angle between the directions of the electron and the proton, is given by*

$$Pd(\cos \varphi) = \frac{G_v^2}{2\pi^3} \int_x^y d\hat{p}_0 F^2 \hat{p} \frac{S\hat{p}_{vv}}{32M_N \sqrt{D}} \cdot d(\cos \varphi). \quad (3.1)$$

In the same approximation as of § 2, i.e. expanding various quantities into the power series of (m/M) and retaining only the first term, the following expressions should be used in (3.1):

$$D = (\omega^2 - 2M_F^2)^2 - 4\hat{p}^2 M_F^2 \sin^2 \varphi \doteq [(\epsilon - \hat{p}_0)^2 - \hat{p}^2 \sin^2 \varphi] 4M_F^2, \quad (3.2)$$

$$\text{and} \quad \omega^2 = (M_N - \hat{p}_0)^2 - \hat{p}^2 + M_F^2 \doteq 2M^2. \quad (3.3)$$

Also for $S\hat{p}$'s the second approximate expressions given in Table I should be used. Furthermore, F must be eliminated by expressing it as the function of \hat{p}_0 and φ by using the energy momentum conservation laws, that is,

$$F = \{(\pm M_N - \hat{p}_0) \sqrt{D} - \omega^2 \hat{p} \cos \varphi\} / 2 \{ (M_N - \hat{p}_0)^2 - \hat{p}^2 \cos^2 \varphi \} \\ \doteq \{ \pm \sqrt{(\epsilon - \hat{p}_0)^2 - \hat{p}^2 \sin^2 \varphi} - \hat{p} \cos \varphi \}. \quad (3.4)$$

A caution must be kept to determine the limits of integration in (3.1): The limits (x and y) are determined by the conditions $D > 0$ and $F > 0$. In the case of $\varphi > 90^\circ$,

* The transition probability is here obtained from the expressions for the emission spectra of electrons at angles φ with energy between \hat{p}_0 and $\hat{p}_0 + d\hat{p}_0$. The same probability could be also obtained from the proton spectra. In the latter case, the corresponding expressions to (3.1)~(3.6) are as follows:

$$Pd(\cos \varphi) = \frac{G_v^2}{2\pi^3} \int_{x'}^{y'} dF \cdot F^2 \hat{p}^2 \frac{S\hat{p}_{vv}}{32M_N F_0 \sqrt{D'}} \cdot d(\cos \varphi) = \frac{G_v^2 d^2 v}{2\pi^3} \Phi'_{vv}(\varphi) d(\cos \varphi), \quad (3.1')$$

$$D' = (\delta^2 - 2m^2)^2 - 4m^2 F^2 \sin^2 \varphi \doteq \{\epsilon^2 - F^2 - m^2\}^2 - 4m^2 F^2 \sin^2 \varphi, \quad (3.2')$$

$$\delta^2 = (M_N - F_0)^2 - F^2 + m^2 \doteq \epsilon^2 + m^2 - F^2, \quad (3.3')$$

$$\hat{p} \doteq \left\{ \begin{array}{l} \pm \epsilon \sqrt{D'} - F \delta^2 \cos \varphi / 2(\epsilon^2 - F^2 \cos^2 \varphi), \\ \pm \epsilon \delta^2 - F \sqrt{D'} \cos \varphi / 2(\epsilon^2 - F^2 \cos^2 \varphi). \end{array} \right\} \quad (3.4')$$

In the case of $\varphi > 90^\circ$ both signs in (3.4') must be used, the limits of integration being $x'=0$ and $y'=\beta_1'$ for the plus sign and $x'=\beta_2'$ and $y'=\beta_1'$ for the minus sign. Then the transition probability is given by the sum of these two representation. In the case of $\varphi \leq 90^\circ$ the minus sign must be abandoned and we have $x'=0$ and $y'=\beta_2'$, where,

$$\beta_1' \doteq \{ \sqrt{\epsilon^2 - m^2 \cos^2 \varphi} - m \sin \varphi \}, \quad (3.5')$$

$$\text{and} \quad \beta_2' \doteq (\epsilon - m). \quad (3.6')$$

both signs in (3.4) must be used, the limits of integration being $x=0$ and $y=\beta_1$ for the plus sign and $x=\beta_2$ and $y=\beta_1$ for the minus sign, where

$$\beta_1 \doteq \frac{M_p}{(M_l \cos^2 \varphi + 2\epsilon)} \cdot \left\{ \epsilon + \frac{(3\epsilon^2 + m^2)}{2M_p} - \sin \varphi \sqrt{\epsilon^2 - m^2 \cos^2 \varphi} + \epsilon(\epsilon^2 - m^2)/M_p \right\}$$

$$\doteq \frac{1}{\cos^2 \varphi} \{ \epsilon - \sin \varphi \sqrt{\epsilon^2 - m^2 \cos^2 \varphi} \}, \quad (3.5)$$

and

$$\beta_2 \doteq \frac{1}{2\epsilon} (\epsilon^2 + m^2). \quad (3.6)$$

Then the transition probability is given by the sum of these two representations. In the case of $\varphi \leq 90^\circ$ the minus sign can not be used and we have $x=0$ and $y=\beta_2$.

When these expressions are used for five interactions, we get

$$Pd(\cos \varphi) = (G_v^2 a_v^2 / 2\pi^3) \Phi_{vv}(\varphi) d(\cos \varphi), \quad (3.7)$$

where

ν	1	2	3	4	5	
$\Phi_{\nu\nu}$	I_a	$2I_a - I_b$	$\frac{1}{3}[4I_a - I_b]$	$\frac{1}{3}[2I_a + I_b]$	I_c	(3.8)

and

$$I_a \doteq \int_x^y d\phi_0 (F^2 \phi / 2 \sqrt{D''}) \phi_0 (\epsilon - \phi_0), \quad (3.9)$$

$$I_b \doteq \int_x^y d\phi_0 (F^2 \phi / 2 \sqrt{D''}) (\epsilon^2 - m^2 - F^2) / 2, \quad (3.10)$$

$$I_c \doteq \int_x^y d\phi_0 (F^4 \phi / 2 m^2 \sqrt{D''}) (\epsilon^2 - m^2 - F^2) / 2, \quad (3.11)$$

and

$$D'' \doteq (\epsilon - \phi_0)^2 - \phi^2 \sin^2 \varphi. \quad (3.12)$$

The integrals of I_a , I_b and I_c are complicated, so they are calculated numerically. For convenience we shall extensively discuss our theoretical results in § 5, with the momentum spectrum of the proton obtained in the next section. As will be shown there, the values of ϕ for different types have enough differences in the neighbourhood of $\varphi=180^\circ$, so that one can in principle decide the type of interaction by correlation experiments. But it might be very difficult actually to carry out such experiments, because for this purpose it is required to detect emitted particles with very high accuracy.

* We wish to express our sincere appreciation to Dr. H. L. Reynolds and Dr. L. C. Biedenharn, Jr. of the Oak Ridge National Laboratory for having pointed out this fact.

§ 4. Proton momentum spectrum

If one compares five approximate expressions in Table I with each other, one can expect that the shapes of the proton momentum spectra drawn from these expressions will be different from each other. These spectra could be obtained by integrating the expression (3.1') first over φ instead of F . However, because it is rather difficult to carry out the integrations over φ , we shall proceed differently. We have succeeded in obtaining the analytical expressions for the electron momentum spectrum and the total transition probability, as has been shown in (2.4) to (2.10). This is due to the fact that the neutrino mass is ignored and its energy (q) arising in (2.3) can be expressed by a simple expression. Thus it is expected that an analogous procedure will be useful also in the calculation of the proton spectrum. We will therefore start from calculating the total transition probability, using as an integration variable the angle denoted by ϕ , the angle between the momenta of the proton and the neutrino.

As has been stated before, we will represent each particle by plane wave and use only the first term of expansion in (m/M) . These approximations are evidently accurate enough in dealing with the neutron decay. The expressions corresponding to (2.1) can then be written as follows:

$$P = \frac{G_v^2}{2\pi^3} \int (Sp_{\nu\nu}/64M_N F_0) \frac{qF^2}{(\dot{p}_0 + q + F \cos \phi)} dF d(\cos \phi). \quad (4.1)$$

The neutrino energy (q) is now expressed as a function of the proton momentum (F) and the angle (ϕ) by using the energy-momentum conservation law:

$$q = \frac{(M_N - M_p)^2 + 2M_N(M_p - F_0) - m^2}{2(M_N - F_0 + F \cos \phi)} \\ \doteq \frac{\epsilon^2 - F^2 - m^2}{2(\epsilon + F \cos \phi)}. \quad (4.2)$$

Substituting $Sp_{\nu\nu}$ given in Table I and q represented by (4.2), the integration over ϕ yields the final results:

$$P = (G_v^2 \alpha_v^2 / 2\pi^3) \int_0^{V_{\bar{\epsilon}\bar{\epsilon} - m^2}} dF M_{\nu\nu}(F), \quad (4.3)$$

where $M_{\nu\nu}$'s are given by the following table:

ν	1	2	3	4	5
$M_{\nu\nu}$	J_a	$2J_a - J_b$	$\frac{1}{3}[4J_a - J_b]$	$\frac{1}{3}[2J_a + J_b]$	J_c

(4.4)

with

$$J_a = \frac{1}{4}(\epsilon^2 - F^2 - m^2)^2 F^2, \quad \frac{1}{6} \left\{ \frac{1}{(\epsilon^2 - F^2)} + \frac{2\epsilon^2 - m^2}{(\epsilon^2 - F^2)^2} + \frac{4\epsilon^2 m^2}{(\epsilon^2 - F^2)^3} \right\}, \quad (4.5)$$

$$J_b = \frac{1}{4} (\epsilon^2 - F^2 - m^2)^2 F^2 \cdot \frac{1}{(\epsilon^2 - F^2)}, \quad (4.6)$$

$$\text{and } J_c = \frac{1}{4} (\epsilon^2 - F^2 - m^2)^2 F^4 \cdot \frac{1}{(\epsilon^2 - F^2) m^2}. \quad (4.7)$$

When the integrals over F are performed, J_a and J_b must, of course, give the same value as (2.9), and J_c the value (2.10). These facts can easily be verified by elementary integrations. The detailed discussion of our results will also be given in the next section.

(To be continued.)

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On a Relativistic Integral Equation for Bound States

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Bound states of two particles are described relativistically by a homogeneous integral equation. Its general properties and reduction to the non-relativistic equation are studied. In the case of scalar particles with scalar photon interaction, if we take the ladder approximation and omit the $D^{(1)}$ part of δ_+ function, energy levels are obtained relativistically as explicit functions of the fine-structure constant. Difficulties which are met in the case of two spinor particles are discussed.

§ 1. Introduction and summary

Energy-levels of bound states of two particles interacting through a quantized field, such as electromagnetic or mesic, are usually determined by solving a Schrödinger equation with potentials which have been calculated in the adiabatic approximation in accordance with the field theory. Such an equation is not relativistically covariant in form, and potentials themselves have meanings well-defined only in the non-relativistic domain. Indeed, even if lowest order potentials exist higher ones turn out to be singular as in the cases of electromagnetic and scalar meson field for two spinor particles, and lowest order potentials themselves are singular in the cases of other meson fields, that is, dynamic interactions through the field can not well be represented by the potentials.¹⁾ With such singular potentials eigen-values of the Schrödinger equation will be indeterminate as shown by Case.²⁾ It is not clear at the present stage whether such difficulties are inherent in the field theory and will not be avoided without a future theory which takes into account the finite size of elementary particles, or the usual perturbation procedure of expansion in power series of coupling constants and v/c is not adequate for the problem. In either case it is desirable, as a first step, to have a relativistic formulation of the bound states in order to make a further approach to the problem, for instance, by studying the nature of the above difficulty.

Following the Feynman's over-all space-time point of view for scattering problem,³⁾ we have derived an integral equation for a wave amplitude which determines eigen-values of bound states four-dimensionally.* A differential equation corresponding to it has formerly been obtained by Nambu¹⁾ and Kita.** Independently of us, Salpeter and Bethe¹⁾ have derived the same integral equation and applied it to the deuteron problem for the scalar meson field. A rigorous derivation of the equation from the field theory has recently been ac-

* The main results given in Secs. 2 to 5 have been reported at the meetings of the physical Society of Japan, held at Tokyo September 1950, held at Osaka November 1950, and held at Kyoto May 1951.

** Report at the meeting of the Physical Society of Japan, held at Tokyo September 1950.

completed by Gell-Mann and Low.⁽⁷⁾ We shall first briefly outline our derivation based on the correspondence to the non-relativistic equation and make clear the distinction between the scattering and bound state problems in Secs. 2 and 3. The former is described by an inhomogeneous integral equation and the latter by a homogeneous one, their kernel being expressed as an infinite series in the expansion of the coupling constant. Some general properties of these equations are also stated in Sec. 2, such as a separation of the center of mass motion, an ambiguity on the definition of the center of mass coordinates, and asymptotic behaviours of the wave amplitude, in the case of "ladder" (c^2) approximation, at a remote distance in relative coordinates, time-like or space-like. In Sec. 3 a reduction to the non-relativistic integral equation and its solutions including the time-like region are considered.

In the general relativistic case, after the separation of the center of mass motion and that of the angular variables in relative coordinates in the rest system, we have a Fredholm's equation of the second kind in two dimensions, relative distance r and time t , which determines the coupling constant if the rest mass of the bound system is given. For the purpose of illustrating such an eigen-value problem we shall first study a system of two scalar particles with scalar photon interaction. Its solutions will be useful for the further treatment of spinor particles and may also serve as a phenomenological model which, embodying the ideas that elementary particles have their own internal structures, gives their mass-spectra and, if possible, decay constants relativistically. If we confine ourselves to the ladder approximation and replace ∂_+ , propagation function of photon, by (i) $\partial(r-t)/r$ or (ii) $\{\partial(r-t) + \partial(r+t)\}/2r$, we have one-dimensional equations on the light-cone. It is found in both cases that, if two particles are treated one-body-theoretically, namely, their propagation functions being I_0 instead of I_+ , the integral equations can be solved, and energy-levels and wave amplitudes are obtained as explicit functions of fine-structure constant $e^2/\hbar c$. These results give the non-relativistic values immediately as $e^2/\hbar c \rightarrow 0$, since the non-relativistic equation is not influenced by the above replacements of ∂_+ and I_+ . In the case (ii) rest masses turn out to take complex values with positive imaginary parts of order $(e^2/\hbar c)^3$ in contrast with the case (i) where they are real. This result may possibly arise partly from the fact that the above replacements of ∂_+ and I_+ have impaired the causality condition as discussed by Stückelberg⁽⁸⁾ and Fierz.⁽⁷⁾ Above results are stated in Sec. 4.

Alternatively, the problem can be expressed in another form dealing with a differential equation and boundary conditions which are contained in the integral equation. In the case (i) above stated where these conditions are found to be simple, it is possible to obtain an one-dimensional differential equation of Schrödinger type by joining wave amplitudes, which satisfy free equations in each region of space-time separated by the light-cone, on the light-cone as shown in Sec. 5.

In sec. 6, we treat a system of two spinor particles interacting through an electromagnetic field, confining ourselves to the ladder approximation. In this case the wave amplitude has sixteen components, but after the separation of the angular variables five of them are coupled together through a system of five integral equations for given j , the total

angular momentum, except for the case $j=0$, where they are further reduced to three. If the one-body-theoretical propagation function K_0 is used, it is found that the kernels of the integral equations have singular terms with factors like $\partial(t-t'-r+r')$, apart from $\delta_+(t'^2-r'^2)$, in contrast with the case of scalar particles. Then, in order to get finite solutions with non-zero eigen-values c'' we must solve the integral equations, in which these singular terms are dropped, under the additional condition that the coefficients of these singular terms are zero. It appears entirely accidental that such a condition is satisfied by the eigen-solutions of the integral equations. Indeed, it is verified in the case of the one-sheet potential, $\partial(r-t)/r$, that this condition is not satisfied actually. This situation is not altered essentially in the case of K_+ , positron-theoretical propagation function, and also irrespective of the sorts of the interacting field, electromagnetic or mesic of any type. Thus, in the case of two spinor particles we meet a fundamental difficulty in treating the bound states relativistically.

§ 2. Derivation and general properties of the integral equation

We shall follow the Feynman's over all space-time treatment⁽³⁾ for the scattering of two spinor particles interacting through, for instance, an electromagnetic field. The same notations as his will be used if not otherwise stated. $\phi_k(1, 2)$ be the wave amplitude that two spinor particles, which are distinguished by the suffices 1 and 2, with original free amplitude $\phi_0(1, 2)$ arrived at space-time points 1 and 2 after k -times scatterings through the mutual electromagnetic interaction. Then, the amplitude after one more scattering (we mean by one scattering a sum of interactions (a), (b), etc., as shown in Fig. 1, which are irreducible in the Feynman-Dyson diagram) is given by

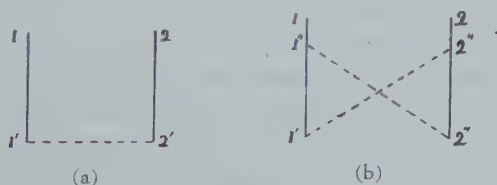


Fig. 1

$$\phi_{k+1}(1, 2) = \int K_+(1, 1'') K_+(2, 2'') G(1'', 2''; 1', 2') \phi_k(1', 2') d\tau_1'' d\tau_2'' d\tau_1' d\tau_2', \quad (2.1)$$

with

$$\begin{aligned} G(1'', 2''; 1', 2') = & -ie_1 e_2 \gamma_{1\mu} \gamma_{2\mu} \delta_+(1'', 2'') \delta(1'', 1') \delta(2'', 2') \\ & + (-ie_1 e_2)^2 \delta_+(1'', 2'') \delta_+(2'', 1') \\ & \times \gamma_{1\mu} K_+(1'', 1') \gamma_{1\nu} \gamma_{2\nu} K_+(2'', 2') \gamma_{2\mu} + \dots, \end{aligned} \quad (2.2)$$

which is an infinite series in the powers of coupling constants, the first, second, ... terms correspond to the graphs (a), (b), ... in Fig. 1. We obtain, for the total amplitude

$$\psi(1, 2) = \sum_{k=0}^{\infty} \phi_k(1, 2) \quad (2.3)$$

an integral equation relativistically invariant

$$\psi(1, 2) = \phi_0(1, 2) + \int K_+(1, 1'') K_+(2, 2'') G(1'', 2''; 1', 2') \psi(1', 2') d\tau_1'' d\tau_2'' d\tau_1' d\tau_2'. \quad (2.4)$$

For the stationary states we should seek for the solution of (2.4) which behaves as a plane wave with respect to center of mass coordinates X_μ ,

$$\psi(1, 2) = e^{-iP_\mu X_\mu} \varphi(x_\mu), \quad (2.5)$$

$$X_\mu = a_1 x_{1\mu} + a_2 x_{2\mu}, \quad x_\mu = x_{1\mu} - x_{2\mu}, \quad (2.6)$$

where a_1 and a_2 are arbitrary constants except for the condition $a_1 + a_2 = 1$, and P_μ are total energy-momentum of the system,

$$P = (P_\mu P_\mu)^{1/2} \quad (2.7)$$

being its rest mass. In general, $G(1'', 2''; 1', 2')$ can be written as Fourier integral of the form

$$\begin{aligned} G(1'', 2''; 1', 2') &= \int g(p_{1\mu}'', p_{2\mu}'', p_{1\mu}', p_{2\mu}') e^{-i(p_{1\mu}'' \cdot x_{1\mu}'' + p_{2\mu}'' \cdot x_{2\mu}'' - p_{1\mu}' \cdot x_{1\mu}' - p_{2\mu}' \cdot x_{2\mu}')} \\ &\quad \times d^4 p_{1\mu}'' d^4 p_{2\mu}'' d^4 p_{1\mu}' d^4 p_{2\mu}' \\ &= \int g(a_1 P_\mu'' + p_\mu'', a_2 P_\mu'' - p_\mu''; a_1 P_\mu' + p_\mu', a_2 P_\mu' - p_\mu') \\ &\quad \times e^{-i(P_\mu'' \cdot X_\mu'' - P_\mu' \cdot X_\mu' + p_\mu'' \cdot x_\mu'' - p_\mu' \cdot x_\mu')} d^4 P_\mu'' d^4 P_\mu' d^4 p_\mu'' d^4 p_\mu', \end{aligned} \quad (2.8)$$

with

$$p_{1\mu} = a_1 P_\mu + p_\mu, \quad p_{2\mu} = a_2 P_\mu - p_\mu, \quad d^4 p = dp_1 dp_2 dp_3 dp_4, \text{ etc.} \quad (2.9)$$

Since $G(1'', 2''; 1', 2')$ is invariant under the total displacements of space and time and then does not depend on X_μ'' if expressed as a function of X_μ'' , $X_\mu'' - X_\mu'$, x_μ'' and x_μ' , it is easily shown that g is a function multiplied by $\delta(P_\mu'' - P_\mu')$ and (2.8) can be written as

$$\begin{aligned} G(1'', 2''; 1', 2') &= (2\pi)^{-4} \int h(a_1 P' + p', a_2 P' - p'; a_1 P' + p', a_2 P' - p') \\ &\quad \times e^{-iP' \cdot (X'' - X') - ip_\mu'' \cdot x_\mu'' + ip_\mu' \cdot x_\mu'} d^4 P' d^4 p_\mu'' d^4 p_\mu'. \end{aligned} \quad (2.10)$$

Then, using (2.5) and (2.10), and expressing K_+ as well-known integral in momentum space, we have from equation (2.4)

$$\begin{aligned} \varphi(x_\mu) &= e^{iP_\mu X_\mu} \phi_0(1, 2) - \int \{\gamma_{1\mu}(a_1 P_\mu + p_\mu) - m_1\}^{-1} \{\gamma_{2\mu}(a_2 P_\mu - p_\mu) - m_2\}^{-1} \\ &\quad \times h(a_1 P_\mu + p_\mu, a_2 P_\mu - p_\mu; a_1 P_\mu + p_\mu', a_2 P_\mu - p_\mu') e^{-ip_\mu'' \cdot x_\mu'' + ip_\mu' \cdot x_\mu'} \varphi(x_\mu') d^4 p_\mu'' d^4 p_\mu'. \end{aligned} \quad (2.11)$$

It is clear that solutions of the type (2.5) are possible in the case of open states, $P > m_1 + m_2$, where incident wave $\phi_0(1, 2)$ can be written as $\exp(-iP_\mu X_\mu) \cdot \varphi_0(x_\mu)$. For bound states $P < m_1 + m_2$, however, solutions (2.5) are possible only if we put $\phi_0 = 0$.

It is shown in the next section that, if we take the non-relativistic limit in equation (2.11), we obtain an integral equation for the largest component φ_1 corresponding to the positive energy states of both particles, which for $t=0$ becomes

$$\varphi_1(x_i) = \varphi_{01}(x_i) - \frac{1}{(2\pi)^3} \int \frac{e^{ip_i(x_i - x_i')}}{\beta + p_i^2/2\mu} \frac{e_1 e_2}{r'} \varphi_1(x_i') d^3 p_i d^3 x_i', \quad (2.12)$$

with

$$\beta = m_1 + m_2 - P, \quad \mu = m_1 m_2 / (m_1 + m_2), \quad (2.13)$$

and φ_{01} being an incident plane wave satisfying $(-\beta + J/2\mu)\varphi_{01} = 0$. This equation is equivalent to a Schrödinger equation for Coulomb potential with appropriate boundary conditions. Indeed, its kernel is expressed as

$$K(x_i) = -\frac{1}{(2\pi)^3} \int \frac{e^{ip_i x_i}}{\beta + p_i^2/2\mu} d^3 p_i = -\frac{\mu}{2\pi r} \times \begin{cases} e^{-\sqrt{2\mu\beta} r}, & (\beta > 0), \\ e^{i\sqrt{-2\mu\beta} r}, & (\beta < 0), \end{cases} \quad (2.14)$$

where for $\beta < 0$ we have used the fact that β defined by (2.13) has a small negative imaginary part originally contained in m_1 and m_2 in the definition of K_+ . With $\varphi_{01} \neq 0$ and $\beta < 0$, equation (2.12) gives the solution for scattering problem, namely a plane wave plus an outgoing spherical wave. Such an integral equation is studied in detail by Jost and Pais³⁾ in the case of Yukawa potential. On the other hand, with $\varphi_{01} = 0$ and $\beta > 0$, it gives the usual solutions for bound states as shown in more detail in the next section. In view of such a correspondence to the non-relativistic case, we shall be able to take the homogeneous equation (2.11) with $\phi_0 = 0$ and $P < m_1 + m_2$ as the fundamental equation which determines the eigen-values of the bound states relativistically, $e_1 e_2$ being determined as a function of P_μ or *vice versa*. On the other hand, the inhomogeneous equation will have solutions for any value of $e_1 e_2$ if P_μ are given, and will give a basis for the relativistic generalization of the work by Jost and Pais. Recently, Kita¹⁰⁾ has shown that the homogeneous equation can be derived on the basis of the *S*-matrix theory of Heisenberg and Møller.

It is shown that eigen-values $e_1 e_2$ for bound states determined by the homogeneous equation (2.11) should not depend on P_μ but on P only and should also be independent of the choice of a_1 and a_2 in (2.6). The latter can be proved as follows: if we put

$$\varphi(x_\mu) = e^{ibP_\mu x_\mu} \varphi'(x_\mu), \quad (2.15)$$

which is a sort of gauge transformation, we have from (2.11) an equation for φ' of the same form except that a_1 and a_2 are replaced by $a_1' = a_1 - b$ and $a_2' = a_2 + b$ with $a_1' + a_2' = 1$ so that the continually varying parameter b does not influence the eigen-values. The former are also proved on the same line by considering the Lorentz transformation. Then, for practical calculation it will be convenient to take the rest system, $P_i = 0$ ($i=1, 2, 3$), and to choose a_1 and a_2 so as to simplify the expressions as exemplified in the following sections. In the rest system kernel of equation (2.11) is invariant with respect to spatial rotations including spins so that it is always possible to obtain the equations for given total angular momentum by separating the angular variables.

We shall in the following confine ourselves to the ladder approximation, that is, we take only the first term in $G(1'', 2''; 1', 2')$ shown in (2.2), the iterations of interaction (a) in Fig. 1 being wholly included in the integral equation. Then, the homogeneous equation (2.11) takes a simple form

$$\varphi(x_\mu) = \frac{ie_1 e_2}{(2\pi)^4} \int \frac{e^{-ip \cdot (x-x')}}{\{\gamma_{1\mu}(a_1 P_\mu + p_\mu) - m_1\} \{\gamma_{2\mu}(a_2 P_\mu - p_\mu) - m_2\}} \\ \times \gamma_{1\mu} \gamma_{2\mu} \delta_+(x_\mu'^2) \varphi(x_\mu') d^4 p d\tau'. \quad (2.16)$$

Its kernel is written as

$$\left\{ \gamma_{1\mu} \left(a_1 P_\mu + i \frac{\partial}{\partial x_\mu} \right) + m_1 \right\} \left\{ \gamma_{2\mu} \left(a_2 P_\mu - i \frac{\partial}{\partial x_\mu} \right) + m_2 \right\} K(x_\mu - x_\mu') \gamma_{1\mu} \gamma_{2\mu} \delta_+(x_\mu'^2),$$

with

$$K(x_\mu) = \frac{i}{(2\pi)^4} \int \frac{e^{-ip_\mu x_\mu} d^4 p}{\{(a_1 P_\mu + p_\mu)^2 - m_1^2\} \{(a_2 P_\mu - p_\mu)^2 - m_2^2\}}, \quad (2.17)$$

which can be expressed in a more closed form convenient to see the asymptotic behaviors of the solutions. Using the expression

$$\{(a_1 P_\mu + p_\mu)^2 - m_1^2\}^{-1} = -i \int_0^\infty e^{i\tau \{(a_1 P_\mu + p_\mu)^2 - m_1^2\}} d\tau,$$

and the same form for particle 2 with a variable s in place of τ , changing variables from τ, s to u, y with $w = u(1+y)/8$ and $s = u(1-y)/8$ and first performing the integrals over p_μ ,⁽¹⁰⁾ we have

$$K(x_\mu) = -\frac{1}{32\pi^2} e^{i(a_1 - a_2)P_\mu x_\mu/2} \int_{-1}^1 e^{iP_\mu x_\mu y/2} k(y) dy, \quad (2.18)$$

$$k(y) = \int_0^\infty e^{-i\zeta(y)u/4 - ix_\mu^2/u} \frac{du}{u} = \begin{cases} -i\pi H_0^{(2)}[(\zeta x_\mu^2)^{1/2}], & (\zeta > 0, x_\mu^2 > 0), \\ 2K_0[(-\zeta x_\mu^2)^{1/2}], & (\zeta x_\mu^2 < 0), \\ i\pi H_0^{(1)}[(\zeta x_\mu^2)^{1/2}], & (\zeta < 0, x_\mu^2 < 0), \end{cases} \quad (2.19)$$

$$\zeta(y) = \frac{1}{4} P^2 y^2 + \frac{1}{2} (m_1^2 - m_2^2) y + \frac{1}{2} (m_1^2 + m_2^2) - \frac{1}{4} P^2. \quad (2.20)$$

For bound states $m_1 + m_2 > P \geq 0$, $\zeta(y)$ is always positive for $1 \geq y \geq -1$ so that in the space-like region, $x_\mu^2 < 0$, $k(y)$ and then $K(x_\mu)$ decrease exponentially as $x_\mu^2 \rightarrow -\infty$ and in the time-like region $K(x_\mu)$ has an oscillatory behavior. On the other hand, for open states $P > m_1 + m_2$, $\zeta(y)$ has two zeros in the range $1 > y > -1$ so that $K(x_\mu)$ has a part which corresponds to the outgoing wave in the space-like region as seen from the asymptotic form of the Hankel function $H^{(1)}$. Such properties of $K(x_\mu - x_\mu')$ will be reflected in the asymptotic behaviors of the solution of equation (2.16). In the case of bound states and $m_1 = m_2 = m$, the following asymptotic expressions for $K(x_\mu)$ are obtained from (2.18), (2.19) and (2.20) by taking $a_1 = a_2 = 1/2$ and the rest system, $P_i = 0$ and $P_4 = P$,

$$K(x_i, t=0) = \begin{cases} \frac{-1}{8\pi^2} \left(\frac{\pi}{2}\right)^{1/2} \frac{e^{-Wr}}{(Wr)^{1/2}} \times \begin{cases} 1, & \left(\frac{8W}{P^2} \gg r \gg \frac{1}{W}\right) \\ \left(\frac{2\pi W}{P^2 r}\right)^{1/2}, & \left(r \gg \frac{8W}{P^2}\right) \end{cases} & \text{for } 2W \gg P, \\ \frac{-1}{8\pi} \frac{e^{-Wr}}{Pr}, & \left(r \gg \frac{1}{W}\right) \text{ for } 2W \ll P, \end{cases} \quad (2.21)$$

$$K(x_i=0, t) = \frac{1}{8\pi^2} \left(\frac{\pi}{2}\right)^{1/2} e^{i\pi/4} \left(\frac{m}{W}\right)^{1/2} \frac{1}{(W|t|)^{3/2}} \left(\cos \frac{Pt}{2} + i \frac{2m}{P} \sin \frac{P|t|}{2}\right) e^{-im|t|}, \quad (|t| \gg 1/W), \quad (2.22)$$

with

$$W = (m^2 - P^2/4)^{1/2}. \quad (2.23)$$

§ 3. Non-relativistic approximation

It will be shown that equation (2.16), in which an inhomogeneous term $\varphi_0(x_\mu)$ is added in the case of open states, is equivalent to the Schrödinger equation with appropriate boundary conditions in the non-relativistic limit, namely, $p/mc \rightarrow 0$ and $r/ci \rightarrow 0$, where p is the relative momentum, $m = m_1$ or m_2 , and r and t are relative distance and time of the two particles. We write 16 components of φ in the form

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}, \quad \begin{aligned} \varphi_1 &= \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}, & \varphi_2 &= \begin{pmatrix} \varphi_{13} & \varphi_{14} \\ \varphi_{23} & \varphi_{24} \end{pmatrix}, \\ \varphi_3 &= \begin{pmatrix} \varphi_{31} & \varphi_{32} \\ \varphi_{41} & \varphi_{42} \end{pmatrix}, & \varphi_4 &= \begin{pmatrix} \varphi_{33} & \varphi_{34} \\ \varphi_{43} & \varphi_{44} \end{pmatrix}, \end{aligned} \quad (3.1)$$

where α and β in $\varphi_{\alpha\beta}$ denote the spinor indices for particles 1 and 2, respectively. Then, the integrand of (2.16) is written as

$$\begin{aligned} & (\gamma_{1\mu} p_{1\mu} + m_1)^{-1} (\gamma_{2\mu} p_{2\mu} + m_2)^{-1} \gamma_{1\nu} \gamma_{2\nu} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \\ &= (p_{1\mu}^2 - m_1^2)^{-1} (p_{2\mu}^2 - m_2^2)^{-1} \left(\begin{aligned} & \left\{ (\not{p}_{14} + m_1)(\not{p}_{24} + m_2) - \mu\nu \right\} \varphi_1 \\ & \left\{ \begin{pmatrix} + & - \\ - & + \end{pmatrix} \right\} \varphi_2 \\ & \left\{ \begin{pmatrix} - & + \\ - & - \end{pmatrix} \right\} \varphi_3 \\ & \left\{ \begin{pmatrix} - & - \\ - & - \end{pmatrix} \right\} \varphi_4 \end{aligned} \right. \\ &+ \{ -\nu(\not{p}_{14} + m_1)(\not{p}_{24} + m_2) + \mu \} \varphi_4 + \{ \pi_1(\not{p}_{24} + m_2) - \pi_2\nu(\not{p}_{14} + m_1) \} \varphi_3 \\ &+ \{ - \begin{pmatrix} + & - \\ - & - \end{pmatrix} \} \varphi_3 + \{ \begin{pmatrix} - & - \\ - & + \end{pmatrix} \} \varphi_4 \\ &+ \{ - \begin{pmatrix} - & + \\ - & + \end{pmatrix} \} \varphi_2 + \{ \begin{pmatrix} + & - \\ - & - \end{pmatrix} \} \varphi_1 \\ &+ \{ - \begin{pmatrix} - & - \\ - & - \end{pmatrix} \} \varphi_1 + \{ \begin{pmatrix} - & - \\ - & - \end{pmatrix} \} \varphi_2 \\ &+ \{ \pi_2(\not{p}_{14} + m_1) - \pi_1\nu(\not{p}_{24} + m_2) \} \varphi_2 \\ &+ \{ \begin{pmatrix} + & - \\ - & - \end{pmatrix} \} \varphi_1 \\ &+ \{ \begin{pmatrix} - & - \\ - & + \end{pmatrix} \} \varphi_4 \\ &+ \{ \begin{pmatrix} - & - \\ - & - \end{pmatrix} \} \varphi_3 \end{aligned} \right), \quad (3.2)$$

with

$$\begin{aligned}\nu &= \sigma_{1i} \sigma_{2i}, \quad \pi_1 = \sigma_{1i} p_{1i}, \quad \pi_2 = \sigma_{2i} p_{2i}, \quad \mu = \pi_1 \pi_2, \\ p_{1\mu} &= a_1 P_\mu + p_\mu, \quad p_{2\mu} = a_2 P_\mu - p_\mu.\end{aligned}\quad (3.3)$$

When the integration over p_i is performed in (2.16), contributions arise from four poles of p_i :

$$\begin{aligned}p_{(1)} &= -a_1 P_4 + (m_1^2 + p_{1i}^2)^{1/2} \simeq (m_1/M) (\beta - P_i^2/2M) + p_{1i}^2/2m_1, \\ p_{(2)} &= -a_1 P_4 - (m_1^2 + p_{1i}^2)^{1/2} \simeq -2m_1, \\ p_{(3)} &= a_2 P_4 - (m_2^2 + p_{2i}^2)^{1/2} \simeq -(m_2/M) (\beta - P_i^2/2M) - p_{2i}^2/2m_2, \\ p_{(4)} &= a_2 P_4 + (m_2^2 + p_{2i}^2)^{1/2} \simeq 2m_2,\end{aligned}\quad (3.4)$$

where we have retained the first two terms only in the expansion of $(m^2 + p^2)^{1/2}$ ($m = m_1$ or m_2 , $p = p_1$ or p_2), used equation (2.13), and taken the usual definition of center of mass

$$a_1 = m_1/M, \quad a_2 = m_2/M, \quad M = m_1 + m_2. \quad (3.5)$$

The residues from each pole should be calculated to the order of $(p/m)^2$ as above and then after some calculations we have by comparing both sides of (2.16)

$$\varphi_2 = 0(p/m) \cdot \varphi_1, \quad \varphi_3 = 0(p/m) \cdot \varphi_1, \quad \varphi_4 = 0(p^2/m^2) \cdot \varphi_1. \quad (3.6)$$

Thus we first obtain an equation for φ_1 only

$$\begin{aligned}\varphi_1(x_\mu) &= \varphi_{01}(x_\mu) - \frac{e_1 e_2}{(2\pi)^3} \int \frac{e^{ip_i(x_i - x_i')}}{\beta + p_i^2/2\mu} d^3p d^3x' \\ &\times \left\{ \int_{-\infty}^{x_4} e^{-ip_{(1)}(x_4 - x_4')} + \int_{x_4}^{\infty} e^{-ip_{(3)}(x_4 - x_4')} \right\} \delta_+(x_\mu'^2) \varphi_1(x_\mu') dx_4',\end{aligned}\quad (3.7)$$

where the denominator of the integrand comes from

$$P_4 - (m_1^2 + p_{1i}^2)^{1/2} - (m_2^2 + p_{2i}^2)^{1/2} \simeq P_4 - M - P_i^2/2M - p_i^2/2\mu \simeq -\beta - p_i^2/2\mu,$$

in which the total momentum P_i has fortunately been separated in this approximation due to the choice (3.5). Next we perform the integration over x_4' and make c in $\delta_+(c^2 t'^2 - r'^2)$ tend to infinity. Contributions from a $I^{(1)}$ part of δ_+ are shown to vanish in this limit and there results an equation

$$\begin{aligned}\varphi_1(x_i, t) &= \varphi_{01}(x_i, t) - \frac{1}{(2\pi)^3} \int \frac{e^{ip_i(x_i - x_i')}}{\beta + p_i^2/2\mu} \frac{e_1 e_2}{r'} \varphi_1(x_i', 0) d^3p d^3x' \\ &\times \begin{cases} e^{-ip_{(1)}t}, & (t \geq 0) \\ e^{-ip_{(3)}t}, & (t \leq 0). \end{cases}\end{aligned}\quad (3.8)$$

If we put $t=0$ in the above equation, we have the equation (2.12). Operating $(-\beta + \Delta/2\mu)$ on both sides of (2.12) we get the Schrödinger equation

$$(-\beta + \Delta/2\mu + e_1 e_2/r) \varphi_1(x_i) = 0. \quad (3.9)$$

Further, putting

$$\varphi_1(x_i) = \sum_{l,m} f_l(r) P_l^m(\cos\theta) e^{im\varphi} \quad (3.10)$$

and separating angular variables (see Appendix A) in equation (2.12) we have for $\beta > 0$ and $\varphi_{01} = 0$

$$f_l(r) = -2e_1 e_2 \mu \left\{ \int_0^r K_{l+1/2}(Wr) I_{l+1/2}(Wr') + \int_r^\infty I_{l+1/2}(Wr) K_{l+1/2}(Wr') \right\} \\ \times (r'/r)^{1/2} f_l(r') dr', \quad W = (2\mu\beta)^{1/2}. \quad (3.11)$$

It can be seen from the asymptotic forms of the modified Bessel functions $K_{l+1/2}$ and $I_{l+1/2}$ that $f_l(r)$ has the asymptotic behaviors required for bound states, that is, $\sim r^l$ for $r \rightarrow 0$, and $\exp(-Wr)$ for $r \rightarrow \infty$. For S -state ($l=0$) equation (3.11) takes a simple form

$$f_0(r) = -e_1 e_2 (\mu/2\beta)^{1/2} r^{-1} \int (e^{-W|r-r'|} - e^{-W(r+r')}) f_0(r') dr', \quad (3.12)$$

whose eigen-functions are well-known Laguerre's polynomials and eigen-values are given by

$$-e_1 e_2 (\mu/2\beta)^{1/2} = n, \quad (n=1, 2, \dots). \quad (3.13)$$

Wave amplitudes for $t \neq 0$ can be obtained from equation (3.8) if their values on $t=0$ have been known. In the rest system we have, for instance, for $t > 0$

$$\varphi_1(x_i, t) = (2\pi)^{-3} \int e^{-i(m_1\beta/M + p_i^2/2m_1)t + ip_i(x_i - x_i')} \varphi_1(x_i', 0) d^3p d^3x' \\ = e^{-it(m_1\beta/M - A/2m_1)} \varphi(x_i, 0).$$

Then, using equation (3.9) we have

$$\varphi_1(x_i, t) = \varphi_1(x_i, 0) \cdot e^{-i(m_1 - m_2)\beta t/M} \times \begin{cases} e^{+ie_1 e_2 \mu t/m_1 r}, & (t > 0), \\ e^{-ie_1 e_2 \mu t/m_2 r}, & (t < 0). \end{cases} \quad (3.14)$$

Such a discontinuity of the wave amplitude arises from the fact that δ_+ in (2.16) has δ -like singularities on the light-cone.

If we take the limit $m_2 \rightarrow \infty$, which means that the particle 2 suffers a negligibly small reaction from the particle 1 and so moves freely, it is shown that equation (2.4) reduces to the Dirac equation for the particle 1 placed in the Coulomb field produced by the particle 2 (see Appendix C).

The above results in this section are not altered at all if we take K_0 , one-body-theoretical propagation function, in place of K_+ in the original equations (2.4) or (2.16). There we have only to consider that P has a small positive imaginary part instead of m_1 and m_2 .

Further, it must be noticed that the reduction of equation (2.16) to the non-relativistic one (2.12) is valid in the asymptotic limit $e_1 e_2 / \hbar c \rightarrow 0$ in view of the nature of approximation, namely, expansion in powers of p/mc and r/ct , so that general equation (2.16) will not always have solutions for finite $e_1 e_2 / \hbar c$, though equation (2.12) has solutions. This problem will be discussed in more detail in Secs. 6 and 7.

§ 4. Scalar particles with scalar photon interaction

Since the integral equation (2.11) for two spinor particles is rather complicated to deal with in a general relativistic case, we shall first study the case of two scalar particles interacting through a scalar photon field. Here interaction Hamiltonian is of the form

$$H' = g_1 \psi^* \psi A + g_2 \phi^* \phi A, \quad (4.1)$$

where ψ and ϕ are field variables for the two particles and A is that for scalar photon field. Then, following the same procedure as in Sec. 2 we have

$$\phi(1, 2) = \int I_+(1, 1'') I_+(2, 2'') G(1'', 2''; 1', 2') \phi(1', 2') d\tau_1'' d\tau_2'' d\tau_1' d\tau_2', \quad (4.2)$$

$$G(1'', 2''; 1', 2') = -ig_1 g_2 \delta_+(1'', 2'') \delta(1'', 1') \delta(2'', 2') + \text{higher terms in } g_1 g_2, \quad (4.3)$$

where the inhomogeneous part has been omitted for bound states from the same reason. This equation can also be obtained on the rigorous ground of the field theory after the procedure of Gell-Mann and Low.⁵⁾

It is apparent that the results on the general properties stated in Sec. 2 are valid unaltered in this case. We shall in the following limit ourselves to the ladder approximation. Then, separating the center of mass motion with (2.5) we have

$$\varphi(x_\mu) = \frac{ik}{(2\pi)^4} \int \frac{e^{-i\phi \cdot (x-x')}}{\{(a_1 P_\mu + p_\mu)^2 - m_1^2\} \{(a_2 l'_\mu - p_\mu)^2 - m_2^2\}} \delta_+(x_\mu'^2) \varphi(x_\mu') d^4 p d\tau', \quad (4.5)$$

$$k = 4m_1 m_2 e_1 e_2, \quad (4.6)$$

where we have put $g_1 = 2m_1 e_1$ and $g_2 = 2m_2 e_2$, e_1 and e_2 being dimensionless.

If we take the non-relativistic limit in (4.5) in the same way as in Sec. 3, the same expressions including the numerical constants are obtained and the rest mass is given by (3.13), that is,

$$P = m_1 + m_2 - \beta = m_1 + m_2 - \mu a^2 / 2n^2, \quad (n=1, 2, \dots), \quad (4.7)$$

$$a = -e_1 e_2. \quad (4.8)$$

Further, in the limit $m_2 \rightarrow \infty$ we have the following Klein-Gordon equation for the particle 1 in the coordinate system where particle 2 is at rest on the origin (see Appendix C):

$$(\square - m_1^2) \psi(1) = 2m_1 e_1 e_2 / r \cdot \psi(1), \quad (4.9)$$

and then eigen-values are given by

$$P = m_2 + m_1 (1 - a^2 / n^2)^{1/2}, \quad (n=1, 2, \dots). \quad (4.10)$$

These approximate values will later be compared with the results which are relativistically obtained.

In the general relativistic case it is possible to separate angular variables in equation (4.5) in the rest system. Putting

$$\varphi(x_\mu) = \sum_{l, m} f_l(r, t) P_l^m(\cos \theta) e^{im\varphi}, \quad (4.11)$$

we have (see Appendix A)

$$f_l(r, t) = \frac{ik}{2\pi} \int_0^\infty r'^2 dr' \int_{-\infty}^\infty dt' \int_0^\infty p dp \int_{-\infty}^\infty p_4 dp_4 \\ \times \frac{e^{-iJ_4(t-t')} J_{l+1/2}(pr) J_{l+1/2}(pr')}{\{(a_1 P + p_4)^2 - p^2 - m_1^2\} \{(a_2 P - p_4)^2 - p^2 - m_2^2\} (rr')^{1/2}} \\ \times \delta_+(t'^2 - r'^2) f_l(r', t'). \quad (4.12)$$

In the following we shall study the case $l=0$ (S -state) :

$$f(r, t) = \frac{ik}{2\pi^2} \int_0^\infty dr' \int_{-\infty}^\infty dt' \int_{-\infty}^\infty dp \cdot dp_4 \frac{e^{-iJ_4(t-t')} \sin pr \sin pr' \delta_+(t'^2 - r'^2) f(r', t')}{\{(a_1 P + p_4)^2 - p^2 - m_1^2\} \{(a_2 P - p_4)^2 - p^2 - m_2^2\}}, \quad (4.13)$$

where we have put $rf_0(r, t) = f(r, t)$. In view of the difficulty in solving such a two-dimensional integral equation analytically, we shall at the present stage reduce it to one-dimensional equations by replacing δ_+ by other invariant functions :

case (i)

$$\delta_+ - 2\pi i D^{(1)} - 2\pi D = \delta(r-t)/r, \quad (4.14a)$$

and case (ii)

$$\delta_+ - 2\pi i D^{(1)} = 4\pi \bar{D} = \{\delta(r-t) + \delta(r+t)\}/2r, \quad (4.14b)$$

where $D^{(1)}$, D , and \bar{D} are defined by Schwinger.¹⁰⁾ If we further treat the particles according to the one-body-theory, namely, replace I_+ by I_0 and then consider P as having a small positive imaginary part, it is shown in the following that explicit solutions of (4.13) can be obtained. The above replacements of δ_+ and I_+ do not alter the non-relativistic equation as seen in its reduction in Sec. 3. In the general relativistic case, however, we shall have a departure from the usual field theory, and especially in the case (i) the two particles are treated in an unsymmetrical way in time with respect to their interactions. Nevertheless, these explicit solutions will be useful to see the characteristic features of the relativistic integral equations so far considered, and may also serve as some relativistic models which embody the ideas that elementary particles are constructed of bare particles.

The case (i) An equation corresponding to (4.13) becomes after integration over p_4

$$f(r, t) = -\frac{k}{4\pi P} \int_0^\infty dr' \int_{-\infty}^\infty dp \left[\int_{-\infty}^t \left\{ \frac{e^{-i(-a_1 P + E_1)(t-t')}}{E_1(E_1 - Q_1/2)} + \frac{e^{-i(-a_1 P - E_1)(t-t')}}{E_1(E_1 + Q_1/2)} \right\} \right. \\ \left. + \int_t^\infty \left\{ \frac{e^{-i(a_2 P - E_2)(t-t')}}{E_2(E_2 - Q_2/2)} + \frac{e^{-i(a_2 P + E_2)(t-t')}}{E_2(E_2 + Q_2/2)} \right\} \right] dt' \sin pr \sin pr' \frac{\delta(r'-t')}{r'} f(r', t'), \quad (4.15)$$

$$Q_1 = P + (m_1^2 - m_2^2)/P, \quad Q_2 = P + (m_2^2 - m_1^2)/P, \quad (4.16)$$

$$E_1 = (p^2 + m_1^2)^{1/2}, \quad E_2 = (p^2 + m_2^2)^{1/2}, \quad (4.17)$$

If we change variables from p to x and from r, t to u, v

$$x = p - E_1 \text{ or } p - E_2, \quad (4.18)$$

$$u = (r + t)/2, \quad v = (r - t)/2, \quad (4.19)$$

and express the integral over p as, for instance,

$$\int_{-\infty}^{\infty} \frac{e^{iE_1(t-t')} \sin pr \sin pr'}{E_1(E_1 + Q_1/2)} dp = \int_{-\infty}^0 \frac{dx}{x^2 - Q_1 x + m_1^2} \{ e^{-ix(u-u') + im_1^2(r-t')/x} \\ - e^{ix(v+u') - im_1^2(u+v')/x} \},$$

the above equation becomes

$$f(u, v) = -\frac{k}{4\pi P} \left[\int_0^{u-v} du' e^{ia_1 P(u-u'-v)} \int_{-\infty}^{\infty} dx \frac{S_1(x; u, u', v)}{(x - Q_1/2)^2 + W^2} \right. \\ \left. + \int_{u-v}^{\infty} du' e^{-ia_2 P(u-u'-v)} \int_{-\infty}^{\infty} dx \frac{S_2(x; u, u', v)}{(x + Q_2/2)^2 + W^2} \right] \frac{f(u', 0)}{u'} \quad \text{for } u \geq v, \quad (4.20a)$$

$$f(u, v) = -\frac{k}{4\pi P} \left[\int_0^{\infty} du' e^{-ia_2 P(u-u'-v)} \int_{-\infty}^{\infty} dx \frac{S_2(x; u, u', v)}{(x + Q_2/2)^2 + W^2} \right] \frac{f(u', 0)}{u'} \\ \text{for } u \leq v, \quad (4.20b)$$

$$S_i(x; u, u', v) = e^{-ix(u-u') + im_i^2 v/x} - e^{ix(v+u') - im_i^2 u/x}, \quad (i=1, 2), \quad (4.20')$$

with

$$W^2 = m_1^2 - Q_1^2/4 = m_2^2 - Q_2^2/4 = \{(m_1 + m_2)^2 - P^2\} \{P^2 - (m_1 - m_2)^2\} / 4P^2, \quad (4.21)$$

where in the integral over x the principal value must be taken at $x=0$ and it is noted that W^2 is positive for bound states, $m_1 + m_2 > P > |m_1 - m_2|$. Putting $v=0$ in (4.20a) we have an one-dimensional equation on the light-cone, in which integration over x can easily be performed by taking the path as shown in Fig. 2,

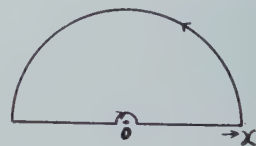


Fig. 2

$$f(u, 0) = -\frac{k}{4PW} \left[\int_0^u du' e^{i(a_1 P - Q_1/2)(u-u')} \{ e^{-iW(u-u')} - e^{-iW(u+u')} \} \right. \\ \left. + \int_u^{\infty} du' e^{-i(a_2 P - Q_2/2)(u-u')} \{ e^{-iW(u'-u)} - e^{-iW(u+u')} \} \right] \frac{f(u', 0)}{u'}. \quad (4.22)$$

This expression can be simplified by choosing a_1 and a_2 as

$$a_1 = Q_1/2P, \quad a_2 = Q_2/2P, \quad (4.23)$$

since eigen-values are independent of α_1 and α_2 as shown in Sec. 2. Indeed, we have the same result if we put $f(u, 0) = \exp(ibPu) \cdot f'(u, 0)$ with $b = \alpha_1 - Q_1/2P = -\alpha_2 + Q_2/2P$. If we express (4.23) in power series of binding energy, $\beta^2 = M - P$, we have

$$\alpha_i = (m_i/M) (1 - \beta/m_i + \beta^2/2m_iM) (1 - \beta/M)^{-2}, \quad (i=1, 2), \quad (4.24)$$

so that they agree with the usual definition (3.5) in the non-relativistic limit $\beta \rightarrow 0$. With (4.23) equation (4.22) has the same form as (3.12), and then eigen-values are given by

$$-k/4PW = n, \quad (n=1, 2, \dots). \quad (4.25)$$

Solving this equation for P by the use of (4.21) we obtain

$$P = \begin{cases} \pm (m_1 + m_2) \{1 - 2m_1m_2(m_1 + m_2)^{-2}\xi\}^{1/2}, \\ \pm (m_1 - m_2) \{1 + 2m_1m_2(m_1 - m_2)^{-2}\xi\}^{1/2}, \end{cases} \quad (4.26)$$

$$\xi = 1 - (1 - u^2/n^2)^{1/2}, \quad (n=1, 2, \dots), \quad (4.27)$$

where u is given by (4.8). The four solutions of P correspond to the cases where free particles 1 and 2 have energies $\pm(p_1^2 + m_1^2)^{1/2}$ and $\pm(p_2^2 + m_2^2)^{1/2}$, and the appearance of negative energies is due to the fact that we have treated the particles one-body-theoretically. In the following we shall consider the first expression of (4.26) only. We have for W which determines the asymptotic behavior of $f(u, 0)$ at large u

$$W = (\mu u/n) (1 - 2\mu\xi/M)^{-1/2}, \quad (n=1, 2, \dots). \quad (4.28)$$

In the case $u/n > 1$, P and W become complex, which will mean that stable bound states are impossible. We have from (4.26) for $u \ll 1$ or $m_1/m_2 \ll 1$

$$P = M - \mu\xi - \frac{1}{2} \frac{(\mu\xi)^2}{M} + \dots, \quad \xi = \frac{1}{2} \left(\frac{u}{n}\right)^2 + \frac{1}{8} \left(\frac{u}{n}\right)^4 + \dots, \quad (u \ll 1), \quad (4.29)$$

$$P = m_2 + m_1(1 - \xi) + m_2 \left(\frac{m_1}{m_2}\right)^2 \left(2\xi - \frac{1}{2}\xi^2\right) - \dots, \quad \left(\frac{m_1}{m_2} \ll 1\right), \quad (4.30)$$

which will be compared with the approximate results (4.7) and (4.10). In calculating $f(u, v)$ for $v \neq 0$ from (4.20a) and (4.20b) it is not easy to integrate first over x since the path as shown in Fig. 2 is not possible in general. However, if we insert the solution of (4.22), for instance, for $n=1$

$$f(u', 0) = u' e^{-Wu'} \quad (4.31)$$

and integrate first over u' , the integration over x along the path is found to be possible in the case $u \geq 0$ and $v \geq 0$, namely, in the space-like region. There appear integrations of the following type

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{-ixu + im^2v/x} dx}{(x - Q/2 + iW)^2 (x - Q/2 - iW)} &= \frac{\pi}{iW} e^{-iQ(u-v)/2 - W(u+v)} \\ &\times \left\{ u + \frac{(Q/2 + iW)^2}{m^2} v + \frac{1}{2W} \right\}, \end{aligned}$$

$$\int_{-\infty}^{\infty} \frac{e^{ixv - im^2 u/x} dx}{(x - Q/2 + iW)^2 (x - Q/2 - iW)} = \frac{\pi}{2iW^2} e^{-iQ(u-v)/2 - W(u+v)},$$

and the result is

$$f(u, v) = \left\{ u + \frac{(Q_2/2 - iW)^2}{m_2^2} v \right\} e^{-W(u+v)}, \quad (u \geq 0, v \geq 0). \quad (4.32)$$

Values of $f(u, v)$ in the time-like region is not easily obtained, but they are expected from the results (2.17) to (2.22) to have an oscillatory character similar to Bessel functions.

The case (ii) Equations corresponding to (4.20a) and (4.20b) are with (4.20')

$$\begin{aligned} f(u, v) = & -\frac{k}{8\pi P} \left[\int_0^{u-v} du' e^{iQ_1(u-u'-v)/2} \int_{-\infty}^{\infty} dx \frac{S_1(x; u, u', v)}{(x - Q_1/2)^2 + W^2} \right. \\ & + \int_{u-v}^{\infty} du' e^{-iQ_2(u-u'-v)/2} \int_{-\infty}^{\infty} dx \frac{S_2(x; u, u', v)}{(x + Q_2/2)^2 + W^2} \left. \right] \frac{f(u', 0)}{u'} \\ & - \frac{k}{8\pi P} \left[\int_0^{\infty} dv' e^{iQ_1(u-v+v')/2} \int_{-\infty}^{\infty} dx \frac{S_1(x; v, v', u)}{(x + Q_1/2)^2 + W^2} \right] \frac{f(0, v')}{v'} \\ & \text{for } u \geq v, \end{aligned} \quad (4.33a)$$

$$\begin{aligned} f(u, v) = & \{ \text{an expression which is obtained from the right side of (4.33a)} \\ & \text{by the simultaneous exchanges: suffices } 1 \longleftrightarrow 2, u \longleftrightarrow v, u' \longleftrightarrow v' \text{ and} \\ & f(u', 0) \longleftrightarrow f(0, v') \} \quad \text{for } u \leq v, \end{aligned} \quad (4.33b)$$

where we have used (4.23). If we put $v=0$ in (4.33a) and $u=0$ in (4.33b) we have a system of two integral equations for $f(u, 0)$ and $f(0, v)$. In view of the symmetry of these equations for u and v , we can make use of the result obtained in the case (i) to solve them. Assuming the solution, for instance, (4.31) for $n=1$

$$f(u, 0) = \alpha u e^{-Wu}, \quad f(0, v) = \beta v e^{-Wv}, \quad (4.34)$$

where α and β are undermined constants, and inserting them into the integrands of (4.33a) and (4.33b), we can calculate $f(u, v)$ in the same way as (4.32) is obtained. The result becomes

$$\begin{aligned} f(u, v) = & -\frac{k}{8PW} \left[\alpha \left\{ u + \frac{(Q_2/2 - iW)^2}{m_2^2} v \right\} \right. \\ & + \beta \left\{ v + \frac{(Q_1/2 - iW)^2}{m_1^2} u \right\} \left. \right] e^{-W(u+v)}, \quad (u \geq 0, v \geq 0). \end{aligned} \quad (4.35)$$

From the condition that (4.35) must be equal to (4.34) for $v=0$ and $u=0$, we have first

$$\alpha/\beta = \pm m_2(Q_1/2 - iW)/m_1(Q_2/2 - iW), \quad (4.36)$$

where the case of minus sign will be discarded since it has no correspondence to the non-relativistic case, and then an equation which determines the eigen-values

$$-(k/8PW)\{1+(Q_1/2-iW)(Q_2/2-iW)/m_1m_2\}=1: \quad (4.37)$$

In the general case $u \neq 1$ similar results are obtained as shown in Appendix D. With (4.21) we can solve (4.37) as follows

$$P=M\{1-4\mu\eta^2/M(1+\eta^2)\}^{1/2}, \quad (4.38)$$

$$W=2\mu\eta(1+\eta^2)^{-1}\{1-4\mu\eta^2/M(1+\eta^2)\}^{1/2}, \quad (4.39)$$

$$\eta=u/(2+ia), \quad (4.40)$$

where we have considered only the case of plus sign of P . It is found that P and W are complex numbers with positive and negative imaginary parts, respectively. Nevertheless, the calculations given above in the case (i) are valid also in this case since it is shown that

$$R_e(W \pm iQ_1/2) > 0, \quad R_e(W \pm iQ_2/2) > 0.$$

For $u \ll 1$, (4.38) and (4.39) are written as

$$\begin{aligned} P/M = & 1 - \mu u^2/2M + (\mu u^4/2M)(1 - \mu/4M) + O(u^6) \\ & + i(\mu u^3/2M)\{1 - u^2(1 - \mu/2M) + O(u^4)\}, \end{aligned} \quad (4.41)$$

$$\begin{aligned} W/\mu u = & 1 - (u^2/2)(1 - \mu/M) + O(u^4) - i(u/2)\{1 - u^2(1 - 3\mu/2M) + O(u^4)\}, \\ & (4.42) \end{aligned}$$

which agree with (4.7) in the non-relativistic limit $u \rightarrow 0$ as it should be. Now the wave amplitude for center of mass motion, $\exp(-iPT)$, increases exponentially with T . This will mean that the system is unstable in a similar way that P given by (4.26) becomes complex if $u/n > 1$, and would presumably result partly from the fact that we have used the uncausal propagation functions I_0 and $4\pi\bar{I}$ instead of I_+ and δ_+ as discussed by Stückelberg⁶⁾ and Fierz⁷⁾ and partly from that the kernel of the integral equation (4.5) in the ladder approximation is not Hermitian. On the other hand, in the case (i) equation (4.22) is found to be Hermitian and eigen-values are real. It will be noted that the result of the separation of center of mass motion from equation (4.2) is not altered in the case where P has a positive imaginary part if we take the rest system initially, since the regions of integrations for t_1' and t_2' are from $-\infty$ to finite times t_1 and t_2 , corresponding to the definition of I_0 which originally means that P has a *small* positive imaginary part.

§ 5. Solutions in a differential form

It is shown that solutions in the case (i) in the preceding section are able to be obtained in a more direct way by solving a differential equation on the light-cone which is reduced from the integral equation. We have from (4.5) a differential equation in four dimensions

$$\{(a_1 P_\mu + i \partial/\partial x_\mu)^2 - m_1^2\} \{(a_2 P_\mu - i \partial/\partial x_\mu)^2 - m_2^2\} \varphi(x_\mu) = ik\delta_+(x_\mu^2) \varphi(x_\mu). \quad (5.1)$$

Separating the angular variables with (4.11) in the rest system, we have for the S -state a differential equation corresponding to (4.13)

$$L_1 L_2 f(r, t) = ik \partial(r^2 - t^2) f(r, t), \quad (5.2)$$

$$L_1 = (a_1 P + i \partial/\partial t)^2 + \partial^2/\partial r^2 - m_1^2, \quad L_2 = (a_2 P - i \partial/\partial t)^2 + \partial^2/\partial r^2 - m_2^2, \quad (5.3)$$

where ∂_+ has been replaced by (4.14b), the case of (4.14a) being treated in the same way. It is convenient to change variables from r, t to u, v defined by (4.19). Then, the equation can be written as

$$L_1 L_2 f(u, v) = ik \{ \partial(u)/4v + \partial(v)/4u \} f(u, v), \quad (5.4)$$

$$L_j = M_j(u) \partial/\partial v + N_j(u) = M_j^*(v) \partial/\partial u + N_j^*(v), \quad (j=1, 2), \quad (5.5)$$

with

$$M_1(x) = \partial/\partial x - ia_1 P, \quad M_2(x) = \partial/\partial x + ia_2 P, \\ N_1(x) = ia_1 P \partial/\partial x + a_1^2 P^2 - m_1^2, \quad N_2(x) = -ia_2 P \partial/\partial x + a_2^2 P^2 - m_2^2, \quad (5.6)$$

where $*$ means complex conjugate. We have from (5.4) three free equations in the regions I, II, and III, shown in Fig. 3, separated by the light-cone $u=0$ and $v=0$

$$L_i L_2 f^{(i)} = 0, \quad (i=I, II, III), \quad (5.7)$$

and the following continuity conditions for them on the lines $u=0$ and $v=0$. For instance, we consider the condition on $v=0$. Note that $L_1 L_2$ can be expressed as

$$L_1 L_2 = M_1(u) M_2(u) \partial^2/\partial v^2 + \{ M_2(u) N_1(u) \\ + M_1(u) N_2(u) \} \partial/\partial v + N_1(u) N_2(u). \quad (5.8)$$

Then, under the assumption that $f(u, 0)$ is a continuous function of u , the term $\partial(v)$ in (5.4) means that $\partial f/\partial v$ is discontinuous on $v=0$, and by integrating both sides of (5.4) from $v=-\varepsilon$ to $+\varepsilon$ for constant u we have

$$[M_1(u) M_2(u) \partial f/\partial v]_{v=+\varepsilon} - [M_1(u) M_2(u) \partial f/\partial v]_{v=-\varepsilon} = ik/4u \cdot [f]_{v=0}, \quad (5.9)$$

$$[f]_{v=+\varepsilon} = [f]_{v=-\varepsilon}. \quad (5.10)$$

On the other hand, (5.7) can be rewritten as

$$f^{(i)} = f_1^{(i)} + f_2^{(i)}, \quad (5.11)$$

$$L_1 f_1^{(i)} = L_2 f_2^{(i)} = 0, \quad (i=I, II, III), \quad (5.12)$$

where the part of $f^{(i)}$ which satisfy $L_1 f^{(i)} = 0$ and $L_2 f^{(i)} = 0$ simultaneously will be included in either $f_1^{(i)}$ or $f_2^{(i)}$. Then, (5.9) can be written as

$$(M_2 N_1 - M_1 N_2) (f_1^{(III)} - f_1^{(I)}) = ik/4u \cdot (f_1^{(I)} + f_2^{(I)}), \quad (v=0), \quad (5.13)$$

with

$$M_2 N_1 - M_1 N_2 = -iP \partial^2/\partial u^2 + \{ -P^2 (a_1^2 - a_2^2) + m_1^2 - m_2^2 \} \partial/\partial u \\ + iP (a_2 m_1^2 + a_1 m_2^2 - a_1 a_2 I^2). \quad (5.14)$$

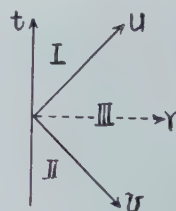


Fig. 3

It will be convenient to choose a_1 and a_2 such that the coefficient of $\partial/\partial u$ in (5.14) vanishes. Then, we have from $a_1 + a_2 = 1$ just the equation (4.23). In this case (5.13) becomes

$$\begin{aligned} (\partial^2/\partial u^2 - W^2)(f_1^{(I)} - f_1^{(III)}) &= kf/4Pu, \\ f &= f_1^{(I)} + f_2^{(I)} = f_1^{(III)} + f_2^{(III)}, \quad (v=0), \end{aligned} \quad (5.15a)$$

where W is given by (4.21). In the same way we have the continuity equations on the line $u=0$

$$\begin{aligned} (\partial^2/\partial v^2 - W^2)(f_1^{(III)} - f_1^{(II)}) &= kf/4Pv, \\ f &= f_1^{(III)} + f_2^{(III)} = f_1^{(II)} + f_2^{(II)}, \quad (u=0). \end{aligned} \quad (5.15b)$$

In order to solve the above equations together with (5.12), boundary conditions, which are included in the original integral equation, are necessary.

In the case of the one-sheet potential (4.14a) these boundary conditions are found to be sufficiently simple. To see this it is convenient to write the equation corresponding to (4.2) as

$$(\square_2 - m_2^2)\psi(1, 2) = ik \int I_0(1, 1') \frac{1}{r(1', 2)} \delta\{r(1', 2) - t_1' + t_2\} \psi(1', 2) d\tau_1', \quad (5.16)$$

and note that the right hand side vanishes if 1 and 2 lie in the space-like region, and in the time-like region where $t = t_1 - t_2 > 0$, since $I_0(1, 1')$ is different from zero only in the time-like region where $t_1 \geq t_1'$. Then, we have

$$L_2 f^{(III)} = L_2 f^{(II)} = 0. \quad (5.17a)$$

In the same way we have

$$L_1 f^{(I)} = 0. \quad (5.17b)$$

In this case we have the continuity equations on the line $v=0$ alone

$$\begin{aligned} (\partial^2/\partial u^2 - W^2)(f_1^{(I)} - f_1^{(III)}) &= kf/2Pu, \\ f &= f_1^{(I)} + f_2^{(I)} = f_1^{(III)} + f_2^{(III)}, \quad (v=0). \end{aligned} \quad (5.18)$$

From equation (5.12), (5.17a), and (5.17b) we have

$$L_1 f_1^{(III)} = L_2 f_1^{(III)}, \quad L_1 f_2^{(I)} = L_2 f_2^{(I)} = 0, \quad (5.19)$$

and then by the use of equation (5.5) and (5.6) it is easily shown that

$$(\partial^2/\partial u^2 - W^2)f_1^{(III)} = (\partial^2/\partial u^2 - W^2)f_2^{(I)} = 0. \quad (5.20)$$

Hence, equation (5.18) reduces to an equation of Schrödinger type

$$(\partial^2/\partial u^2 - W^2)f = kf/2Pu, \quad (v=0), \quad (5.21)$$

which is just the differential form of (4.22). To solve equation (5.21) we must further impose on it a boundary condition that $f(u)$ vanishes for large u , which is contained in the integral equation.

§ 6. Spinor particles with electromagnetic interaction

We start from the equation (2.16), where the center of mass coordinates are separated already, and choose the rest system for simplicity. We classify the wave amplitude according to the two energy states of the both particles as in (3.1). Each $\varphi_i (i=1, 2, 3, 4)$ has four components and can be expanded as follows, by using the four independent spin eigen-functions (I), (II), (III) and (IV) given in Appendix A.

$$\varphi_i(x_\mu) = A_i(r, t) \text{ (I)} + B_i(r, t) \text{ (II)} + C_i(r, t) \text{ (III)} + D_i(r, t) \text{ (IV)}, \quad (i=1, 2, 3, 4). \quad (6.1)$$

Here, (I), (III), (IV) correspond to three triplet states and (II) corresponds to singlet one. Substituting (6.1) in (3.2) and (2.16), performing the angular integrations and comparing the factor of the spin eigen-functions on both sides of (2.16), we get the following integral equations for the radial functions for the given total angular momentum j , (detailed calculations are found in Appendix A).

$$\begin{aligned} A_1 = & \frac{ie_1e_2}{\pi^2} \int \dots \int \frac{dp_4 p^2 dp dt' r'^2 dr'}{(p_{1\mu}^2 - m_1^2)(p_{2\mu}^2 - m_2^2)} \delta_+(x_\mu'^2) e^{-ip_4(t-t')} \\ & \times \left[\left\{ (\omega_1 + m_1)(\omega_2 + m_2) + \frac{p^2}{2j+1} \right\} F_{-1/2, -1/2}(A_1' - A_4') \right. \\ & + \frac{2(j+1)p^2}{2j+1} F_{-1/2, 3/2}(D_1' - D_4') \\ & + \frac{ip(\omega_1 + m_1)}{2j+1} F_{-1/2, 1/2} \{B_2' + 3B_3' - (j+1)(C_2' - C_3')\} \\ & \left. + \frac{ip(\omega_2 + m_2)}{2j+1} F_{-1/2, 1/2} \{3B_2' + B_3' - (j+1)(C_2' - C_3')\} \right], \\ D_1 = & \frac{ie_1e_2}{\pi^2} \int \dots \int \frac{dp_4 p^2 dp dt' r'^2 dr'}{(p_{1\mu}^2 - m_1^2)(p_{2\mu}^2 - m_2^2)} \delta_+(x_\mu'^2) e^{-ip_4(t-t')} \\ & \times \left[\frac{2jp^2}{2j+1} F_{3/2, -1/2}(A_1' - A_4') \right. \\ & + \left\{ (\omega_1 + m_1)(\omega_2 + m_2) - \frac{p^2}{2j+1} \right\} F_{3/2, 3/2}(D_1' - D_4') \\ & - \frac{ip(\omega_1 + m_1)}{2j+1} F_{3/2, 1/2} \{B_2' + 3B_3' + j(C_2' - C_3')\} \\ & \left. - \frac{ip(\omega_2 + m_2)}{2j+1} F_{3/2, 1/2} \{3B_2' + B_3' + j(C_2' - C_3')\} \right], \quad (6.2) \\ B_2 = & \frac{ic_1c_2}{\pi^2} \int \dots \int \frac{dp_4 p^2 dp dt' r'^2 dr'}{(p_{1\mu}^2 - m_1^2)(p_{2\mu}^2 - m_2^2)} \delta_+(x_\mu'^2) e^{-ip_4(t-t')} \\ & \times [ijp(\omega_2 - m_2 - \omega_1 - m_1) F_{1/2, -1/2}(A_1' - A_4') \\ & + i(j+1)p(\omega_1 + m_1 - \omega_2 + m_2) F_{1/2, 3/2}(D_1' - D_4') \\ & + (\omega_1 + m_1)(\omega_1 - m_2) F_{1/2, 1/2}(B_2' + 3B_3') + p^2 F_{1/2, 1/2}(3B_2' + B_3')], \end{aligned}$$

$$\begin{aligned}
 C_2 = & \frac{ie_1e_2}{\pi^2} \int \dots \int \frac{d^3p_4 p^2 dp dt' r'^2 dr'}{(p_{1\mu}^2 - m_1^2)(p_{2\mu}^2 - m_2^2)} \delta_+(x_{\mu}^{\prime 2}) e^{-ip_4(t-t')} \\
 & \times [ip(\omega_1 + m_1 + \omega_2 - m_2) F_{1/2, -1/2}(A_1' - A_4') \\
 & + ip(\omega_1 + m_1 + \omega_2 - m_2) F_{1/2, 3/2}(D_1' - D_4') \\
 & + \{(\omega_1 + m_1)(\omega_2 - m_2) + p^2\} F_{1/2, 1/2}(C_2' - C_3')],
 \end{aligned}$$

with other four equations denoted by $(6.2)'$ which are given from (6.2) by the operation :

“Exchanges of the suffices of the wave amplitudes 1 and 4 (e.g. $A_1 \longleftrightarrow A_4$) ; 2 and 3, respectively, with simultaneous changes of the signs of m_1 and m_2 ($m_1 \longleftrightarrow -m_1$, $m_2 \longleftrightarrow -m_2$).”

In the above

$$\omega_1 = a_1 P + p_4, \quad \omega_2 = a_2 P - p_4, \quad (6.3)$$

$$A_1' = A_1(r', t'), \text{ etc.}, \quad A_1 = A_1(r, t), \text{ etc.},$$

and $F_{m,n}$ abbreviates the following function :

$$F_{m,n} \equiv F_{m,n}(pr, pr') = (\pi/2pr)^{1/2} J_{j+n}(pr) (\pi/2pr')^{1/2} J_{j+n}(pr'). \quad (6.4)$$

It should be noted that, on account of the property of the spatial parity, only 8 of 16 functions are coupled together. Other set of equations are given from (6.2) and $(6.2)'$ by the operation :

“Exchanges of the suffices of the wave amplitudes 1 and 3 ; 2 and 4, respectively, with simultaneous change of the sign of m_1 .”

In the case of $j=0$, we are left with only four equations for B and D , namely, the second and the third ones in (6.2) and $(6.2)'$ with $A_1 = A_4 = C_2 = C_3 = 0$.

Further we observe that in the right of (6.2) triplet functions A , C and D appear always in the form $A_1 - A_4$ etc., hence we can split the system of eight equations into two parts, the first part involves only five functions (6.5) below and includes also the eigen-value problem, the second part merely expresses the other three functions as the functions of (6.5) .^{*} We shall consider the former alone in the following discussions.

$$\begin{aligned}
 A_1 - A_4 & \equiv f_1, \quad D_1 - D_4 \equiv f_2, \quad B_2 - B_3 \equiv f_3, \\
 B_2 + B_3 & \equiv f_4, \quad C_2 - C_3 \equiv f_5.
 \end{aligned} \quad (6.5)$$

Hereafter we consider the case of Λ_0 . We observe from (6.2) that the kernels are singular on the light-cone in contrast with that of scalar case as shown in (4.12) . After the p_4 -integration the highest singular terms are found to be of the form

$$\int_0^\infty p dp J_{j+1/2}(pr) J_{j+1/2}(pr') \cos E(t-t'), \quad \int_0^\infty p^2 dp J_{j+3/2}(pr) J_{j+1/2}(pr') \frac{\sin E(t-t')}{E}, \quad (6.6)$$

^{*} This reduction of the number of equations is not possible in the case of the other interactions, for instance, scalar or pseudo-scalar meson interactions.

which have the δ -like singularities on the light-cone $(t-t')^2 = (r-r')^2$. To make more clear these points, we take the appropriate linear combinations of (6.5) so as to eliminate these singular terms, and then get the following three equations (6.7) which are free from singularities and two others (6.8) which contain the singular kernels given by (6.10) below.

$$\begin{aligned}
 f_1 - f_2 = & \frac{u}{2} \iint_{(r-r')^2 \geq (t-t')^2} dt' r'^2 dr' \delta_+(x_\mu'^2) \frac{1}{\sqrt{rr'}} \left[\left\{ \frac{P^2 - m^2}{P} G_{-1/2, -1/2} \right. \right. \\
 & + \frac{4jW^2}{(2j+1)P} (G_{-1/2, -1/2} - G_{3/2, -1/2}) \left. \right\} f_1' \\
 & - \left\{ \frac{P^2 - m^2}{P} G_{3/2, 3/2} + \frac{4(j+1)W^2}{(2j+1)P} (G_{3/2, 3/2} - G_{-1/2, 3/2}) \right\} f_2' \\
 & + \frac{2iVM}{(2j+1)P^2} (G_{3/2, 1/2} - G_{-1/2, 1/2}) (2Pf_4' - mf_3') \\
 & + \frac{2iW}{(2j+1)} \left\{ jG_{3/2, 1/2} + (j+1)G_{-1/2, 1/2} \right\} f_5' \left. \right] + R_1, \quad (6.7a)
 \end{aligned}$$

$$\begin{aligned}
 f_4 = & \frac{u}{2} \iint_{(r-r')^2 \geq (t-t')^2} dt' r'^2 dr' \delta_+(x_\mu'^2) \frac{1}{\sqrt{rr'}} \left[-\frac{2ijMW}{P} G_{1/2, -1/2} f_1' \right. \\
 & - \frac{2i(j+1)MW}{P} G_{1/2, 3/2} f_2' + \frac{(P^2 - M^2)}{P^2} G_{1/2, 1/2} (2Pf_4' - mf_3') \left. \right] + R_2, \quad (6.7b)
 \end{aligned}$$

$$\begin{aligned}
 f_5 = & \frac{u}{2} \iint_{(r-r')^2 \geq (t-t')^2} dt' r'^2 dr' \delta_+(x_\mu'^2) \frac{1}{\sqrt{rr'}} \left[2iWG_{1/2, -1/2} f_1' \right. \\
 & - 2iWG_{1/2, 3/2} f_2' + \frac{(P^2 - M^2)}{P} G_{1/2, 1/2} f_5' \left. \right] + R_3, \quad (6.7c)
 \end{aligned}$$

$$\begin{aligned}
 f_3 = & \frac{u}{2} \iint_{(r-r')^2 \geq (t-t')^2} dt' r'^2 dr' \delta_+(x_\mu'^2) \frac{1}{\sqrt{rr'}} \left[\frac{4jW^2}{(2j+1)P} (G_{3/2, -1/2} f_1' - G_{3/2, 3/2} f_2') \right. \\
 & + \frac{(P^2 - m^2)M^2}{P^3} G_{3/2, 3/2} f_2' - \frac{2iWM}{(2j+1)P^2} G_{3/2, 1/2} (2Pf_4' - mf_3') \\
 & - \frac{2ijW}{(2j+1)} G_{3/2, 1/2} f_5' \left. \right] \\
 & - \frac{1}{2j+1} \cdot \frac{u}{Pr} \iint dt' r' \delta_+(x_\mu'^2) dr' \left[H_1^+(r, t; r', t') \{ jf_1' + (j+1)f_2' \} \right. \\
 & + H_2^+(r, t; r', t') f_3' \left. \right] + R_4, \quad (6.8a)
 \end{aligned}$$

$$\begin{aligned}
 f_3 = & \frac{u}{2} \iint_{(r-r')^2 \geq (t-t')^2} dt' r'^2 dr' \delta_+(x_\mu'^2) \frac{1}{\sqrt{rr'}} \left[-\frac{2iWmM}{P} \left\{ jG_{1/2, -1/2} f_1' \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& + (j+1) G_{1/2, 3/2} f_2' \Big\} + \frac{m(P^2 - M^2)}{P^3} G_{1/2, 1/2} (2P f_4' - m f_3') \Big] \\
& + \frac{a}{P r} \iint dt' r' dr' \delta_+(x_\mu'^2) \Big[H_2^-(r, t; r', t') \{ j f_1' + (j+1) f_2' \} \\
& + H_1^-(r, t; r', t') f_3' \Big] + R_5, \tag{6.8b}
\end{aligned}$$

$$\begin{aligned}
G_{m, n} \equiv G_{m, n}(Wr, Wr') &= \frac{1 + \epsilon(r - r')}{2} K_{j+m}(Wr) I_{j+n}(Wr') \\
&+ (-)^{m-n} \frac{1 - \epsilon(r - r')}{2} I_{j+m}(Wr) K_{j+n}(Wr'), \tag{6.9}
\end{aligned}$$

$$\begin{aligned}
H_1^\pm(r, t; r', t') &= \left(\frac{1 + \epsilon(t - t')}{2} e^{i a_1(t - t')P} + \frac{1 - \epsilon(t - t')}{2} e^{-i a_2(t - t')P} \right) \\
&\times \left[\delta(t - t' + r - r') + \delta(t - t' - r + r') \right. \\
&\left. + (-)^{j-1/2 \pm 1/2} \left\{ \delta(t - t' + r + r') + \delta(t - t' - r - r') \right\} \right], \tag{6.10}
\end{aligned}$$

$$\begin{aligned}
H_2^\pm(r, t; r', t') &= \left(\frac{1 + \epsilon(t - t')}{2} e^{i a_1(t - t')P} + \frac{1 - \epsilon(t - t')}{2} e^{-i a_2(t - t')P} \right) \\
&\times \left[\delta(t - t' + r - r') - \delta(t - t' - r + r') \right. \\
&\left. + (-)^{j+1/2 \pm 1/2} \left\{ \delta(t - t' + r + r') - \delta(t - t' - r - r') \right\} \right], \quad (\epsilon(x) = |x|/x), \\
m &= m_1 - m_2, \quad M = m_1 + m_2, \quad W^2 = -(P^2 - M^2)(P^2 - m^2)/4P^2, \quad a = -e_1 e_2, \tag{6.11}
\end{aligned}$$

(for $j=0$, the three equations (6.7b), (6.8a) and (6.8b) hold with $f_1=f_5=0$).

In the above formulae, we have already performed the p -integrations in (6.2) in the space-like region $(r - r')^2 \geq (t - t')^2$ (See Appendix B.), and denoted the part of the right hand side of (6.2) contributing from the region $(r - r')^2 < (t - t')^2$ by R_i which can not be calculated in the explicit form. It should be remarked that if we take a_1, a_2 as given by (4.16) and (4.23), the kernels do not depend on relative time t in the space-like region. R_i 's vanish, if we replace $\delta_+(x_\mu'^2)$ by $\delta(r' - t')/r'$ and put $r = t$.

The second integrals in the right of (6.8a) and (6.8b) represent the singular contributions from the light-cone. Inserting (6.10) in (6.8) and performing the integrations, we see that the wave amplitudes $f_2(r, t)$ and $f_3(r, t)$ themselves contain the terms multiplied by the singular factors $\delta(r - t)$, $\delta(r + t)$, $1/(r + t)$, and $1/(r - t)$, provided that the conditions (6.12) below are not satisfied. Substituting these functions in the right of (6.7) and (6.8), we observe that these singularities just coincide with that of $\delta_+(x_\mu'^2)$ and the equations have no finite solutions, namely, the eigen-value of a must be zero.

In order that the wave amplitudes do not contain the above singular terms which prevent the existence of the finite solutions, the following conditions must be satisfied

$$j f_1(r, r) + (j+1) f_2(r, r) = f_3(r, r), \quad (6.12a)$$

$$j f_1(r, -r) + (j+1) f_2(r, -r) = -f_3(r, -r), \quad (6.12b)$$

since the factor of, for instance, $\delta(r-t)$ appearing in the right of (6.8a) is apart from a common factor given by

$$\begin{aligned} & \int_0^\infty dr' \left[\left(\frac{1 + \epsilon(r-r')}{2} e^{ia_1(r-r')P} + \frac{1 - \epsilon(r-r')}{2} e^{-ia_2(r-r')P} \right) \right. \\ & \quad \times \{ j f_1(r', r') + (j+1) f_2(r', r') - f_3(r', r') \} \\ & \quad \left. + (-)^j e^{ia_1(r+r')P} \{ j f_1(r', -r') + (j+1) f_2(r', -r') + f_3(r', -r') \} \right], \end{aligned}$$

and by putting it equal to zero identically and differentiating with respect to r twice we get three equations from which (6.12a) is deduced.

The conditions (6.12) impose additional restrictions on the five wave amplitudes which must satisfy the five equations (6.7) and (6.8). In the case of one-sheet potential $\delta(r-t)/r$, it can be shown in the following that actually there is no solutions satisfying (6.7), (6.8) and (6.12a). We replace $\delta_+(x_\mu'^2)$ in (6.7) and (6.8) by $\delta(r'-t')/r'$, and put $r=t$, then, using (6.12a) we have finite contributions from the singular terms in (6.8),

$$\begin{aligned} & 2 \iint dt' r' \delta(r'-t')/r' \cdot \left[H_1^+(r, r; r', t') \{ j f_1' + (j+1) f_2' \} + H_2^+(r, r; r', t') f_3' \right] \\ & = j f_1(r, r) + (j+1) f_2(r, r) + f_3(r, r), \quad (6.13) \\ & 2 \iint dt' r' \delta(r'-t')/r' \cdot \left[H_2^-(r, r; r', t') \{ j f_1' + (j+1) f_2' \} + H_1^-(r, r; r', t') f_3' \right] \\ & = j f_1(r, r) + (j+1) f_2(r, r) + f_3(r, r). \end{aligned}$$

Thus we get the one dimensional equations for $f_i(r, r)$ ($i=1, 2, 3, 4, 5$) which are given from (6.7) and (6.8) by the following operations

- (i) $\iint \frac{dt' r'^2 dr'}{(r-r')^2 \geq (t-t')^2} \delta_+(x_\mu'^2) \frac{1}{\sqrt{rr'}} \dots$ is replaced by $\int_0^\infty \sqrt{\frac{r'}{r}} dr' \dots$,
- (ii) R_i 's are dropped,
- (iii) (6.13) is substituted into (6.8).

By using the following formulae for $G_{m,n}(x, x')$,

$$\begin{aligned} & \left(\frac{d}{dx} + \frac{j+m+1/2}{x} \right) \frac{1}{\sqrt{x}} G_{m,n}(x, x') = -\frac{1}{\sqrt{x}} G_{m-1,n}(x, x') \\ & \quad + \frac{1}{\sqrt{x}} \delta(x-x') \left\{ K_{j+m}(x) I_{j+n}(x') + (-)^{m-n-1} I_{j+m}(x) K_{j+n}(x) \right\}, \\ & \left(\frac{d}{dx} - \frac{j+m-1/2}{x} \right) \frac{1}{\sqrt{x}} G_{m,n}(x, x') = -\frac{1}{\sqrt{x}} G_{m+1,n}(x, x') \end{aligned}$$

$$+\frac{1}{\sqrt{x}}\delta(x-x')\left\{K_{j+m}(x)I_{j+n}(x')+(-)^{m-n-1}I_{j+m}(x)K_{j+n}(x)\right\}.$$

above equations can be transformed into a system of simultaneous first order differential equations (6.14);

$$\begin{aligned} & \left(\frac{d}{dx}-\frac{j-1}{x}\right)f_1-\left(\frac{d}{dx}+\frac{j+2}{x}\right)f_2-\frac{i(P^2-m^2)}{2WP}f_3 \\ &= \frac{u}{x}\left[\frac{W}{2Px}\left\{f_3-3(jf_1+(j+1)f_2)\right\}+if_3\right], \\ & j\left(\frac{d}{dx}-\frac{j-1}{x}\right)f_1+(j+1)\left(\frac{d}{dx}+\frac{j+2}{x}\right)f_2+\frac{i(P^2-m^2)M}{2mWP}f_3 \\ &= u\left[-\frac{W}{2P}\left(\frac{d}{dx}+\frac{2}{x}\right)\frac{1}{x}\left\{jf_1+(j+1)f_2+f_3\right\}\right. \\ & \quad \left.+\frac{2Wj(j+1)}{Px^2}(f_1-f_2)-\frac{iMW}{P^2x}(2Pf_4-mf_3)\right], \\ & \left(\frac{d}{dx}+\frac{j+1}{x}\right)\left((j+1)f_3-\frac{M}{m}f_3\right)-\frac{i(2j+1)(P^2-M^2)}{2WP}f_1 \\ &= u\left[-\frac{MW}{2mP}\left(\frac{d}{dx}+\frac{j+1}{x}\right)\frac{1}{x}\left\{jf_1+(j+1)f_2+f_3\right\}\right. \\ & \quad \left.+\frac{i(j+1)}{x}(f_1-f_2)+\frac{iM^2}{P^2x}\left\{jf_1+(j+1)f_2\right\}\right], \\ & \left(\frac{d}{dx}-\frac{j}{x}\right)\left(jf_3+\frac{M}{m}f_3\right)+\frac{i(2j+1)(P^2-M^2)}{2WP}f_2 \\ &= u\left[\frac{MW}{2mP}\left(\frac{d}{dx}-\frac{j}{x}\right)\frac{1}{x}\left\{jf_1+(j+1)f_2+f_3\right\}\right. \\ & \quad \left.+\frac{ij}{x}(f_1-f_2)-\frac{iM^2}{xP^2}\left\{jf_1+(j+1)f_2\right\}\right], \\ & \frac{m}{P}f_4-f_3=-\frac{uW}{2Px}\left\{jf_1+(j+1)f_2+f_3\right\}, \quad x\equiv Wr. \end{aligned} \quad (6.14)$$

Thus five wave amplitudes f_i must satisfy the six equations (6.14) and (6.12a). Eliminating two functions, we finally get four equations of the following type for three functions $f_1-f_2\equiv g_1$, $f_3\equiv g_2$, $f_5\equiv g_3$;

$$\begin{aligned} dg_k(x)/dx &= \sum_{i=1}^3 L_{ki}(x)g_i(x), \quad (k=1, 2, 3), \\ 0 &= \sum_{i=1}^3 M_i(x)g_i(x), \end{aligned} \quad (6.15)$$

($L_{ik}(x)$, $M_i(x)$ are rational functions of x). Differentiating repeatedly the last equations of (6.15) and using other three ones, we get unlimited numbers of independent linear homogeneous algebraic equations for g_i , so g_i or f_i must be equal to zero.

In the more general case, even if there exists a finite solution satisfying accidentally the conditions (6.12), such a solution has no physical significance, since in the non-relativistic approximation ($11'/P \rightarrow 0$, $\partial_+(x_\mu'^2) \rightarrow \partial(t')/r'$) of (6.7) and (6.8), f_1 and f_2 become the radial functions for different orbital angular momenta $j=1/2$ and $j=3/2$, (f_3 vanishes in this approximation,) and so do not satisfy (6.12).

The above result is not altered in the case of K_\perp , which has the same singular character on the light-cone as K_0 . For K_\perp , the kernels cannot be calculated explicitly even in the space-like region, but their singularities on the light-cone are easily estimated. Besides the ∂ -like singularities as given by (6.10) there appear the ones like $1/(t-t'-r+r')$ and $\log |t-t'-r+r'|$. The appearance of the latter makes it impossible to write down the singularities-free equations like (6.7). In the case of $j=0$, if we take the highest singular terms only (logarithmic terms are dropped), we get the following equations from (6.2) (See Appendix B).

$$\begin{aligned} f_2(r, t) &= \frac{u}{Pr} \iint dt' r' dr' \partial_+(x_\mu'^2) (e^{-ia_2 P(t-t')} - e^{ia_1 P(t-t')}) \\ &\quad \times \{H_3^+(r, t; r', t') f_2(r', t') + H_4(r, t; r', t') f_3(r', t')\} + \dots, \\ f_3(r, t) &= \frac{u}{Pr} \iint dt' r' dr' \partial_+(x_\mu'^2) (e^{-ia_2 P(t-t')} - e^{ia_1 P(t-t')}) \\ &\quad \times \{H_4(r', t; r, t') f_2(r', t') - H_3^-(r, t; r', t') f_3(r', t')\} + \dots, \end{aligned} \quad (6.16)$$

(the equation for $f_4(r, t)$ does not contain the highest singular terms,) with

$$\begin{aligned} H_\gamma^\pm(r, t; r', t') &= (t-t')(\partial_+(\lambda) \pm \partial_+(\lambda')), \\ \lambda &= (t-t')^2 - (r-r')^2, \quad \lambda' = (t-t')^2 - (r+r')^2, \quad \partial_+(x) = \partial(x) + 1/i\pi x, \\ 4H_4(r, t; r', t') &= (1 + \epsilon(t-t')) \left\{ \partial_+(t-t' + r-r') - \partial_+(t-t' - r+r') \right. \\ &\quad \left. - \partial_+(t-t' + r+r') + \partial_+(t-t' - r-r') \right\} \\ &\quad - (1 - \epsilon(t-t')) \left\{ \partial_-(t-t' + r-r') - \partial_-(t-t' - r+r') \right. \\ &\quad \left. - \partial_-(t-t' + r+r') + \partial_-(t-t' - r-r') \right\}. \end{aligned} \quad (6.17)$$

As in the previous case there will be no finite solution satisfying (6.16) in general, and this result is not changed essentially in the case of the meson interaction of any type, where $\partial_+(x_\mu'^2)$ is replaced by $2\pi i \Delta_x(x_\mu'^2)$, for instance, in the vector case.

Above difficulties have their origin partly in the fact that the singularities of the propagation functions of particles and photon coincide with each other. To make this point more clear we shall separate the above coincidence by replacing $\partial_+(x_\mu'^2)$ in (6.7) and (6.8) by

$$\partial(r'/c' - t')/r', \quad (6.18)$$

where c' is a constant larger than unity (note that we use the natural units $\hbar=c=1$). $c' \rightarrow \infty$ represents the instantaneous interaction at $t'=0$, i.e., adiabatic approximation. In order to get the one dimensional integral equations on the cone $r/c'=t$, we put $t=r/c'$ in (6.7) and (6.8) and perform the t' -integrations. Now the contributions from the time-like region vanish, since the condition $(r-r')^2 \geq (t-t')^2 = (r-r')^2/c'^2$ is satisfied. On the cone $r'/c'=t'$ there are finite contributions from the singular terms in (6.8), namely,

$$\begin{aligned} & -\frac{1}{2} \iint dt' r'^2 dr' \frac{\partial(r'/c' - t')}{r'} [\{jf_1' + (j+1)f_2'\} H_1^+(r, r/c'; r', t') \\ & \quad + H_2^+(r, r/c'; r', t') f_3'] = \frac{c'^2}{c'^2-1} \{jf_1 + (j+1)f_2\} - \frac{c'}{c'^2-1} f_3, \\ & -\frac{1}{2} \iint dt' r'^2 dr' \frac{\partial(r'/c' - t')}{r'} [\{jf_1' + (j+1)f_2'\} H_2^-(r, r/c'; r', t') \\ & \quad + H_1^-(r, r/c'; r', t') f_3'] = \frac{c'^2}{c'^2-1} f_3 - \frac{c'}{c'^2-1} \{jf_1 + (j+1)f_2\}, \end{aligned} \quad (6.19)$$

with $f_1=f_1(r, r/c')$ etc.

Thus we have the equations, denoted by (6.7)' and (6.8)', which are given from (6.7) and (6.8) by the following operation:

- (i) $\iint_{(r-r')^2 \geq (t-t')^2} dt' r'^2 dr' \partial_+(x_\mu'^2) \frac{1}{\sqrt{rr'}} \dots$ is replaced by $\int_0^\infty \sqrt{\frac{r'}{r}} dr' \dots$,
- (ii) R_t 's are dropped,
- (iii) (6.19) is substituted into (6.8).

These equations will describe the bound system properly, since they give the correct asymptotic behavior of the wave amplitudes at $r \rightarrow \infty$ and $r \rightarrow 0$, though they are too complicated to be solved analytically. To reinstate the relativistic invariance we must take the limiting case $c' \rightarrow 1$, then, infinities arise in the integral equations (6.8)' as can be seen from (6.19) and the eigen-values of u will accumulate to zero.

7. Discussions

We have treated the bound states of two particles in the complete relativistic form mainly under the restriction to the ladder approximation. In this respect the effects of higher terms in the kernel $G(1, 2; 1', 2')$ must be examined. Provided that the integral equation with complete G is solved and if we consider the case $m_1=m_2=M/2$ for simplicity, it can be seen from the consideration of dimension that the eigen-value will be of the form

$$P/M = f(c^2/\hbar c). \quad (7.1)$$

Under the assumption of the validity of the weak coupling perturbation theory, which

forms the basis of the present investigation as seen in the expansion of G , f in (7.1) will be expanded in powers of $e^2/\hbar c$ as

$$f(e^2/\hbar c) = 1 - (e^2/\hbar c)^2/8 + \text{higher terms in } (e^2/\hbar c)^2, \quad (7.2)$$

where the first two terms are determined by the non-relativistic solution ($v \rightarrow 0$ approximation). Since the binding energy is comparable with the relative kinetic energy $mv^2/2$ of the particles in this approximation, we have a relation

$$e^2/\hbar c \sim v/c. \quad (7.3)$$

Thus, in contrast with the scattering problem where $e^2/\hbar c$ and v/c are independent parameters, it will be necessary to take into account the higher terms in G in order to make a consistent relativistic treatment of the bound states; for instance, in order to obtain a result to the order $(v/c)^2$ we must include the terms up to e^6 in G , the similar results having been confirmed by Salpeter and Bethe⁴⁾ and Levy¹¹⁾ in the case of a scalar meson interaction.

The complex energy eigen-values obtained in Sec. 4 will have their origins partly in the uncausal character of propagation functions I_0 and \bar{D} and partly in the ladder approximation. As has been shown by Gell-Mann and Low,³⁾ eigen-values P_μ of the integral equation must be real if we use the causal functions, I_+ and δ_+ , and the complete G . Then, it will be probable that, as the higher terms in G are included successively, each term of (7.2) turns out to be real up to the order considered.

The problem concerning the convergence of G , which seems to be intractable at the present stage, will be left out of consideration, though it will be more serious in the case of pseudoscalar meson interaction than the other cases. We confine ourselves to the finite terms of G . In the spinor case there will be no finite solution of (2.4) as shown in Sec. 6. Such a difficulty has its origin in the singularities of the propagation functions K_+ or K_0 on the light-cone in contrast with I_+ or I_0 in the scalar case where finite solutions are obtained as shown in Secs. 4 and 5. One of us (C. H.) has calculated the Fredholm's determinant $D(\lambda)$, $\lambda = e, c_2$, of the integral equation and confirmed that each term of it diverges in the case of two spinor particles whatever the interaction may be, electromagnetic or mesic of any type (these results will be published later). This together with the results in Sec. 6 suggests strongly that in the spinor cases the integral equations are essentially singular and there is no eigen-value corresponding to a finite solution except for $\lambda=0$. Thus, it seems to be possible that the radius of convergence of the S -matrix in the scattering problem is zero, that is, though each term of it is finite, its infinite series corresponding to the iterated interactions, say, of the graph (a) in Fig. 1, may diverge. Nevertheless, the non-relativistic equation (2.12) has finite solutions as shown in Sec. 3 and Appendix C. This results from the fact that the reduction to (2.12) from (2.16) is valid in the asymptotic sense, $p/mc \rightarrow 0$ or $e^2/\hbar c \rightarrow 0$, namely, the expansion such as (7.2) will not always converge for finite $e^2/\hbar c$, so that equation (2.4) by itself would not describe the bound states properly.

Concerning the above difficulty there is an interesting suggestion¹²⁾ that the propagation function including the radiative corrections of the self-energy type, i. e., S_F' of Dyson,¹³⁾

is less singular than S_F . We are now studying the possibility of removing the difficulty in this way. In the other way, the invariant regularization procedure of Pauli and Villars⁽¹⁾ will be applied to this problem conveniently for the phenomenological treatment of the theory. At any rate, the invariant formulation (2.4) of bound states will be more convenient at the present stage than other methods, such as the Hamiltonian formalism in the adiabatic approximation or the non-adiabatic treatment given by Dancoff,⁽¹⁵⁾ in separating out the infinite terms of self-interaction and in trying to remove the above difficulty in the spinor case.

In conclusion, the authors would like to thank Messrs. H. Kita, H. Tanaka and Professor Y. Nambu for many stimulating discussions, Professors Z. Shirogane and T. Inoue for their continual encouragements during the work, and Miss M. Kawabata for the preparation of the manuscript.

Appendix A. Separation of angular variables

(i) Scalar equation

Here we consider the separation of the angular variables in the equation of the form

$$\varphi(x_i, t) = \int g(r, r'; t, t'; p_i^2) e^{i p_i(x_i - x_i')} h(r', t') \varphi(x_i', t') d^3 p d\tau'. \quad (\text{A} \cdot 1)$$

We have

$$\begin{aligned} e^{i p_i(x_i - x_i')} &= \sum_{l'=0}^{\infty} (2l'+1) i^{l'} \left(\frac{\pi}{2\rho r'} \right)^{1/2} J_{l'+1/2}(\rho r) P_{l'}(\cos \gamma) \\ &\times \sum_{l''=0}^{\infty} (2l''+1) (-i)^{l''} \left(\frac{\pi}{2\rho r'} \right)^{1/2} J_{l''+1/2}(\rho r') P_{l''}(\cos \gamma'), \end{aligned} \quad (\text{A} \cdot 2)$$

$$\cos \gamma = \cos u \cos \theta + \sin u \sin \theta \cos(\beta - \varphi),$$

$$\cos \gamma' = \cos u \cos \theta' + \sin u \sin \theta' \cos(\beta - \varphi'), \quad (\text{A} \cdot 3)$$

$$p_i = (p, u, \beta), \quad x_i = (r, \theta, \varphi), \quad x_i' = (r', \theta', \varphi'), \quad (\text{A} \cdot 3)'$$

and

$$\frac{2l+1}{2} P_l(\cos \gamma) = \sum_{m=-l}^l \bar{P}_l^m(\cos u) \bar{P}_l^m(\cos \theta) e^{im(\beta - \varphi)}, \quad (\text{A} \cdot 4)$$

where \bar{P}_l^m denote the normalized Legendre functions, and satisfy

$$\int_0^\pi \bar{P}_l^m(\cos \theta') \bar{P}_l^m(\cos \theta') \sin \theta' d\theta' = \delta_{ll'}. \quad (\text{A} \cdot 5)$$

Expanding

$$\varphi(x_i, t) = \sum_{l,m} f_l(r, t) P_l^m(\cos \theta) e^{im\varphi}, \quad (\text{A} \cdot 6)$$

and inserting this in the right of (A.1), we have

$$\int e^{i p_i(x_i - x_i')} P_l^m(\cos \theta') e^{im\varphi'} \sin \theta' d\theta' d\varphi' \sin u du d\beta$$

$$\begin{aligned}
&= (4\pi) \sum_{l'=0}^{\infty} (2l'+1) l' \left(\frac{\pi}{2pr} \right)^{1/2} J_{l'+1/2}(pr) (-i)^l \left(\frac{\pi}{2pr'} \right)^{1/2} J_{l+1/2}(pr') \\
&\quad \times \int \sin u du d\beta P_{l'}(\cos \gamma) P_l^m(\cos \alpha) e^{im\beta} \\
&= (4\pi)^2 \left(\frac{\pi}{2pr} \right)^{1/2} J_{l+1/2}(pr) \left(\frac{\pi}{2pr'} \right)^{1/2} J_{l+1/2}(pr') P_l^m(\cos \theta) e^{im\varphi}. \quad (\text{A} \cdot 7)
\end{aligned}$$

We compare the factor of $P_l^m(\cos \theta) e^{im\varphi}$ on both sides of (A.1), we find

$$\begin{aligned}
f_i(r, t) &= (4\pi)^2 \int g(r, r'; t, t'; p_i^2) h(r', t') \left(\frac{\pi}{2pr} \right)^{1/2} \\
&\quad \times J_{l+1/2}(pr) \left(\frac{\pi}{2pr'} \right)^{1/2} J_{l+1/2}(pr') p^2 dp dt' r'^2 dr'. \quad (\text{A} \cdot 8)
\end{aligned}$$

For the non relativistic equation the integral over p can be performed using (B. 2a) with $t=t'$, and the equation (3.11) is obtained.

(ii) *Spinor equation*

First of all, sixteen components of the wave amplitude ψ are classified as in (3.1) according to the two energy states of the both particles. Each φ_i has four components. There exist four independent spin eigen-functions belonging to given j, μ , where j is the magnitude of the total angular momentum $\mathbf{M} = (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)/2 + (\mathbf{r} \times \mathbf{p})$, and μ is its z -component. We adopt the following four bases throughout in Sec. 6.

$$\begin{aligned}
(\text{I}) &\equiv \frac{1}{\sqrt{2j-1}} \begin{pmatrix} -\sqrt{j+\mu-1} & \sqrt{j+\mu} & Y_{j-1}^{\mu-1} & \sqrt{j+\mu} & \sqrt{j-\mu} & Y_{j-1}^{\mu} \\ \sqrt{j+\mu} & \sqrt{j-\mu} & Y_{j-1}^{\mu} & -\sqrt{j-\mu-1} & \sqrt{j-\mu} & Y_{j-1}^{\mu+1} \end{pmatrix}, \\
(\text{II}) &\equiv \frac{1}{\sqrt{2j+1}} \begin{pmatrix} 0 & -Y_j^{\mu} \\ Y_j^{\mu} & 0 \end{pmatrix}, \\
(\text{III}) &\equiv \frac{1}{\sqrt{2j+1}} \begin{pmatrix} \sqrt{j+\mu} & \sqrt{j-\mu+1} & Y_j^{\mu-1} & \mu Y_j^{\mu} \\ \mu Y_j^{\mu} & -\sqrt{j-\mu} & \sqrt{j+\mu+1} & Y_j^{\mu+1} \end{pmatrix}, \\
(\text{IV}) &\equiv \frac{1}{\sqrt{2j+3}} \begin{pmatrix} \sqrt{j-\mu+1} & \sqrt{j-\mu+2} & Y_{j+1}^{\mu-1} & \sqrt{j+\mu+1} & \sqrt{j-\mu+1} & Y_{j+1}^{\mu} \\ \sqrt{j+\mu+1} & \sqrt{j-\mu+1} & Y_{j+1}^{\mu} & \sqrt{j+\mu+1} & \sqrt{j+\mu+2} & Y_{j+1}^{\mu+1} \end{pmatrix}, \quad (\text{A} \cdot 9)
\end{aligned}$$

where (I), (III), (IV) correspond to three triplet states and (II) to singlet one, and they satisfy following relations

$$\begin{aligned}
\nu(\text{I}) &= (\text{I}), \quad \nu(\text{III}) = (\text{III}), \quad \nu(\text{IV}) = (\text{IV}), \\
\nu(\text{II}) &= -3(\text{II}), \quad \nu = (\sigma_{1i} \sigma_{2i}). \quad (\text{A} \cdot 10)
\end{aligned}$$

We expand the four components wave amplitudes φ_i as follows

$$\varphi_i = A_i(r, t) (\text{I}) + B_i(r, t) (\text{II}) + C_i(r, t) (\text{III}) + D_i(r, t) (\text{IV}), \quad (i=1, 2, 3, 4). \quad (\text{A} \cdot 11)$$

Following relations are easily verified :

$$\begin{aligned}
\pi_1(\text{I})_p &= -p \{ j(\text{II})_p + (\text{III})_p \}, \quad \pi_2(\text{I})_p = -p \{ j(\text{II})_p - (\text{III})_p \}, \\
\mu(\text{I})_p &= -p^2 \{ (\text{I})_p - 2j(\text{IV})_p \} / (2j+1), \\
\pi_1(\text{II})_p &= \pi_2(\text{II})_p = -p \{ (\text{I})_p + (\text{IV})_p \} / (2j+1), \quad \mu(\text{II})_p = p^2 (\text{II})_p, \\
\pi_1(\text{III})_p &= -\pi_2(\text{III})_p = -p \{ (j+1)(\text{I})_p - j(\text{IV})_p \} / (2j+1), \quad \mu(\text{III})_p = -p^2 (\text{III})_p, \\
\pi_1(\text{IV})_p &= -p \{ (j+1)(\text{II})_p - (\text{III})_p \}, \quad \pi_2(\text{IV})_p = -p \{ (j+1)(\text{II})_p + (\text{III})_p \}, \\
\mu(\text{IV})_p &= p^2 \{ 2(j+1)(\text{I})_p + (\text{IV})_p \} / (2j+1),
\end{aligned} \tag{A.12}$$

with

$$\pi_1 = \sigma_{1i} \dot{p}_{1i} = \sigma_{1i} \dot{p}_i, \quad \pi_2 = \sigma_{2i} \dot{p}_{2i} = -\sigma_{2i} \dot{p}_i, \quad \mu = \pi_1 \pi_2,$$

where $(\text{I})_p$, etc. denote the functions of (A.9) with arguments u, β in (A.3)'. From (A.10), (A.12), (A.7) and (6.4) we get

$$\begin{aligned}
\int (\text{I})' &= F_{-1/2, -1/2}(\text{I}), \quad \int \pi_1(\text{I})' = -ip F_{1/2, -1/2} \{ (\text{III}) + j(\text{II}) \}, \\
\int \pi_2(\text{I})' &= ip F_{1/2, -1/2} \{ (\text{III}) - j(\text{II}) \}, \\
\int \mu(\text{I})' &= -p^2 \{ F_{-1/2, -1/2}(\text{I}) + 2j F_{3/2, -1/2}(\text{IV}) \} / (2j+1), \\
\int (\text{II})' &= F_{1/2, 1/2}(\text{II}), \\
\int \pi_1(\text{II})' &= \int \pi_2(\text{II})' = ip \{ F_{-1/2, 1/2}(\text{I}) - F_{3/2, 1/2}(\text{IV}) \} / 2j+1, \\
\int \mu(\text{II})' &= p^2 F_{1/2, 1/2}(\text{II}), \\
\int (\text{III})' &= F_{1/2, 1/2}(\text{III}), \\
\int \pi_1(\text{III})' &= -\pi_2(\text{III})' = ip \{ (j+1) F_{-1/2, 1/2}(\text{I}) + j F_{3/2, 3/2}(\text{IV}) \} / 2j+1, \\
\int \mu(\text{III})' &= -p^2 F_{1/2, 1/2}(\text{III}), \\
\int (\text{IV})' &= F_{3/2, 3/2}(\text{IV}), \\
\int \pi_1(\text{IV})' &= -ip F_{1/2, 3/2} \{ (\text{III}) - (j+1)(\text{II}) \}, \\
\int \pi_2(\text{IV})' &= ip F_{1/2, 3/2} \{ (\text{III}) + (j+1)(\text{II}) \}, \\
\int \mu(\text{IV})' &= -p^2 \{ 2(j+1) F_{-1/2, 3/2}(\text{I}) - F_{3/2, 3/2}(\text{IV}) \} / (2j+1),
\end{aligned}$$

where \int means

$$(4\pi)^{-2} \int e^{ip_i(x_i - x_i')} \sin \theta' d\theta' d\varphi' \sin \alpha d\alpha d\beta$$

and (I)' etc. denote the functions of (A.9) with arguments θ' , φ' in (A.3)'. Inserting these expressions in (3.2) and comparing the factors of (I), (II), (III) and (IV) we get the equation (6.2).

The above formulae are also sufficient to perform the separation of the angular variables in the case of meson interaction of any type.

Appendix B. Integrals in the p -space

We shall consider the various integrals appearing in (6.2). A typical integral in the p_4 -plane is as follows:

$$\begin{aligned} & \frac{1}{i\pi} \int d p_4 \frac{e^{-i p_4(t-t')}}{\{(a_1 P + p_4)^2 - p^2 - m_1^2\} \{(a_2 P - p_4)^2 - p^2 - m_2^2\}} \\ & \left\{ \begin{aligned} &= \frac{1 + \epsilon(t-t')}{2} \frac{e^{i a_1 P(t-t')}}{p^2 + W^2} \left\{ -i a_1 \sin E_1(t-t')/E_1 + \cos E_1(t-t')/P \right\} \\ &+ \frac{1 - \epsilon(t-t')}{2} \frac{e^{-i a_2 P(t-t')}}{p^2 + W^2} \left\{ i a_2 \sin E_2(t-t')/E_2 + \cos E_2(t-t')/P \right\} \quad (\text{for } K_0^*), \\ &= \frac{1}{2p(W^2 + p^2)} \left[e^{i a_1 P(t-t')} \left\{ \{ a_1 P \cos E_1(t-t')/E_1 - i \sin E_1(t-t') \} \right. \right. \\ &\quad \left. \left. + \epsilon(t-t') \{ \cos E_1(t-t') - i a_1 P \sin E_1(t-t')/E_1 \} \right\} \right. \\ &\quad \left. + e^{-i a_2 P(t-t')} \left\{ \{ a_2 P \cos E_2(t-t')/E_2 + i \sin E_2(t-t') \} \right. \right. \\ &\quad \left. \left. - \epsilon(t-t') \{ \cos E_2(t-t') + i a_2 P \sin E_2(t-t')/E_2 \} \right\} \right] \quad (\text{for } K_+), \end{aligned} \right. \quad (\text{B.1}) \end{aligned}$$

where $\epsilon(x) = x/|x|$, $E_1 = (p^2 + m_1^2)^{1/2}$, $E_2 = (p^2 + m_2^2)^{1/2}$; and W , a_1 and a_2 are given by (4.16), (4.23) and (4.21). The integration paths are as follows:

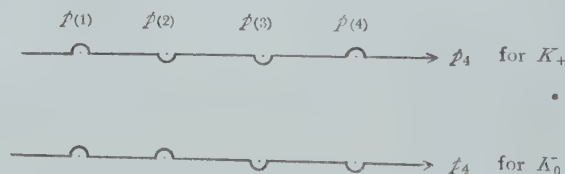


Fig. 4

where $p_{(1)}$, $p_{(2)}$, $p_{(3)}$ and $p_{(4)}$ are given by (3.4).

In the case of K_0 the following formulae are sufficient to evaluate the kernels appearing in (6.2) in the space-like region $(r-r')^2 > (t-t')^2$,

$$\begin{aligned} & \int_0^\infty p dp \left\{ J_{j+1/2}(pr) J_{j+1/2}(pr') / (W^2 + p^2) \right\} \begin{cases} \sin E(t-t')/E \\ \cos E(t-t') \end{cases} \\ &= G_{1/2, 1/2}(Wr, Wr') \begin{cases} \sin(m^2 - W^2)^{1/2}(t-t') / (m^2 - W^2)^{1/2}, \\ \cos(m^2 - W^2)^{1/2}(t-t'), \end{cases} \quad (\text{B.2a}) \end{aligned}$$

$$\int_0^\infty p^2 dp \{ J_{j+3/2}(pr) J_{j+1/2}(pr') / (W^2 + p^2) \} \begin{cases} \sin E(t-t')/E \\ \cos E(t-t') \end{cases} \\ = WG_{3/2, 1/2}(Wr, Wr') \begin{cases} \sin(m^2 - W^2)^{1/2}(t-t') / (m^2 - W^2)^{1/2}, \\ \cos(m^2 - W^2)^{1/2}(t-t'), \end{cases} \quad (\text{B} \cdot 2b)$$

$$\int_0^\infty p^3 dp \{ J_{j+3/2}(pr) J_{j-1/2}(pr') / (W^2 + p^2) \} \sin E(t-t')/E \\ = W^2 G_{3/2, -1/2}(Wr, Wr') \sin(m^2 - W^2)^{1/2}(t-t') / (m^2 - W^2)^{1/2}, \quad (\text{B} \cdot 2c)$$

with $E = (p^2 + m^2)^{1/2}$ and for $(r-r')^2 \geq (t-t')^2$,

$$\int_0^\infty p dp J_{j+1/2}(pr) J_{j+1/2}(pr') \sin E(t-t')/E = 0, \quad \text{for } (r-r')^2 > (t-t')^2, \quad (\text{B} \cdot 3)$$

with $G_{m,n}(rW, r'W)$ given by (6.9). To prove (B.2a), we remark the following formulae (see Watson¹⁶, p. 367 and 431)

$$\frac{J_{j+1/2}(pr)}{(pr)^{j+1/2}} \frac{J_{j+1/2}(pr')}{(pr')^{j+1/2}} = \frac{1}{2^{j+1/2} \sqrt{\pi} j!} \int_0^\pi \frac{J_{j+1/2}(pR)}{(pR)^{j+1/2}} \sin^{2j+1} \phi d\phi, \quad (\text{B} \cdot 4)$$

with $R = (r^2 + r'^2 - 2rr' \cos \phi)^{1/2}$,

$$\int_0^\infty \frac{p^{j+3/2}}{W^2 + p^2} \frac{J_{1/2}(E(t-t'))}{E^{1/2}} J_{j+1/2}(pR) dp = \frac{J_{1/2}((m^2 - W^2)^{1/2}(t-t'))}{(m^2 - W^2)^{1/4}} W^{j+1/2} K_{j+1/2}(WR), \quad (\text{B} \cdot 5)$$

for $R \geq (t-t') > 0$ and for $-3/2 < j < 2$,

$$\int_0^\pi \frac{K_{j+1/2}(R)}{R^{j+1/2}} \sin^{2j+1} \phi d\phi = 2^{j+1/2} \sqrt{\pi} j! \frac{K_{j+1/2}(r)}{r^{j+1/2}} \frac{I_{j+1/2}(r')}{r'^{j+1/2}}, \quad \text{for } r > r'. \quad (\text{B} \cdot 6)$$

Substituting (B.4) into the left of (B.2a), changing the order of integrations of p and ϕ , we get the right hand side by using (B.5) and (B.6). The restriction of $j < 2$ in (B.5) is removed by the analytic continuation with respect to j . Others are given by differentiation of (B.2a) with respect to r and t with the aid of the well-known recurrence formulae for the Bessel functions. To prove (B.3) we substitute (B.4) in the left hand side. The left hand side is

$$\frac{1}{2^{j+1/2} \sqrt{\pi} j!} (rr')^{j+1/2} \int_0^\pi d\phi \int_0^\infty dp \frac{\sin E(t-t')}{E} \frac{p^{j+2/3}}{R^{j+1/2}} J_{j+1/2}(pR) \sin^{2j+1} \phi,$$

but by Watson¹⁶ (p. 415) for $R \geq |r-r'| > |t-t'| \equiv a$

$$\int_0^\infty J_{j+1/2}(pR) \frac{J_{1/2}(aE)}{E^{1/2}} p^{j+3/2} dp = 0, \quad \text{for } 0 > j > -3/2, \quad (\text{B} \cdot 7)$$

therefore (B.3) follows for $0 > j > -3/2$, but for $j > 0$ (B.3) will be justified by analytic continuation.

Next, we evaluate the singularities of the kernels on the light-cone. In the case of K_0 , following integrals are to be evaluated,

$$\int_0^\infty p dp J_{j+1/2}(pr) J_{j+1/2}(pr') \cos E(t-t'), \quad \int_0^\infty p^2 dp J_{j+3/2}(pr) J_{j+1/2}(pr') \frac{\sin E(t-t')}{E}.$$

These integrals are intractable to be estimated except for $j=0$. Using the transformations (4.18) we get (see the reference 10, Appendix)

$$\begin{aligned} \int_0^\infty p dp J_{1/2}(pr) J_{1/2}(pr') \cos E(t-t') &= \frac{1}{2\sqrt{rr'}} \left[\delta(t-t'+r-r') + \delta(t-t'-r+r') \right. \\ &\quad \left. - \delta(t-t'+r+r') - \delta(t-t'-r-r') - m^2(t-t') \left(Re \frac{H_1^{(1)}(m\lambda^{1/2})}{m\lambda^{1/2}} - Re \frac{H_1^{(1)}(m\lambda'^{1/2})}{m\lambda'^{1/2}} \right) \right], \end{aligned} \quad (\text{B} \cdot 8)$$

with

$$\lambda^{1/2} = \sqrt{(t-t')^2 - (r-r')^2} \quad \text{for} \quad (t-t')^2 > (r-r')^2,$$

$$\lambda'^{1/2} = i \sqrt{(r-r')^2 - (t-t')^2} \quad \text{for} \quad (t-t')^2 < (r-r')^2,$$

similarly for $\lambda' = (t-t')^2 - (r+r')^2$.

The second term of (B.8) is finite at $\lambda=0$ or $\lambda'=0$. If we confine ourselves to the estimation of the singularities alone we get through the similar calculations,

$$\begin{aligned} \int_0^\infty p dp J_{j+1/2}(pr) J_{j+1/2}(pr') \cos E(t-t') &= \frac{1}{2\sqrt{rr'}} \left[\delta(t-t'+r-r') + \delta(t-t'-r+r') \right. \\ &\quad \left. + (-)^{j-1} \left\{ \delta(t-t'+r+r') + \delta(t-t'-r-r') \right\} \right] + \dots, \\ \int_0^\infty p^2 dp J_{j+3/2}(pr) J_{j+1/2}(pr') \frac{\sin E(t-t')}{E} &= \frac{1}{2\sqrt{rr'}} \left[-\delta(t-t'+r-r') + \delta(t-t'-r+r') \right. \\ &\quad \left. + (-)^j \left\{ \delta(t-t'+r+r') - \delta(t-t'-r-r') \right\} \right] + \dots. \end{aligned} \quad (\text{B} \cdot 9)$$

Using these results, we derive (6.10). We note that these integrals vanish for $(r-r')^2 > (t-t')^2$ on account of (B.3). In the case of K_+ , besides (B.9) there appear singular integrals of other types, which are for $j=0$

$$\begin{aligned} \int_0^\infty p dp J_{1/2}(pr) J_{1/2}(pr') \sin E(t-t') \\ &= -\frac{1}{2\sqrt{rr'}} (t-t') m^2 \left(I_m \frac{H_1^{(1)}(m\lambda^{1/2})}{m\lambda^{1/2}} - I_m \frac{H_1^{(1)}(m\lambda'^{1/2})}{m\lambda'^{1/2}} \right) \\ &= -\frac{(t-t')}{\pi\sqrt{rr'}} \left(-\frac{1}{\lambda} + \dots + \frac{1}{\lambda'} + \dots \right), \\ \int_0^\infty p^2 dp J_{3/2}(pr) J_{1/2}(pr') \frac{\cos E(t-t')}{E} \end{aligned} \quad (\text{B} \cdot 10)$$

$$= -\frac{1}{2\sqrt{rr'}} \left[(r-r') \tilde{m}^2 I_m \frac{H_1^{(1)}(m\lambda^{1/2})}{m\lambda^{1/2}} - (r+r') \tilde{m}^2 I_m \frac{H_1^{(1)}(m\lambda'^{1/2})}{m\lambda'^{1/2}} \right. \\ \left. + \frac{1}{r} \left\{ I_m H_0^{(1)}(m\lambda^{1/2}) - I_m H_0^{(1)}(m\lambda'^{1/2}) \right\} \right]. \quad (\text{B} \cdot 11)$$

(B·11) contains logarithmic singular terms at $\lambda=0$ or $\lambda'=0$. We have discarded these logarithmic terms and considered the λ^{-1} and δ -like singularities alone to derive (6·16) and (6·17).

Appendix C. The " $m_2 \rightarrow \infty$ " approximation

(i) Spinor particles

Operating $(i\gamma_{1\mu}\partial/\partial x_{1\mu} - m_1)$ on both sides of equation (2·4) in the ladder approximation, we have

$$(i\gamma_{1\mu}\partial/\partial x_{1\mu} - m_1)\psi(1, 2) = e_1 e_2 \int K_+(2, 2') \gamma_{1\mu} \gamma_{2\mu} \partial_+(1, 2') \psi(1, 2') d\tau_2'. \quad (\text{C} \cdot 1)$$

In the approximation $m_2 \rightarrow \infty$ the particle 2 suffers a negligible reaction from the particle 1 and so moves freely. In this case the wave amplitude can be written as the product of wave amplitudes of each particle

$$\psi(1, 2) = \varphi_1(1) \varphi_2(2). \quad (\text{C} \cdot 2)$$

In the following we shall omit the suffices 1 and 2 in φ_1 and φ_2 . $\varphi(2)$ satisfies the following Feynman's equations⁽³⁾

$$\varphi(2) = - \int \frac{\partial}{\partial x_{2\mu}} \{ K_+(2, 2') \gamma_{2\mu} \varphi(2') \} d\tau_2' = \int K_+(2, 2') \gamma_{24} \varphi(2') d^3 x_2', \quad (\text{C} \cdot 3)$$

$$\varphi^+(2) = \int \frac{\partial}{\partial x_{2\mu}} \{ \varphi^+(2') \gamma_{2\mu} K_+(2, 2') \} d\tau_2' = \int \varphi^+(2') \gamma_{24} K_+(2', 2) d^3 x_2', \quad (\text{C} \cdot 4)$$

where the four-dimensional volume integrals extend over a region containing the point 2, and the three-dimensional surface integrals are performed on the plane $t_2' < t_2$ for (C·3) and $t_2' > t_2$ for (C·4), respectively, since $\varphi(2)$ belongs to a positive energy state. Substituting (C·2) into (C·1) and multiplying both sides by $\varphi^+(2) \gamma_{24}$ from the left, we perform the surface integral on the plane $t_2 = +\infty$. Then, with the normalization

$$\int \varphi^+(2) \gamma_{24} \varphi(2) d^3 x_2 = 1, \quad (\text{C} \cdot 5)$$

and by the use of (C·4), we have

$$\left(i\gamma_{1\mu} \frac{\partial}{\partial x_{1\mu}} - m_1 \right) \varphi(1) = e_1 \gamma_{1\mu} A_\mu(1) \varphi(1), \quad (\text{C} \cdot 6)$$

with

$$A_\mu(1) = \int j_\mu(2') \partial_+(1, 2') d\tau_2', \quad j_\mu(2) = e_2 \varphi^+(2) \gamma_{2\mu} \varphi(2), \quad (\text{C} \cdot 7)$$

where $A_\mu(1)$ represent the electromagnetic field at a point 1 produced by the current j_μ

of the particle 2. In the limit $m_2 \rightarrow \infty$ the classical approximation can also be taken for the particle 2, and if we choose the coordinate system such that the particle 2 is at rest on the origin, we have

$$j_4(2) = e_2 \delta(x_2) \delta(y_2) \delta(z_2), \quad j_i(2) = 0 \quad (i=1, 2, 3). \quad (\text{C} \cdot 8)$$

Then, $A_\mu(1)$ in (C.7) become

$$A_4(1) = e_2/r_1, \quad A_i(1) = 0, \quad (\text{C} \cdot 9)$$

where a contribution has arisen only from D part of δ_+ function. Thus, equation (C.6) becomes the Dirac equation for the particle 1 in the Coulomb field of the fixed particle 2. It is noted that the same results are obtained if we take K_0 instead of K_+ in the equation (2.4).

(ii) Scalar particles

In the same way as above we start from the equation (4.2). With (C.2) we have

$$(\square_1 - m_1^2) \varphi(1) \varphi(2) = ik \varphi(1) \int I_+(2, 2') \delta_+(1, 2') \varphi(2') d\tau_2'. \quad (\text{C} \cdot 10)$$

An equation for the free amplitude $\varphi(2)$ corresponding to (C.4) is

$$\int \left\{ \frac{\partial \varphi^*(2)}{\partial x_{24}} I_+(2, 2') - \varphi^*(2) \frac{\partial I_+(2, 2')}{\partial x_{24}} \right\} d^3 x_2 = \varphi^*(2'), \quad (\text{C} \cdot 11)$$

where the surface integral is performed on the plane $t_2 > t_2'$. Then, we have from (C.10)

$$\begin{aligned} (\square_1 - m_1^2) \varphi(1) \int \left\{ \frac{\partial \varphi^*(2)}{\partial x_{24}} \varphi(2) - \varphi^*(2) \frac{\partial \varphi(2)}{\partial x_{24}} \right\} d^3 x_2 \\ = ik \varphi(1) \int \varphi^*(2') \varphi(2') \delta_+(1, 2') d\tau_2', \end{aligned} \quad (\text{C} \cdot 12)$$

where the integral over x_2 is performed on the surface $t_2 = +\infty$. If we choose the rest system for the particle 2, its current density j_μ is given by

$$\begin{aligned} j_4(2) &= \frac{e_2}{4\pi i} \left\{ \frac{\partial \varphi^*(2)}{\partial x_{24}} \varphi(2) - \varphi^*(2) \frac{\partial \varphi(2)}{\partial x_{24}} \right\} \simeq \frac{e_2 m_2}{2\pi} \varphi^*(2) \varphi(2), \\ j_i(2) &= 0. \end{aligned} \quad (\text{C} \cdot 13)$$

For the scalar particle we should normalize $\varphi(2)$ with respect to charge. Then, taking the classical limit (C.8) and using the definition (4.6), we have from (C.12) a Klein-Gordon equation for the particle 1

$$(\square_1 - m_1^2) \varphi(1) = 2m_1 \frac{e_1 e_2}{r} \varphi(1). \quad (\text{C} \cdot 14)$$

Appendix D. Eigen-values for $n \neq 1$ in the case of scalar particles

(i) One-sheet potential (general l)

We start from (4.12) and replace $\delta_+(x_\mu'^2)$ by $\delta(r' - l')/r'$. Using the formulae (B.1), (B.2a), we get at once

$$f_l(r, r) = -\frac{k}{4P} \int_0^\infty \left(\frac{r'}{r} \right)^{1/2} \left(\frac{1 + \epsilon(r-r')}{2} K_{l+1/2}(Wr) I_{l+1/2}(Wr') \right. \\ \left. + \frac{1 - \epsilon(r-r')}{2} I_{l+1/2}(Wr) K_{l+1/2}(Wr') \right) f_l(r' r') dr', \quad (\text{D} \cdot 1)$$

where we have used the values of α_1, α_2 given by (4.16) and (4.23), and put $r=l$. (D.1) is equivalent to the non-relativistic equation given by (3.11) except for the definitions of W and k . Eigen-values are given again by (4.25).

(ii) *Two-sheets potential* ($l=0$)

We follow the same procedure as in Sec. 4, namely, assuming the solution of (4.33) as

$$\dot{f}_n(u, 0) = aue^{-Wu} L_n^1(2Wu), \quad f_n(0, v) = bve^{-Wv} L_n^1(2Wv), \quad (\text{D} \cdot 2)$$

we insert them into the integrands of (4.33a) and (4.33b). To facilitate the calculations we use the generating function of the Laguerre's polynomials

$$\sum_{n=0}^{\infty} \frac{L_n^1(2Wu) t^n}{n!} = -\frac{te^{-2Wut/(1-t)}}{(1-t)^2}, \quad (\text{D} \cdot 3)$$

and replace $f(u', 0)/u'$ in (4.33) by

$$ae^{-Wu'} \left(-\frac{te^{-2Wu't/(1-t)}}{(1-t)^2} \right) = -\frac{at e^{-Wu'(1+t)/(1-t)}}{(1-t)^2},$$

and similarly for $f(0, v')$. After the u', v' and x integrations in (4.33), which can be performed exactly in the same manner as in Sec. 4, we get the functions of the following type,

$$\exp \left\{ -\frac{1+t}{1-t} Wu - i \frac{m_2^2 v}{a_2 P + iW(1+t)/(1-t)} \right\}$$

To pick up the factor of t^n , we use the following relations

$$e^{-Wu(1+t)/(1-t)} = e^{-Wu} e^{-2Wut/(1-t)} = e^{-Wu} \sum_{n=0}^{\infty} 2Wu L_n^1(2Wu) \frac{t^n}{n \cdot n!},$$

$$e^{-i \frac{m_2^2 v}{a_2 P + iW(1+t)/(1-t)}} = e^{-i(a_2 P - iW)v} e^{-\frac{2Wvt}{\gamma_2 - t}} \\ = e^{-ia_2 P v - Wv} 2Wv \sum_{n=0}^{\infty} L_n^1(2Wv) \frac{t^n \gamma_2^{-n}}{n \cdot n!},$$

with

$$\gamma_1 = \frac{a_1 P + iW}{a_1 P - iW}, \quad \gamma_2 = \frac{a_2 P + iW}{a_2 P - iW},$$

where we have used the value of α_1, α_2 given by (4.23). Thus we get the following result corresponding to (4.35)

$$f_n(u, v) = -\frac{k}{16PIV^2} e^{-W(u+v)} \\ \times \left[a \sum_{m=0}^n \binom{n}{m} \frac{2Wu L_m^1(2Wu) \cdot 2Wv L_{n-m}^1(2Wv)}{m(n-m) \gamma_2^{n-m}} \right]$$

$$+ b \sum_{n=0}^{\infty} \binom{n}{m} \frac{2WuL_m^1(2Wv) \cdot 2WvL_{n-m}^1(2Wv)}{m(n-m)\gamma_1^{n-m}} \Big], \quad u, v > 0. \quad (\text{D} \cdot 4)$$

In (D·4) the term $m=0$ in the first series, for example, means

$$\alpha \frac{2WvL_n^1(2Wv)}{n\gamma_1^n}.$$

From the conditions that (D·4) must be equal to (D·2) for $v=0$ and $u=0$ we have an equation which determines the eigen-values

$$-\frac{k}{4P|Wn} \left\{ 1 + \left(\frac{(Q_1/2 - iW)(Q_2/2 - iW)}{m_1 m_2} \right)^n \right\} = 1, \quad (\text{D} \cdot 5)$$

which is the generalization of (4·37) for $n=1$. We can not give the explicit forms of P and W by solving (D·5) with (4·16) and (4·21) for the general value of n except for $n=1$. We give a few terms of P/M and W in the power series of α ,

$$\begin{aligned} P/M &= 1 - (\mu\alpha^2/Mn^2) (1/2 - i\alpha/2 + p_2\alpha^2 + O(\alpha^3)), \\ W &= (\mu\alpha/n) (1 - i\alpha/2 + q_2\alpha^2 + O(\alpha^3)), \end{aligned} \quad (\text{D} \cdot 6)$$

with

$$\begin{aligned} 2p_2 &= -5/4 + 1/4n^2 + \mu/4n^2M, \\ 2q_2 &= \mu/n^2M - 1. \end{aligned}$$

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Generalization of Statistics

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The possibility and necessity for a series of statistics, in which a state may at most be occupied with n particles, are discussed and the corresponding distribution function

$$W(\epsilon) = \frac{\sum_{m=1}^n m! e^{-m(\epsilon - \eta)/kT}}{\sum_{m=0}^n e^{-m(\epsilon - \eta)/kT}}$$

and the formulas for second quantization

$$\sum_{\text{Perm. among } \alpha, \beta, \dots, \pi} X_\alpha X_\beta \dots X_\pi = 0 \quad \text{for all } \alpha, \beta, \dots, \pi (n+1 \text{ letters}),$$

$$X_\gamma X_\alpha^{*n} X_\beta - X_\alpha^{*n} X_\beta X_\gamma = \delta_{\alpha\gamma} X_\beta,$$

$$X_\alpha^n X_\alpha^{*n} + n X_\alpha^{*n} X_\alpha^n = n X_\alpha^{*n-1}, \quad \sum_{m=0}^n X_\alpha^{*m} X_\alpha^n X_\alpha^{*n-m} = n!$$

and their Hermit conjugate relations,

where X_α denotes the annihilation operator and its conjugate X_α^{*} the creation operator, are established.

§ 1. Introduction

There exist two sorts of statistics, one of which is Fermi's statistics characterized by the fact that a state is allowed to be occupied with one particle, and the other of which is Bose's statistics, (characterized by the fact) that a state is allowed to be occupied with any number of particles.

There seems at present to be no need for statistics of different types. However there remains the possibility of a series of statistics, in one of which a state may at most be occupied with n particles. Mathematically speaking, since Bose's and Fermi's statistics are related respectively to the identical and the alternating representation of the symmetric group, the other statistics will be related to the other representations of that group.

It is believed that the particles composed of two Fermions are subject to Bose's statistics. For instance, let us consider an ensemble of N particles composed of two Fermions. As the wave function of the ensemble, we put

$$\Psi = c \sum D(I) P \phi_1(x_1, y_1) \phi_2(x_2, y_2) \dots \phi_N(x_N, y_N)^* \quad (1)$$

where x_i, y_i are coordinates of Fermions of each group respectively, ϕ_i is wave function of the i -th compound particle, P is a permutation operator acting on the suffices of ϕ_i , and D is representation matrix of the symmetric group, c being a normalization factor.

* See next article.

Operating on Ψ with an arbitrary permutation Q , we have

$$Q\Psi = D(Q)\Psi.$$

The operation Q is equivalent to the simultaneous permutation of x 's and y 's. Namely $Q = Q_x Q_y$, consequently $D(Q) = D_x(Q) D_y(Q)$.

As the particles are all Fermions, $D(Q)$ are $+1$ or -1 according to either Q is even permutation or odd. Consequently $D(Q)$ is always $+1$ whatever Q is. This is the very reason why the law above mentioned is believed.

However, the state $\phi(x_2, y_1)$, for instance, represents also a compound state and so the summation of P 's in (1) should be taken all the elements of the direct product $S_x \times S_y$, if we take into account for not only the simultaneous interchange, but also the partial interchange between Fermions. (Where S_x and S_y are the symmetric group of order N .)

As we show in the last section, in this case, we can verify by means of the group theory that the ensemble obeys to the statistics with the capacity 2 for a state; viz.

$$\Psi \text{ (more than 3 Fermions fall into the same state)} \equiv 0,$$

$$\Psi \text{ (2 Fermions at most fall into the same state)} \neq 0.$$

Generally we can conclude that the ensemble of the compound particles composed of n Fermions is subject to the statistics with capacity n .

In application, for instance, to the analysis of band spectra the customary theory may be correct, because the partial interchange between nucleons may be negligible in the phenomena.

However, for strong interaction, the new stand point should be taken into account in such a phenomenon as star-producing process, which may be treated field-theoretically as if the nucleus were a point.

The conception of "elementary particle" is temporary and changes with the age. The elementary particle at an age will turn out to be regarded as the compound in a later age. Therefore, there is no elementary particle in the absolute sense, but only in the relative sense. It is desirable that a theory of elementary particles may be established which works well whether the internal structure of the particle may be negligible or not. The method of second quantization which is usually believed to be based on the character of elementary particles as mathematical points, a theoretical procedure which may be called "punctiformalism", is rather general if we accept a more general formalism of second quantization which displays complementarily the structure of the particle lost by the punctiformalism.

§ 2. Wave function of the ensemble

Let us treat an ensemble with N particles of the same kinds, which have the same Hamiltonians. If there is a wave function of the ensemble $\phi(r_1, r_2, \dots, r_N)$, $P\phi(r_1, r_2, \dots, r_N)$ is also an allowed wave function, where $P = \begin{pmatrix} 1 & 2 & 3 & \dots & N \\ 1' & 2' & 3' & \dots & N' \end{pmatrix}$ is a permutation operator acting upon the subscripts distinguishing each state.¹⁾ Also the linear combination of these is allowed. In general, we have for the wave function

$$\Psi = \sum_P a_P P\psi(r_1, r_2, \dots, r_N), \quad (2)$$

where a_P is a constant.

Operated by the transposition (i, k) between the i -th and the k -th state, it represents again physically the same state as the original Ψ apart from the case of accidental degeneracy. Hence, we have

$$(i, k)\Psi = c\Psi, \quad (3)$$

where c is a const., subject to the condition $|c|=1$.

Repeating the transposition (i, k) , we have

$$(i, k)(i, k)\Psi = c^2\Psi = \Psi.$$

Therefore, it follows that $c^2=1$. If we limit ourselves to the case that Ψ is merely a scalar, we have as only possibilities $c=\pm 1$ corresponding to the Bose's and the Fermi's statistics respectively. If, then, we may have other possibilities of statistics, we are at once led to the idea that we should extend Ψ to be of matrix type (or operator type), but not the wave function ψ . Spur $\Psi^*\Psi$ may be taken as the definition of probability of Ψ -state.²⁾

Though what has been done above would mean a case of degeneration from the current view of quantum mechanics, we shall show later that it does not necessarily coincide with the case of degeneration.

Along with (2), we may write $c\Psi = \Psi c$, so we have two cases

$$P\Psi = c\Psi \quad \text{or} \quad \Psi c, \quad (4)$$

Ψ being now considered to be a matrix. We can easily see that c must be a representation of the permutation (i, k) , or generally P or P^{-1} . Putting the representation $D(P^{-1})$ of the inverse P in (1) instead of a_P , we have

$$\Psi = c' \sum_Q D(Q^{-1}) Q \psi(r_1, r_2, \dots, r_N), \quad (5)$$

where c' is a number.

This specification satisfies the second type of the equation (4).³⁾ Among the representations of the symmetric group, we should search for one which satisfies our conditions, just like the identity or alternating representations correspond to the Bose's or the Fermi's statistics respectively.

§ 3. Conditions necessary for the capacity= n

The conditions necessary for the capacity= n in the true sense may be expressed that the wave function of the ensemble must identically vanish when more than n particles

1) At first sight, it seems rather natural to take the permutation among the particle-suffices instead of the state-suffices, but it is not convenient for our present purpose.

2) Originally, the correspondence from complex state ψ to real number $\psi^*\psi$ is quite artificial, and we can define probability of the generalized Ψ adequately according to characters of the operators contained in it.

3) It is convenient to take $D(P^{-1})$ and not to take $D(P)$ as a_P , when we proceed from the theory of the configuration space to that of second quantization.

occupy one and the same state,⁴⁾ and otherwise must not identically vanish.

Written in formula,

$$\Psi \text{ (when the number of particles in the same state is greater than } n+1) \equiv 0, \quad (6)$$

$$\Psi \text{ (when the number of particles in the same state is equal to or less than } n) \neq 0. \quad (6')$$

In the case of capacity $=n$ in a wider sense, as, for example, resulting from the degeneration of Fermi particles, formula (6') would be abandoned.

Let us examine if there are such representations as to satisfy the above conditions among the functions of type (5).

For simplicity, we first confine ourselves to the case $n=2$, but the following arguments do not spoil the generality, as the case $n > 2$ may be treated in much the same way.

§ 4. The case $n=2$

First of all, we consider the case where the total number of particles $N=3$, for this is the fundamental case. In this case, Ψ is linearly composed of $3!=6$ terms;

$$\text{viz.} \quad \Psi = c' \sum D(P^{-1}) P\psi(r_1, r_2, r_3). \quad (7)$$

According to (6), if the three particles are in the same state,

$$\Psi = c' (\sum D(P^{-1})) \psi(r_1, r_2, r_3) \equiv 0,$$

$$\text{since} \quad P\psi(r_1, r_2, r_3) = \psi(r_1, r_2, r_3), \quad \text{for all } P.$$

Accordingly we are led to demand

$$\sum_{\text{all } P} D(P^{-1}) = \sum D(P) = I \equiv 0.$$

This means that the sum of the representation matrices of the third order symmetric group must be the zero matrix. For an arbitrary permutation P , we have in general $D(I')V=V$, so that $\text{spur } DV = \text{spur } V$, in particular.

On the other hand, the quotient of $\text{spur } V$ divided by the total number of elements of the symmetric group is the number of times how often the identity representation is contained in the representation D . Therefore, on the restriction that the representation should not contain identity representation, we have always

$$\text{spur } V = 0$$

so that $\text{spur } DV = 0$ for each arbitrary element $D(I')$ in the group. Finally we see it is ensured that $V \equiv 0$ by virtue of Burnside's theorem on the assumption of the irreducibility of the representation D . Thus we have seen that $D(I')$ may be any irreducible⁵⁾ representation other than the identity one.

In the third order symmetric group, there are three sorts (corresponding to the number of the classes of the group) of irreducible representations, of which the identity representa-

4) From the condition that the i th and the j th are the same state follows the identity, $(i, j)\Psi = \Psi$.

5) One sees easily that the irreducibility condition is not always necessary.

tion $\{3\}^{(1)}$ and the alternating representation $\{1, 1, 1\}$ are one dimensional, and the third representation $\{2, 1\}$ is two dimensional.

As the alternating representation is weeded out by the second formula (6') of our statistical conditions, the only one which survives is the representation $\{2, 1\}$. This is immediately recognized upon calculation⁽¹⁾ of characters of the group. That is to say, for two arbitrary permutations P and Q differing in one transposition only, it may be readily confirmed for this representation that it does not hold

$$D(P^{-1}) + D(Q^{-1}) \equiv 0.$$

In fact, the representation matrices are as follows;

$$\begin{aligned} (1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & (1, 2) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & (1, 3) &= \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \\ (1, 2, 3) &= \begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{3}}{2} & \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, & (1, 3, 2) &= \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{3}}{2} & \\ +\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, & (2, 3) &= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}. \end{aligned}$$

At any rate, we have seen that there is a function with such coefficients as to satisfy the statistical conditions desired. As remarked above it is not always necessary that the representation be irreducible, but they must contain at least such a representation as $\{2, 1\}$ more than once, among other representations except the identity one.

After these preparations, we shall shift ourselves to the case where the total number is N . The first thing to be done is to ensure if there are such D 's as to make $\Psi \equiv 0$ when arbitrary three states in Ψ turn out to be the same. For this purpose, it is sufficient to consider only the case where the first three states are equal:

Ψ (when the first three states are equal)

$$= c' \sum_{P'} \{ (\sum_{P'} D(P'^{-1})) \psi(r_{1'}, r_{2'}, r_{3'}, \dots, r_{N'}) \}, \quad (8)$$

where P' run over $3! = 6$ elements in which the permutations among $4' 5' \dots N'$ are common and different only with respect to $1' 2' 3'$. For instance, we have

$$P'_1 = \begin{pmatrix} 1 & 2 & 3 & \dots & N \\ 1' & 2' & 3' & \dots & N' \end{pmatrix}, \quad P'_{12} = \begin{pmatrix} 1 & 2 & 3 & \dots & N \\ 2' & 1' & 3' & \dots & N' \end{pmatrix}, \text{ etc.} \quad (9)$$

In order that (8) vanishes identically, it is sufficient that

$$\sum_{6 \text{ terms}} D(P'^{-1}) \equiv 0 \quad (10)$$

for all sets of such P' 's.

6) See, for instance; Murnaghan, *The theory of group representation*.
Weyl, *Classical group*.
Weyl, *Gruppentheorie und Quantenmechanik*.

On the other hand, the symmetric group G of the N -th order can be developed by the left cosets of the third order subgroup F consisting of 1, 2, 3. Thus we have

$$G = \sum P_i F$$

where F stands for (1), (12), (13), (23), (123), (132).

Taking arbitrarily $P_1 = \begin{pmatrix} 1 & 2 & 3 & \cdots & N \\ 1' & 2' & 3' & \cdots & N' \end{pmatrix}$ not belonging to F , and forming the product $P_1 F$, we obtain

$$P_1 = \begin{pmatrix} 1 & 2 & 3 & \cdots & N \\ 1' & 2' & 3' & \cdots & N' \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & \cdots & N \\ 2' & 1' & 3' & \cdots & N' \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & N \\ 3' & 2' & 1' & 4' & \cdots & N' \end{pmatrix} \text{ etc.}$$

Next, let us take an arbitrary P_2 which belongs neither to F nor to $P_1 F$, and form the product $P_2 F$. Proceeding in this way we get a sequence of $P_i F$, each coset $P_i F$ corresponding to each set of P 's referring to (8). The left hand side of (10) may be written as

$$\sum_{P'} D(P'^{-1}) = \sum_{f_k} D[(P_i f_k)^{-1}] = \left\{ \sum_{f_k} D(f_k^{-1}) \right\} D(P_i^{-1}) = \left\{ \sum_{f_k} D(f_k) \right\} D(P_i^{-1}),$$

where f_k is an element of F . Therefore (10) demands that

$$\sum_{f_k} D(f_k) \equiv 0.$$

In fact, referring to suitably chosen axis, it being possible to arrange the matrix elements of the representation of the subgroup, so that it contains the irreducible representations $\{2, 1\}$, $\{1, 1, 1\}$ in diagonal block form, it is sure that $\sum_k D(f_k) \equiv 0$, if we choose a representation D not containing the identical representation.

Thus the statistical conditions (6) and (6') will be valid by the choice of the representation $\{2, 1\}$ more than once.

However, it is not uniquely determined, there remaining a certain number of possibilities.

The practical method of construction of such representations can be obtained easily by the repeated use of the branching rule.⁶⁾

What the branching rule implies is that the $N-1$ th order subgroup of the irreducible representation of the N th order symmetric group $\{\lambda_1, \lambda_2, \cdots, \lambda_k\} \rightarrow \{\lambda_i\}$ is a partition of N and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, $N = \sum \lambda_i$ can be reduced to a sum of the irreducible representations of the $N-1$ th order $\{\lambda_1, \cdots, \lambda_i - 1, \cdots, \lambda_k\}$ where i runs from 1 to k , and the partition of $N-1$ is so chosen as to recover its descending order.

We can see, for instance, the third order subgroup of the representation $\{2, 1, 1, \cdots, 1\} \rightarrow 2 + 1 + 1 + \cdots + 1 = N$ contains irreducible representation $\{2, 1\}$ once and $\{1, 1, 1\}$ $N-3$ times. Similarly $\{2, 2, 1, \cdots, 1\}$ contains $\{2, 1\}$ $N-3$ times and $\{1, 1, 1\} \times (N-4)(N-3)/2$ times.

Finally we arrive at the conclusion that we have to take the representation of which partition contains 2 at least more than once but never contains the number larger than 2.

Therefore, in the case of capacity $n=2$ there are

$$\{2, 1, 1, \cdots, 1\} \quad \{2, 2, 1, \cdots, 1\} \cdots \{2, 2, \cdots, 2, 1\}, \quad (N-1)/2 \text{ kinds}$$

for N odd,

$$\{2, 1, 1, \dots, 1\} \quad \{2, 2, 1, \dots, 1\} \dots \{2, 2, \dots, 2, 2\}, \quad N/2 \text{ kinds}$$

for N even.

Quite similarly in the case of capacity n , we shall have to choose representations of which the partition contain n necessarily and never to contain the number larger than n in its partitions.

§ 5. Statistics and degeneration

Let us now inquire the relations between the statistics above discussed and the degenerations caused by neglecting the certain physical quantities of the Fermi particle; for they seem to be the same thing on superficial observation.

For instance, the characteristics of the ensemble of electrons, explicitly neglecting their spin and taking account of the space coordinates only, seems to be identical with that of the ensemble of the particles with capacity=2.

Actually in such a case, there appears the representation $\{1, 1, \dots, 1\}$ when all spins are parallel, the representation $\{2, 1, \dots\}$ when a pair of spins is antiparallel and others all parallel, and so on, while in the case of capacity=2 in our true sense there never occurs the representation $\{1, 1, \dots, 1\}$. Originally, the Fermi statistics is not necessarily deduced from quantum mechanics but established by the postulate of the exclusion principle devised ad hoc by Pauli. Our statistics with capacity=2 corresponds to such a case that the possible occurrence of the spin parallel state by neglecting the physical quantity with two degrees of freedom of the Fermi particles is on compulsion unallowed. Generally speaking, the possible states are $\{1, \dots, 1\}$, $\{2, 1, \dots, 1\} \dots \{2, 2, \dots, 2\} \dots \{3, 1, \dots, 1\} \dots \{3, 2, \dots\} \dots \{4, 1, \dots\} \dots \{n, 1, \dots\} \dots$ etc., if we neglect implicitly the physical quantity with n degrees of degeneracy of the Fermi particles.

Therefore, the statistics with capacity= n might be said to prohibit all states not containing n in their partitions from those possible states of the ensemble of the Fermi particles.

Now the conception of the so-called statistics seems to force us to one particular symmetry; the Fermi statistics to alternating representation and the Bose's to identical representation, so that it is desirable to press our case into a particular representation.

To treat statistically the ensemble of electrons or photons, account must actually be taken of all the particles in the universe and it needs to construct the antisymmetrical or symmetrical wave functions for all of them.

However, if we take out any number of electrons or photons among from all of them, the symmetric character of wave functions inherits, and it is enough to take account of the antisymmetrical or symmetrical character referring to only those particles concerned.

The utility of the Fermi's statistics, indeed, depends upon this practical inheritance. In our case this inheritance is also indispensable. It is not adequate if other kinds of statistics appear by addition or reduction of particles.

In fact, in the case of representation $\{2, 2, 1, \dots, 1\}$, for instance, reducing the number

of particles by one, it is split into $\{2, 1, 1, \dots, 1\} + \{2, 2, \dots, 1\}$ by the branching rule, and reducing the number by one, once more, into $\{1, 1, \dots, 1\} + 2\{2, 1, \dots, 1\} + \{2, 2, \dots, 1\}$. Moreover, reducing any number of particles, there remains the numeral 2 necessarily in the partition.⁷⁾ All these representations satisfy the statistical condition (6) and (6').

Thus we may conclude that any representation having 2 in its partition can not be statistically distinguishable and has equal right in this respect. This conglomeration of the representations is necessary in order to ascertain the inheritance of our statistics. However, it is to be noted that this inheritance is not one of irreducible as in the case of the Fermi's and the Bose's statistics.

Physically speaking, to include the numerous representations in one category amounts to postulate the impossibility of detecting a hidden physical quantity, which serves to distinguish the states corresponding to the representations.

Therefore, the particle with capacity= n in the true sense has only a proper function corresponding to a state, while the degenerate particle accompanies a certain number of proper functions belonging to a state.

§ 6. The distribution function of the statistics with capacity= n in the true sense

Now let us shift our discussion to the problem concerning the distribution function. Of course, in this case we must be careful in distinguishing between the case of the true sense and that of degeneration. We shall treat the former only.

According to Fowler⁸⁾ it is convenient to introduce the generating function. In the statistics of the true sense, the proper function is unique for any value of the capacity and the generating function is

$$g(x^{\varepsilon}) = 1 + x^{\varepsilon} + x^{2\varepsilon} + \dots + x^{n\varepsilon}.$$

The distribution function is

$$W(\varepsilon) = \lambda \frac{\partial}{\partial \lambda} \log g(\lambda \vartheta^{\varepsilon}), \text{ where } \vartheta = e^{-\frac{1}{\lambda T}} \text{ and } \lambda \text{ are provided for the condition}$$

$$N = \sum \lambda \frac{\partial}{\partial \lambda} \log g(\lambda \vartheta^{\varepsilon}).$$

$$\text{Putting } \lambda = e^{\eta(T)/kT},$$

$$W(\varepsilon) = \frac{e^{-(\varepsilon-\eta)/kT} + 2e^{-2(\varepsilon-\eta)/kT} + \dots + ne^{-n(\varepsilon-\eta)/kT}}{1 + e^{-(\varepsilon-\eta)/kT} + \dots + e^{(n-1)(\varepsilon-\eta)/kT}} \quad (11)$$

or

$$W(\varepsilon) = \frac{n + (n-1)e^{(\varepsilon-\eta)/kT} + \dots + e^{(n-1)(\varepsilon-\eta)/kT}}{1 + e^{(\varepsilon-\eta)/kT} + \dots + e^{n(\varepsilon-\eta)/kT}} \quad (11')$$

In general η is a function of temperature T and equal to ε_0 at zero point of temperature,

7) The case when the total number of the particles is less than 2 has no grave meaning, just as the case of one particle in the Fermi statistics.

8) Fowler, *Statistical Mechanics*, 2nd edition.

whose physical meaning corresponds to the fact that the energy levels beneath ε_0 are entirely occupied and the levels above it are entirely empty as seen in (11).

§ 7. Equation of motion of the ensemble in the configuration space

Before we enter into the second quantization formalism obeying our statistics, we shall begin with the equation of motion in the configuration space. Since we have to do with the particles of one and the same kind, the Hamiltonians must be of the same form for each of the particles. Furthermore we may assume, for simplicity, the Hamiltonian not to contain the mutual interaction. Then for each particle the equation of motion is given by

$$-i\hbar \cdot dx_\alpha/dt = \sum_\beta H_{\alpha\beta} x_\beta$$

where x_α stands for the proper function in α state. As the total Hamiltonian of the ensemble is given by the sum of single particle Hamiltonians, we can take as the wave function of the ensemble a linear combination of the products of the proper functions, for all particles, whose coefficients are the inverse representations of such permutations⁹⁾ among the state-suffices as determined under the condition that the particle-suffices are arranged in a certain fixed order.

For example, when three particles 1, 2, 3 occupy the 5th, the 8th and the 9th states respectively, we may take the form

$$\begin{aligned} \Psi \sim & D^{-1}(1)x_5(1)x_8(2)x_9(3) + D^{-1}(5,8)x_8(1)x_5(2)x_9(3) \\ & + D^{-1}(8,9)x_5(1)x_9(2)x_8(3) + D^{-1}(5,9)x_9(1)x_8(2)x_5(3) \\ & + D^{-1}(5,8,9)x_8(1)x_9(2)x_5(3) + D^{-1}(5,9,8)x_9(1)x_5(2)x_8(3). \end{aligned}$$

First we must mention an important character of the wave function thus defined. For instance, when the first and the second states in the wave function are identical, even if the states (1st and 2nd) are exchanged, it remains unchanged as a whole, because of relation

$$(1,2)P(x_{\alpha_1}^1 x_{\alpha_2}^2 \cdots x_{\alpha_N}^N) = P(x_{\alpha_2}^1 x_{\alpha_1}^2 \cdots x_{\alpha_N}^N). \quad (12)$$

Such a pair as in (12) of the wave functions of the ensemble belongs to a left coset of the symmetric group of the N th order developed by the subgroup of the second order $H\{(1), (12)\}$ after the manner of § 3, viz.

$$\Psi = c' \sum_{P'} [D(1) + D(12)] D^{-1}(P') P' x_{\alpha_1}^1 x_{\alpha_2}^2 \cdots x_{\alpha_N}^N.$$

Consequently, operating on Ψ from the left, $D(1,2)$ has the character of unit too, besides $D(1)$ itself. Similarly when n states are identical, Ψ can be developed into the coset by the symmetric subgroup of the n th order, and therefore there are $n!$ virtual

9) Though it seems rather natural to take the permutations among the particle suffices fixing the order of state-suffices, it is not convenient to proceed from the configuration space to the second quantization.

unit, operating to the left of Ψ . Thus, the normalization factor C' in (5) must be chosen as

$$C' = 1/\sqrt{n_1! n_2! \dots}$$

where $N = \sum n_i$, and n_i is the occupation number in the i -th state. There must be a factor depending on the dimensions of the representations which may, however, be ignored because of its constancy.

Now, let us consider the time-variation of the wave function. With

$$\Psi = 1/\sqrt{n_1! n_2! \dots} \sum D^{-1}(P) P x_{\alpha_1}(1) x_{\alpha_2}(2) \dots x_{\alpha_N}(N),$$

we have

$$\begin{aligned} -\hbar \frac{d}{dt} \Psi &= -i\hbar / \sqrt{n_1! n_2! \dots} \left[\left\{ D(1) \frac{dx_{\alpha_1}(1)}{dt} x_{\alpha_2}(2) \dots x_{\alpha_N}(N) \right. \right. \\ &\quad \left. \left. + D^{-1}(a_1 a_2) \frac{dx_{\alpha_2}}{dt} x_{\alpha_1}(2) \dots x_{\alpha_N}(N) + \dots \right\} \right. \\ &\quad \left. + \left\{ D(1) x_{\alpha_1}(1) \frac{dx_{\alpha_2}(2)}{dt} \dots x_{\alpha_N}(N) \right. \right. \\ &\quad \left. \left. + D^{-1}(a_1 a_2) x_{\alpha_2}(1) \frac{dx_{\alpha_1}(2)}{dt} \dots x_{\alpha_N}(N) + \dots \right\} + \dots \right] \\ &= 1/\sqrt{n_1! n_2! \dots} \left[\left\{ D(1) \sum_{\beta_1} H_{\alpha_1 \beta_1} x_{\beta_1}(1) x_{\alpha_2}(2) \dots x_{\alpha_N}(N) \right. \right. \\ &\quad \left. \left. + D^{-1}(a_1 a_2) \sum_{\beta_2} H_{\alpha_2 \beta_2} x_{\beta_2}(1) x_{\alpha_1}(2) \dots x_{\alpha_N}(N) + \dots \right\} \right. \\ &\quad \left. + \left\{ D(1) \sum_{\beta_2} H_{\alpha_2 \beta_2} x_{\alpha_1}(1) x_{\beta_2}(2) \dots x_{\alpha_N}(N) \right. \right. \\ &\quad \left. \left. + D^{-1}(a_1 a_2) \sum_{\beta_1} H_{\alpha_1 \beta_1} x_{\alpha_2}(1) x_{\beta_1}(2) \dots x_{\alpha_N}(N) + \dots \right\} + \dots \right] \\ &= 1/\sqrt{n_1! n_2! \dots} \left[H_{\alpha_1 \alpha_1} \sum_P D^{-1}(P) P x_{\alpha_1}(1) x_{\alpha_2}(2) \dots x_{\alpha_N}(N) \right. \\ &\quad \left. + H_{\alpha_1 \beta_1} \sum_{P \beta_1'} D^{-1}(P) P'_{\beta_1} x_{\beta_1}(1) x_{\alpha_2}(2) \dots x_{\alpha_N}(N) + \dots \right. \\ &\quad \left. + H_{\alpha_2 \alpha_2} \sum_P D^{-1}(P) P x_{\alpha_1}(1) \dots x_{\alpha_N}(N) \right. \\ &\quad \left. + H_{\alpha_2 \beta_2} \sum_{P \beta_2'} D^{-1}(P) P'_{\beta_2} x_{\alpha_1}(1) x_{\beta_2}(2) \dots x_{\alpha_N}(N) + \dots \right], \quad (13) \end{aligned}$$

where P'_{β_1} means a permutation which has β_1 in place of α_1 in P .

To express each term in (13) by Ψ defined by (5), we must now study the relation between P and P'_{β_i} . At the beginning we have arranged the sequence of states $\{\alpha_i\}$ in a certain order and we have labeled the number 1, 2, 3 etc...to these states, as the procedure of constructing Ψ . Accordingly we must rearrange P'_{β_i} in accordance with this prescription. For instance, we assume that all states under consideration are occupied by

one particle, namely $n_i=1$ for all these states, and instead of u_i in P , β_i which ought to be labeled to the j -th state appears in P'_{β_i} , where $j > i$, $n_{\beta_i}=1$. Then, denoting the rearranged P'_{β_i} by P_{β_i} , we have the following relations

$$P = P_{\alpha_i} = (u_i, \beta_i) P'_{\beta_i} (u_i, \beta_i),$$

$$P'_{\beta_i} = P_{\beta_i} f_{\alpha_i}^{\beta_i},$$

where

$$\begin{aligned} P_{\alpha_i} &= \begin{pmatrix} \alpha_1 \cdots \alpha_i \cdots \alpha_{j-1}, & \beta_i, & \alpha_{j+1} \cdots \alpha_N \\ \alpha'_1 \cdots \alpha'_i \cdots \alpha'_{j-1}, & \beta_i, & \alpha'_{j+1} \cdots \alpha'_N \end{pmatrix}, & P'_{\beta_i} &= \begin{pmatrix} \alpha_1 \cdots \beta_i \cdots \alpha_{j-1}, & u_i, & \alpha_{j+1} \cdots \alpha_N \\ \alpha'_1 \cdots \beta'_i \cdots \alpha'_{j-1}, & u_i, & \alpha'_{j+1} \cdots \alpha'_N \end{pmatrix}, \\ P_{\beta_i} &= \begin{pmatrix} \alpha_1 \cdots \alpha_{i-1}, & u_i, & \alpha_{i+1} \cdots \alpha_{j-1}, & \beta_i, & \alpha_{j+1} \cdots \alpha_N \\ \alpha'_1 \cdots \alpha'_{i-1}, & u_i, & \alpha'_{i+1} \cdots \alpha'_{j-1}, & \beta_i, & \alpha'_{j+1} \cdots \alpha'_N \end{pmatrix}, \\ f_{\alpha_i}^{\beta_i} &= \begin{pmatrix} \alpha_1 \cdots \alpha_{i-1}, & u_i, & \beta_i, & \alpha_{i+1} \cdots \alpha_{j-1}, & \alpha_{j+1} \cdots \alpha_N \\ \alpha'_1 \cdots \alpha'_{i-1}, & u_i, & \alpha_{i+1}, & \alpha_{i+2} \cdots \alpha_{j-1}, & \beta_i, & \alpha_{j+1} \cdots \alpha_N \end{pmatrix}. \end{aligned}$$

And consequently we have

$$P^{-1} = (u_i, \beta_i) f_{\beta_i}^{\alpha_i} P_{\beta_i}^{-1} (u_i, \beta_i),$$

$$f_{\beta_i}^{\alpha_i} = (f_{\alpha_i}^{\beta_i})^{-1} = (u_{j-1}, \beta_i) \cdots (u_{i+2}, u_{i+3}) (u_{i+1}, u_{i+2}). \quad (14)$$

Since the process of inserting $(f_{\beta_i}^{\alpha_i} P_{\beta_i}^{-1})$ between the transpositions (u_i, β_i) is an operation replacing β_i in $(f_{\beta_i}^{\alpha_i} P_{\beta_i}^{-1})$ by u_i , we denote it simply by $I_{\alpha_i \beta_i}$, which operates from the left only.

Generally in the case of the capacity $= n$, a state ought to be considered to split into the virtually distinguished n states which can not be physically distinguishable. And so a state is labeled n -fold. For instance, in the case $n=2$, the i -th state is labeled virtually by $2i-1$ as well as by $2i$. In such an expedient, as the capacity for each virtual state is 1, the formula (14) is available.

In spite of such distinction and independency of the states in the configuration space, we must identify them in the space of occupants numbers. When $n=2$, since the occupation number n are limited to 0, 1, 2, a matrix referring to a state is, in general, expressed in the form

$$\alpha(n'n'') = \begin{pmatrix} \alpha(00) & \alpha(01) & \alpha(02) \\ \alpha(10) & \alpha(11) & \alpha(12) \\ \alpha(20) & \alpha(21) & \alpha(22) \end{pmatrix}.$$

To obtain the general form of f_{ij} , we shall introduce for convenience a matrix in a direct product from. However, not the individual factors in the direct product but only the product itself has the mathematical meaning.

$$f_{i,j} = \cdots \times E \times E \times \cdots \times \begin{pmatrix} I & 0 \\ 0 & I_{2j, 2j-1} \end{pmatrix} \begin{pmatrix} 0 \\ (2j, 2j-1) & 0 \\ 0 & 0 & I \end{pmatrix} \times E_{j=1} \times E_{j=2} \times \cdots \times E_{i+1}$$

$j > i$

$$\times \begin{pmatrix} 0 & 0 & 0 \\ \begin{pmatrix} I & 0 \\ 0 & I_{2i+1, 2i} \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} (2i+1, 2i) & 0 \\ 0 & I \end{pmatrix} & 0 \end{pmatrix} \times E \times E \times \dots$$

where

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_i = \begin{pmatrix} I_{2i+1, 2i} & I_{2i, 2i-1} & 0 & 0 \\ 0 & \begin{pmatrix} I_{2i+1, 2i} (2i, 2i-1) & 0 \\ 0 & (2i+1, 2i) I_{2i, 2i-1} \end{pmatrix} & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} (2i+1, 2i) & 0 \\ 0 & I \end{pmatrix} \end{pmatrix},$$

$$f_{i,j}^* = \dots \times E \times E \times \dots \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times E \times E \times \dots,$$

$$f_{i,j} = \dots \times E \times E \times \dots \times \begin{pmatrix} 0 & \begin{pmatrix} I & 0 \\ 0 & I_{2j, 2j+1} \end{pmatrix} & 0 \\ 0 & 0 & \begin{pmatrix} (2j, 2j+1) & 0 \\ 0 & I \end{pmatrix} \\ 0 & 0 & 0 \end{pmatrix} \times E'_{j+1} \times E'_{j+2} \times \dots \times E'_{\infty-1}$$

$$\times \begin{pmatrix} 0 & 0 & 0 \\ \begin{pmatrix} I & 0 \\ 0 & I_{2i, 2i-1} \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} (2i-2, 2i-1) & 0 \\ 0 & I \end{pmatrix} & 0 \end{pmatrix} \times E \times E \times \dots$$

where

$$E'_i = \begin{pmatrix} I_{2i-2, 2i-1} I_{2i-1, 2i} & 0 & 0 \\ 0 & \begin{pmatrix} I_{2i-2, 2i-1} (2i-1, 2i) & 0 \\ 0 & (2i-2, 2i-1) I_{2i-1, 2i} \end{pmatrix} & 0 \\ 0 & 0 & \begin{pmatrix} (2i-2, 2i-1) & 0 \\ 0 & I \end{pmatrix} \end{pmatrix}$$

where I denotes the usual unit matrix, and $I_{i, i-1}$ is so defined that

$$I_{i, i-1} (i-1, i-2) = (i, i-2) \quad \text{and} \quad I_{i+1, i} I_{i, i-1} = I_{i+1, i-1}, \quad I_{i, i-1} I_{i-1, i} = I,$$

when operating from the left and acts as the usual unit when operating from the right. Of course, the introduction of these operators is not essential but an expedient to write the matrix in a direct product from.

After these preparations, we shall rewrite the equation (13) in the configuration space to that in the occupation number space. Then we have

$$-i\hbar \frac{d}{dt} \Psi(n_1 n_2 \dots) = \sum_i n_i H_{ii} \Psi(n_1 n_2 \dots) + \sum_{i \neq j} \sqrt{n_i (n_j + 1)} H_{ij} I_{ij} f_j^i \Psi(n_1 \dots n_{i-1}, n_{j+1})$$

where

$$f_j^i = (n_1 n_2 \dots | f_{ij} | n_1 \dots n_i - 1, \dots n_j + 1, \dots)$$

or, namely

$$-i\hbar \frac{\partial}{\partial t} \Psi = H \Psi,$$

where

$$H = \sum_{i,j} H_{ij} I_{ij} f_{ij} N_{ij} = \sum H_{ij} O_{ij},$$

$$O_{ij} = I_{ij} N_{ij} f_{ij} \quad (15)$$

and

$$N_{ij} = \dots \times \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \times \dots \times \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \times \dots \times \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \times \dots$$

and Ψ is a vector whose component is $\psi(n_1 n_2 \dots)$.

§ 8. The second quantization for the capacity 2

To proceed to the second quantization formula, we must split O_{ij} into such a product form as $O_{ij} = X_i^* X_j$. Hence, if we introduce X 's such that

$$X_i = \dots \times E \times E \times \begin{pmatrix} 0 & \begin{pmatrix} S_{2i} I_{2i, 2i-1} & 0 \\ 0 & S_{2i-1} \end{pmatrix} & 0 \\ 0 & 0 & \begin{pmatrix} \sqrt{2} S_{2i} (2i, 2i-1) & 0 \\ 0 & \sqrt{2} S_{2i-1} \end{pmatrix} \\ 0 & 0 & 0 \end{pmatrix} \\ \times E_{i-1} \times E_{i-2} \times \dots \times E_2 \times E_1 \times \dots$$

or

$$\bar{X}_i = \dots \times E \times E \times \begin{pmatrix} 0 & \begin{pmatrix} S_{2i}^* I_{2i-1, 2i-2}^* & 0 \\ 0 & S_{2i-1}^* \end{pmatrix} & 0 \\ 0 & 0 & \begin{pmatrix} \sqrt{2} S_{2i}^* (2i-1, 2i-2) & 0 \\ 0 & \sqrt{2} S_{2i-1}^* \end{pmatrix} \\ 0 & 0 & 0 \end{pmatrix}, \\ \times E_{i-1}^* \times E_{i-2}^* \times \dots \times E_2^* \times E_1^* \times \dots$$

where

$$E_i^* = \begin{pmatrix} I_{2i, 2i-1}^* I_{2i-1, 2i-2}^* & 0 & 0 \\ 0 & \begin{pmatrix} (2i, 2i-1) I_{2i-1, 2i-2}^* & 0 \\ 0 & I_{2i, 2i-1}^* (2i-1, 2i-2) \end{pmatrix} & 0 \\ 0 & 0 & (2i, 2i-1) (2i-1, 2i-2) \end{pmatrix}$$

and

$$X_i^* = \dots \times E_1^* \times E_2^* \times \dots \times E_{i-1}^* \times \begin{pmatrix} 0 & 0 & 0 \\ I_{2i-1, 2i}^* S_{2i}^* & 0 & 0 \\ 0 & \begin{pmatrix} \sqrt{2} (2i-1, 2i) S_{2i}^* & 0 \\ 0 & \sqrt{2} S_{2i-1}^* \end{pmatrix} & 0 \end{pmatrix}, \\ \times E \times E \times \dots$$

where

$$E_i^* = \begin{pmatrix} I_{2i-1, 2i}^* I_{2i, 2i+1}^* & 0 & 0 \\ 0 & \begin{pmatrix} (2i-1, 2i) I_{2i, 2i+1}^* & 0 \\ 0 & I_{2i-1, 2i}^* (2i, 2i+1) \end{pmatrix} & 0 \\ 0 & 0 & (2i-1, 2i) (2i, 2i+1) \end{pmatrix}$$

$$\bar{X}^* = \cdots \times E_1' \times E_2' \times \cdots \times E_{i-1}' \times \left(\begin{array}{ccc} 0 & 0 & 0 \\ I_{2i-2, 2i-1} S_{2i}' & 0 & 0 \\ 0 & \left(\sqrt{2} (2i-2, 2i-1) S_{2i}' \right) & 0 \\ 0 & 0 & \sqrt{2} S_{2i-1}' \end{array} \right) \\ \times E \times E \times \cdots$$

where $I_{i, i+1}^*$ is an operator so defined that $(i-1, i)I_{i, i+1}^* = (i-1, i+1)$, $I_{i, i+1}^* I_{i+1, i+2}^* = I_{i, i+2}^*$, operating from the right, as usual unit from the left.

We have

$$\left. \begin{array}{l} X_i^* X_j = O_{ij}, \quad \text{for } j < i \\ X_i^* X_j = O_{ij}, \quad \text{for } i > j \\ \text{and} \\ X_i^* X_i = X_i^* X_i = O_{ii}, \quad \text{for } i=j \end{array} \right\} \quad (16)$$

assuming that $S_i = I_{-\infty, i} q_i$, $S_i^* = q_i^* I_{i, -\infty}$, and $S_i' = q_i' I_{i, -\infty}$, $S_i'^* = I_{-\infty, i} q_i'^*$; where $I_{i, -\infty} = \prod_{x=i}^{-\infty} I_{x, x-1}$ and $I_{-\infty, i}$ being the inverse of $I_{i, -\infty}$, operator q_i being defined as $q_i I_{i+1, i} = I$ only acting from the left as well as $I_{i+1, i} q_i^* = I$ from the right, and also as $q_i' I_{i, i-1} = I$, $I_{i, i-1} q_i'^* = I$.

(Generally operator I_{ij} acts on the products in the whole and so the position of the operator in the product is not significant but the order, while for the operators q 's not only the order but also the position must be taken into account).¹¹⁾

Furthermore, we have the following relations,¹⁰⁾

$$\begin{aligned} (A) \quad & \left\{ \begin{array}{l} \sum_{\text{Perm. in } \alpha, \beta, \gamma} X_\alpha X_\beta X_\gamma = 0, \\ \text{and also} \\ \sum X_\alpha \bar{X}_\beta \bar{X}_\gamma = 0, \end{array} \right. \quad \begin{array}{l} \text{for all } \alpha, \beta, \gamma, \\ \\ \text{for all } \alpha, \beta, \gamma, \end{array} \\ (B) \quad & \left\{ \begin{array}{l} X_\gamma (X_\alpha^* X_\beta) = (X_\alpha^* X_\beta) X_\gamma, \\ X_\gamma (X_\alpha^* X_\beta) = (\bar{X}_\alpha^* X_\beta) X_\gamma, \end{array} \right. \quad \begin{array}{l} \text{for } \alpha \leq \beta \text{ and all } \gamma's \neq \alpha, \\ \text{for } \alpha \geq \beta \text{ and all } \gamma's \neq \alpha, \end{array} \\ (B') \quad & \left\{ \begin{array}{l} X_\gamma (X_\alpha^* X_\beta) = (X_\alpha^* X_\beta) X_\gamma, \\ X_\gamma (X_\alpha^* X_\beta) = (\bar{X}_\alpha^* X_\beta) X_\gamma, \end{array} \right. \quad \begin{array}{l} \text{for } \alpha \leq \beta \text{ and all } \gamma's \neq \alpha, \\ \text{for } \alpha \geq \beta \text{ and all } \gamma's \neq \alpha, \end{array} \\ (A') \quad & \left\{ \begin{array}{l} \sum_{\text{Perm. } \alpha, \beta, \gamma} X_\alpha^* X_\beta^* X_\gamma^* = 0, \\ \sum_{\text{Perm. } \alpha, \beta, \gamma} \bar{X}_\alpha^* X_\beta^* X_\gamma^* = 0, \end{array} \right. \\ (B) \quad & \left\{ \begin{array}{l} X_\gamma X_\alpha X_\alpha^* = X_\alpha X_\alpha^* X_\gamma, \\ X_\gamma X_\alpha \bar{X}_\alpha^* = \bar{X}_\alpha \bar{X}_\alpha^* X_\gamma, \end{array} \right. \quad \gamma \neq \alpha, \end{aligned}$$

10) These relations among X_α 's have their meanings only when operating to Ψ . In calculation, we use the relation that the sum of the representatives of the symmetric subgroups for permutations among 1, 2, 3 identically vanishes.

11) The self consistency for the strange operators S 's should be verified; see the last part of this §.

$$\begin{aligned}
 (\bar{B}') \quad & \begin{cases} X_\gamma^* X_\alpha X_\alpha^* = X_\alpha X_\alpha^* \bar{X}_\gamma^*, & \gamma \neq \alpha, \\ X_\gamma^* X_\alpha X_\alpha^* = \bar{X}_\alpha X_\alpha^* X_\gamma^*, \end{cases} \\
 (C) \quad & X_\alpha (X_\alpha^* X_\beta) - (X_\alpha^* X_\gamma) X_\alpha = X_\beta, \quad \text{for all } \beta, \\
 (C') \quad & X_\alpha^* (X_\beta^* X_\alpha) - (X_\beta^* X_\alpha) X_\alpha^* = -X_\beta^*.
 \end{aligned}$$

Furthermore

$$\begin{aligned}
 (D) \quad & X_\alpha^2 X_\alpha^* + 2X_\alpha^* X_\alpha^2 = 2X_\alpha^*, \\
 (D') \quad & X_\alpha X_\alpha^{*2} + 2X_\alpha^{*2} X_\alpha = 2X_\alpha^*, \\
 (E) \quad & X_\alpha^{*2} X_\alpha^2 + X_\alpha X_\alpha^* X_\alpha + X_\alpha^2 X_\alpha^{*2} = 2!, \\
 (E') \quad & X_\alpha^{*2} X_\alpha^2 + X_\alpha^* X_\alpha^2 X_\alpha^* + \bar{X}_\alpha^2 \bar{X}_\alpha^{*2} = 2!, \\
 (F) \quad & X_\alpha^* X_\alpha = \bar{X}_\alpha^* X_\alpha, \quad X_\alpha X_\alpha^* = \bar{X}_\alpha \bar{X}_\alpha^*, \quad \text{etc.}
 \end{aligned}$$

Since the relations above mentioned are deduced from the equation in the configuration space in which the same physically indistinguishable states are forced to be distinguish X_α from X_α (or X_α^* from X_α^*). However, there is physically no distinction between them. Accordingly if we identify them, we can see that equations (A), (B), (C), (D), (E) and (A'), (B'), (C'), (D'), (E') are their Hermite conjugates respectively.

Let us forget all the statements discussed in the configuration space and let us afresh introduce operators X_α and their Hermite conjugate operators X_α^* , and we are going to set up finally the followings second quantization formula (the relations between the algebraized quantities)¹²⁾;

$$(J) \quad \begin{cases} \sum_{\text{Perm. in } \alpha, \beta, \gamma} X_\alpha X_\beta X_\gamma = 0, \\ (G) \quad \begin{cases} X_\gamma X_\alpha^* X_\beta = X_\alpha^* X_\beta X_\gamma & \text{and } X_\gamma X_\alpha X_\alpha^* = X_\alpha X_\alpha^* X_\gamma, \text{ if } \alpha \neq \gamma \\ X_\alpha X_\alpha^* X_\beta - X_\alpha^* X_\beta X_\alpha = X_\beta, & \text{for } \alpha = \gamma \text{ and all } \beta. \end{cases} \\ X_\alpha^2 X_\alpha^* + 2X_\alpha^* X_\alpha^2 = 2X_\alpha^*, \\ X_\alpha^{*2} X_\alpha + X_\alpha X_\alpha^{*2} X_\alpha + X_\alpha^2 X_\alpha^{*2} = 2 \end{cases}
 \end{cases}$$

and their Hermite conjugate relations.

The relations (G) and their Hermite conjugates are valid in all statistics, while the other relations are proper for our statistics with the capacity=2. For these equations we can verify that the proper values of $X_\alpha^* X_\alpha$ are 0, 1, 2, viz.

$$X_\alpha^* X_\alpha (X_\alpha^* X_\alpha - 1) (X_\alpha^* X_\alpha - 2) = 0.$$

Furthermore

$$[X_\alpha^*, H] = -\partial H / \partial X_\alpha, \quad [X_\alpha, H] = \partial H / \partial X_\alpha^*$$

where

$$H = \sum_{\alpha\beta\gamma} H_{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma.$$

12) Associative law is not held in (B), (B') and (C) (C'). But we assume it is held in (G).

In deriving the relations (A), (B), (C) etc., the strange operators S , etc. have played essential roles, and these operators were to be introduced to split O_{ij} forcibly to the product form $X_i^* X_j$. These circumstances might have been caused by the facts that the same states physically indistinguishable were forced to be distinguishable and moreover we have used the finite symmetry group theory labelling the states concerned with the positive integers, while they are ranged from $-\infty$ to $+\infty$. In spite of these, the final results may be correct, as is seen from the followings.

To avoid these operators, we may take the density operator O_{ij} itself, in which the strange operators are not introduced essentially. Then we can derive directly from (16) the following relations

$$O_{\alpha\gamma} O_{\gamma\delta} = O_{\gamma\delta} O_{\alpha\gamma}$$

except the cases when more than three of $\alpha, \beta, \gamma, \delta$ are equivalent, and

$$O_{\alpha\beta} O_{\beta\beta} - O_{\beta\beta} O_{\alpha\beta} = O_{\alpha\beta}, \quad \text{for } \alpha \neq \beta.$$

Furthermore

$$\begin{aligned} & \sum_{\text{Perm. } \alpha', \beta', \gamma' \text{ fixing the order } \alpha, \beta, \gamma} O_{\alpha'\alpha} O_{\beta'\beta} O_{\gamma'\gamma} \\ &= 2 \sum_{\text{Perm. } \alpha', \beta', \gamma' \text{ fixing } \alpha, \beta, \gamma} (\partial_{\alpha'\alpha} O_{\beta'\beta} O_{\gamma'\gamma} + O_{\alpha'\alpha} \partial_{\beta'\beta} O_{\gamma'\gamma} + O_{\alpha'\alpha} O_{\beta'\beta} \partial_{\gamma'\gamma}) \\ & \quad - (\partial_{\alpha\beta} + \partial_{\beta\gamma} + \partial_{\gamma\alpha}) (\partial_{\alpha'\alpha} \sum_{\text{Perm. } \beta', \gamma' \text{ fixing } \beta, \gamma} O_{\beta'\beta} O_{\gamma'\gamma} + \partial_{\beta'\beta} \sum_{\text{Perm. } \alpha', \gamma'} O_{\alpha'\alpha} O_{\gamma'\gamma} + \partial_{\gamma'\gamma} \sum_{\text{Perm. } \alpha', \beta'} O_{\alpha'\alpha} O_{\beta'\beta}) \\ & \quad - 4 (\partial_{\alpha\beta} \partial_{\alpha'\alpha} \partial_{\beta'\beta} O_{\gamma'\gamma} + \partial_{\beta\gamma} O_{\alpha'\alpha} \partial_{\beta'\beta} \partial_{\gamma'\gamma} + \partial_{\gamma\alpha} \partial_{\alpha'\alpha} O_{\beta'\beta} \partial_{\gamma'\gamma}). \end{aligned}$$

Moreover, interchanging the roles of α, β, γ and that of α', β', γ' , we have the relations

$$\begin{aligned} & \sum_{\text{Perm. } \alpha, \beta, \gamma \text{ fixing } \alpha', \beta', \gamma'} O_{\alpha'\alpha} O_{\beta'\beta} O_{\gamma'\gamma} \\ &= 2 \sum_{\text{Perm. } \alpha, \beta, \gamma} (\partial_{\alpha'\alpha} O_{\beta'\beta} O_{\gamma'\gamma} + O_{\alpha'\alpha} \partial_{\beta'\beta} O_{\gamma'\gamma} + O_{\alpha'\alpha} O_{\beta'\beta} \partial_{\gamma'\gamma}) \\ & \quad - (\partial_{\alpha'\beta'} + \partial_{\beta'\gamma'} + \partial_{\gamma'\alpha'}) (\partial_{\alpha'\alpha} \sum_{\text{Perm. } \beta, \gamma} O_{\beta'\beta} O_{\gamma'\gamma} + \partial_{\beta'\beta} \sum_{\text{Perm. } \alpha, \gamma} O_{\alpha'\alpha} O_{\gamma'\gamma} + \partial_{\gamma'\gamma} \sum_{\text{Perm. } \alpha, \beta} O_{\alpha'\alpha} O_{\beta'\beta}) \\ & \quad - 4 (\partial_{\alpha'\beta'} \partial_{\alpha'\alpha} \partial_{\beta'\beta} O_{\gamma'\gamma} + \partial_{\beta'\beta} O_{\alpha'\alpha} \partial_{\beta'\beta} \partial_{\gamma'\gamma} + \partial_{\gamma'\gamma} \partial_{\alpha'\alpha} O_{\beta'\beta} \partial_{\gamma'\gamma}), \end{aligned}$$

which can be reduced to

$$6O_{\alpha\alpha}(O_{\alpha\alpha}-1)(O_{\alpha\alpha}-2)=0$$

when

$$\alpha=\alpha'=\beta=\beta'=\gamma=\gamma'.$$

These formulae are just the same as those obtained from (J).

§ 9. The second quantization formula for the general case

In the general case of the capacity $=n$, we can similarly obtain the following relations

$$\sum_{\text{Perm. } \alpha, \beta, \dots, \pi} \underbrace{X_\alpha X_\beta \dots X_\pi}_{n+1} = 0, \quad \text{for all } \alpha, \beta, \dots, \pi.$$

$$(I) \quad X_T X_\alpha^* X_\beta - X_\alpha^* X_\beta X_T = X_\alpha \delta_{\alpha\beta}, \quad (\text{even if } \alpha = \beta),$$

$$X_\alpha^n X_\alpha^* + n X_\alpha^* X_\alpha = n X_\alpha^{n-1},$$

$$X_\alpha^n X_\alpha^* + X_\alpha^* X_\alpha^n + X_\alpha^* X_\alpha^n X_\alpha^* + \dots + X_\alpha^* X_\alpha^n X_\alpha^* X_\alpha^* + \dots + X_\alpha^* X_\alpha^n X_\alpha^* X_\alpha^* X_\alpha^* + \dots$$

$$+ X_\alpha^* X_\alpha^n X_\alpha^* X_\alpha^* + \dots + X_\alpha^* X_\alpha^n X_\alpha^* X_\alpha^* X_\alpha^* = n!, \quad \text{for } n = \text{even},$$

$$X_\alpha^n X_\alpha^* + X_\alpha^* X_\alpha^n + X_\alpha^* X_\alpha^n X_\alpha^* + X_\alpha^* X_\alpha^n X_\alpha^* X_\alpha^* + \dots + \dots$$

$$+ X_\alpha^* X_\alpha^n X_\alpha^* X_\alpha^* + X_\alpha^* X_\alpha^n X_\alpha^* X_\alpha^* X_\alpha^* = n!, \quad \text{for } n = \text{odd},$$

$$\text{or} \quad \sum_{m=0}^n X_\alpha^* X_\alpha^n X_\alpha^* X_\alpha^{n-m} = n!$$

and their Hermite conjugate relations. From (I) we can verify successively the following relations

$$X_\alpha^2 X_\alpha^* X_\alpha - X_\alpha^* X_\alpha^3 = 2 X_\alpha^2,$$

$$X_\alpha^2 X_\alpha^* X_\alpha - X_\alpha^* X_\alpha^4 = 3 X_\alpha^3,$$

$$\dots \dots \dots$$

$$\dots \dots \dots$$

$$\dots \dots \dots$$

$$X_\alpha^{n-1} X_\alpha^* X_\alpha - X_\alpha^* X_\alpha^n = (n-1) X_\alpha^{n-1},$$

and their Hermite conjugates. By virtue of these, we can verify easily that

$$X_\alpha^* X_\alpha (X_\alpha^* X_\alpha - 1) (X_\alpha^* X_\alpha - 2) \dots (X_\alpha^* X_\alpha - n) = 0,$$

which indicates that $X_\alpha^* X_\alpha$ has the proper values 0, 1, 2, ..., n .

We emphasize that these second quantization formulas in the case $n = \infty$ are quite different from the Bose's. Because of our statistical condition (6), Ψ is identically vanished, but in Bose not the case. However, the distribution function is apparently quite same from (11).

§ 10. Conservation law of statistical capacity

As we showed in § 1, in the wave function of the ensemble of the compound particles composed of two Fermions, the summation about P should be taken over all the elements of the direct product $S_x \times S_y$.

Since the direct product $S_x \times S_y$ is a subgroup of the symmetric group of order $2N!$, the probability spur $\Psi^* \Psi$ which is simple characteristics of $S_x \times S_y$ is not the simple (irreducible) but the compound of the symmetric group of order $2N!$ from the standpoint where the ensemble is composed from $2N$ Fermions. In this case, since Fermion corresponds to a representation of the symmetric group characterized by the partition $\{1, 1, 1, 1, 1, 1, \dots\}$ we have the relation¹³⁾

13) Murnaghan, *The theory of group representation*, p. 155.

direct product $\{1, 1, \dots, 1\} \times \{1, 1, \dots\} = \{2, 1, 1, \dots\} + \{1, 1, \dots\} + \dots$, where $\{ \}$ in left member referring to the irreducible representation of symmetric group of the $N!$ th order, and $\{ \}$ in right member to that of $2N!$ th order. As mentioned in § 3 the statistical behavior of the ensemble are completely determined by the largest number contained in partitions of the representation. And therefore we are ensured that the compound particles composed of two Fermions are subject to statistics with capacity 2.

Similarly we can verify that the compound particles composed of n Fermions obey the statistics with capacity n .

Furthermore the general assertion may be correct that particles composed from the particles obeying to statistics with capacity n_i are subject to the statistics with capacity $n = \sum_i n_i$.

Thus we may assert the conservation law of the statistical capacities.

* * *

To cease my pen, I will express my thanks to Prof. K. Husimi for his kind discussions and also to Prof. M. Kobayasi for his good will.

Excitation Function for Meson Production by γ -Ray

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The excitation function for the production of π -meson by γ -ray is calculated by the Lorentz-covariant perturbation method, taking into account of the effect of higher orders. For the π^0 -meson production, various features have been obtained according to the theories adopted, while for the charged π -meson production, nearly similar dependence of cross section upon the incident photon energy has unanimously been obtained.

§ 1. Introduction and summary

As mentioned in the paper $[I]^*$, taking into consideration of the effect of higher orders the results obtained are considerably at variance with those obtained in the lowest order calculation for the π^0 -meson production, but rather in fair accord with the experimental evidences. There are left some problems untouched, among which the excitation function for the γ - π process is the most important one.

Many experimental data about this problem have been reported, the main issue of which may be summarized thus:

- (1) The "neutral" cross section rises much steeply with energy than does the charged²⁾.
- (2) Up to about 300 Mev of photon energy, $\sigma^+(90^\circ)$ rises almost linearly with energy³⁾.

Our object in this paper is to study this problem in the same way as in $[I]**$. In § 2, we shall show the results obtained with the matrix elements calculated up to the order e^3 for γ - π^0 process. When P. S. coupling is adopted, the symmetrical theory seems to be favourable. In § 3, introducing the Pauli-type interaction in order to estimate the effect of orders higher than e^3 and to make the calculated values of the static anomalous magnetic moments which play a role in our γ - π process coincident with the measured ones as in § 4 and § 5 of $[I]$, we have calculated the excitation functions not only for γ - π^0 process but also for γ - π^\pm in the case of P. S. coupling. As for the former, the symmetrical theory is again promising, but for the latter any remarkable difference between the symmetrical and the (charged+neutral) theory is not revealed. Thus, it is found that the theory which has shown the good agreement with the experimental data for the neutral as well as for the charged meson production with regard to the angular distribution and to the magnitude of differential cross sections turns out to be favourable also with respect to the

* We refer to the paper of "Neutral-Meson Production by γ -Ray"¹⁾ by $[I]$ throughout this paper.

** The meson type adopted is of course pseudoscalar.

the excitation functions of $\gamma\pi$ processes.

§ 2. Excitation function for the $\gamma\pi^0$ process

As the differential cross section at 90° in the laboratory system can be easily measured, we confine our attention in this value and investigate the excitation function of $\sigma(90^\circ)$ for the energy range from 164 to 364 Mev of incident photons throughout this paper.

The four independent gauge-invariant fundamental expressions U_l , U_m , $U_A^{(1)*}$ and $U_A^{(2)*}$ in (4) of [I] are adopted, accordingly χ_1, \dots, χ_6 in (39) of [I] are monotonously increasing functions with respect to K . Thus, the aspects of the excitation functions are essentially determined by the magnitudes and the relative signs of G_m , G'_m , $G_A^{(1)}$, $G_A^{(1)'}$, $G_A^{(2)}$, $G_A^{(2)'}$, G_C , and G'_C . $\sigma^0(90^\circ)$ is given by (35) and (36) in [I], namely

$$\sigma^0(90^\circ) = (e^2/4\pi)(f^0)^2/4\pi \cdot (1/4MK_0) [I + II + III] \quad \text{for } P + \gamma \rightarrow P + \pi^0,$$

$$\sigma^0(90^\circ) = (e^2/4\pi)(f^0)^2/4\pi \cdot (1/4MK_0) [III'] \quad \text{for } N + \gamma \rightarrow N + \pi^0.$$

The values of coupling constants used in each case are respectively the same with those of [I] throughout this paper. The results of tedious computation are shown in Fig. 1 and 2.

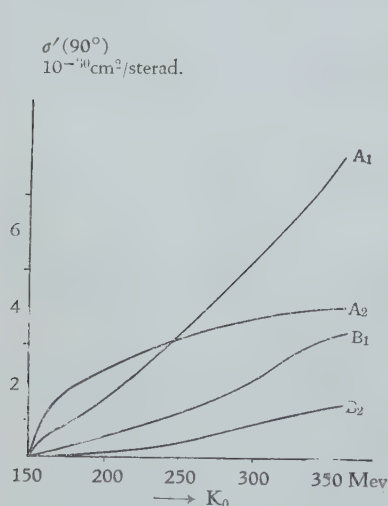


Fig. 1. Excitation function for $P + \gamma \rightarrow P + \pi^0$.

A_1 : $\sigma'(90^\circ)**$ in the case of P. S. coupling of symmetrical theory with $a/4\pi=2$.

A_2 : $\sigma'(90^\circ)$ in the case of P. S. coupling of (charged+neutral) theory with $a/4\pi=1$.

B_1 : $\sigma'(90^\circ)$ in the case of P. V. coupling of symmetrical theory with $(g')^2 a/4\pi=3$.

B_2 : $\sigma'(90^\circ)$ in the case of P. V. coupling of (charged+neutral) theory with $(g')^2 a/4\pi=3$.

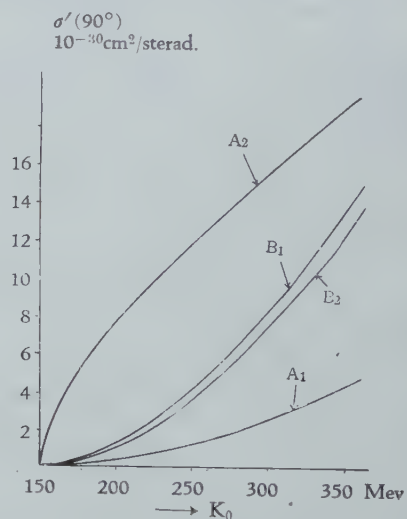


Fig. 2. Excitation function for $N + \gamma \rightarrow N + \pi^0$.

A_1 , A_2 , B_1 and B_2 are the same as in the case of $P + \gamma \rightarrow P + \pi^0$ respectively.

* As for the notation used without any explanations, see the paper [I].

** In order to show the difference of result which is produced by adopting the symmetrical and the (charged+neutral) theory, we describe hereafter $\sigma'(90^\circ) = (K_0/2\mu)\sigma(90^\circ)$ instead of $\sigma(90^\circ)$.

§ 3. Excitation functions for the γ - π process with the Pauli-type interactions

Since the underlying assumptions and the method of calculation has been explained in detail in § 4 and § 5 of [I], the results only are given in this section. We show in Fig. 3 and 4 the excitation functions both for γ - π^0 and for γ - π^\pm obtained with the value $f^2/4\pi=15$. The suffices 1 and 2 mean that the symmetrical and (charged + neutral) theory

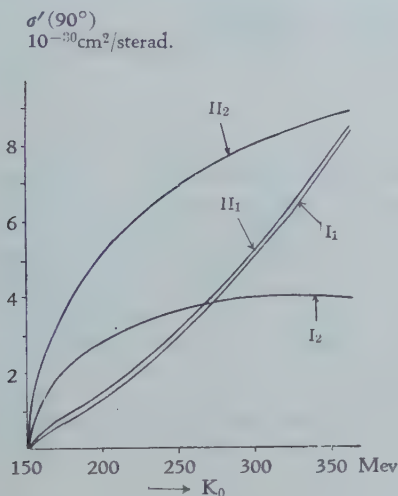


Fig. 3. Excitation function for γ - π^0 .

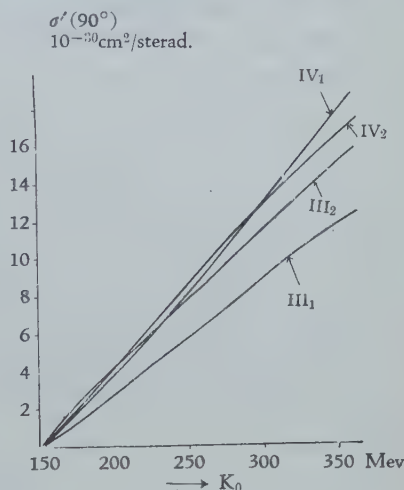


Fig. 4. Excitation function for γ - π^\pm .

are adopted respectively, and moreover I, II, III and IV indicate the processes $P+\gamma\rightarrow P+\pi^0$, $N+\gamma\rightarrow N+\pi^0$, $P+\gamma\rightarrow N+\pi^+$ and $N+\gamma\rightarrow P+\pi^-$ respectively.

§ 4. Discussions

First of all, we examine the results in the case of P. S. coupling. As for the γ - π^0 production, the magnetic interaction plays the most important role as in [I], and furthermore the considerably different properties are obtained depending on whether the symmetrical or the (charged + neutral) theory is adopted. This difference is due to the opposite signs of the contributions from the diagrams in which, aside from the interaction with electromagnetic field, the nucleon emits a virtual charged meson in the first, emits a real neutral meson in the second and reabsorbs the virtual charged meson in the last step. So far as the excitation functions are concerned it may be said, comparing the results obtained in § 2 and § 3 with the experimental data, that the symmetrical theory rather than the (charged + neutral) theory* will describe successfully the aspect of the disturbance of the virtual charged meson cloud by the incident photon, while for the charged meson production, the excitation functions

* In the (charged + neutral) theory, however, there are left some rooms for evasion to the effect that f^0 is not necessarily equal to $f/\sqrt{2}$.

obtained from both theories show the similar behaviors. So that we are unable to decide the superiority or inferiority about these theories only from this point.

Recalling the results of [1] and looking at these excitation functions both for the neutral and for the charged meson production, on the whole we see that the experimental facts can be explained to some extent by using P. S. coupling of the symmetrical theory.

Besides, we cannot overlook the fact that the P. V. coupling gives rise to a fairly reasonable result for the excitation function for γ - π^0 process although it failed to give satisfactory results for the angular distribution.

The author should like to express his gratitude to Prof. K. Husimi and Asst. Prof. Z. Koba for their valuable advices.

References

- 1) S. Minami, Prog. Theor. Phys. **7** (1952), 69.
- 2) A. Silverman and M. Stearns, Phys. Rev. **83** (1951), 853.
- 3) It is rumored that the results of the experiments which have been done by J. Steinberger *et al.*, show such properties.

On the Spin Wave Theory of Magnetic Susceptibility and Resonance Absorption in Antiferromagnetics

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A theory of resonance absorption in antiferromagnetics based on the spin wave theory developed by Anderson and Kubo is presented. The obtained resonance frequencies are given by two sharp ones, which are compared with the results of the classical theory by Nagamiya. In both cases the anisotropy energy has uniaxial symmetry. The absorption coefficient is also computed. Our calculations of the resonance frequencies are then extended to the antiferromagnetics having the anisotropy energy of orthorhombic symmetry. In the course of these calculations we obtained expressions for the magnetic susceptibility, using the obtained energy levels of the spin system which is subjected to a static magnetic field. They are in agreement with those which Kubo has already given by use of the method of statistical thermodynamics.

§ 1. Introduction

Recently, P. W. Anderson¹⁾ evolved the spin wave theory of antiferromagnetics based on Heller and Kramers' semi-classical approximation. Even in the case of the one-dimensional lattice with spins equal to one half, which corresponds to the worst case in Anderson's theory, the value of the lowest energy level obtained by him shows good agreement with the result by Hulthén²⁾ and Bethe.³⁾ Using more rigorous method originated by Holstein and Primakoff,⁴⁾ R. Kubo⁵⁾ developed the statistical thermodynamical theory of the spin wave system and obtained expressions for the thermodynamical quantities of this at low temperatures for different kinds of crystal structures. Instead of diagonalizing the Hamiltonian of the spin system subjected to a static magnetic field, he used an ingenious method to obtain the magnetic susceptibility. In the present paper, however, this Hamiltonian is diagonalized in order that we can treat the antiferromagnetic resonance absorption. Furthermore, the magnetic susceptibility is computed from this diagonalized Hamiltonian.

A theory of antiferromagnetic resonance absorption has already been presented by Nagamiya⁶⁾ from the point of view of the classical theory of Weiss. But, it seems worth while to reconsider this problem on the basis of quantum theoretical treatment: we expect, on the one hand, that the shape of the absorption band can be interpreted from a detailed knowledge of the structures of energy levels of the spin system and, on the other hand, that the quantum theoretical considerations of elementary processes are necessary for the analysis of the collision broadening of the absorption band. Although we see that the result of the present calculation is contrary to the former expectation and the latter problem

does not be considered here, we shall still have an interesting comparison of our result with that of the classical theory.

In § 2 the approximate energy levels of the spin system subjected to a static magnetic field are obtained from the method of Anderson and Kubo. By use of these energy levels the partition function of the spin system is constructed, from which the magnetic susceptibility at low temperatures is calculated (§ 3). In § 4 the resonance frequencies are determined, and in § 6 these are compared with that of the classical theory. Furthermore, following Karplus and Schwinger's theory of saturation effect in microwave spectroscopy,⁷⁾ the absorption coefficient is obtained in § 5.

§ 2. Diagonalization of the spin wave Hamiltonian

Let us confine our considerations to the simple cubic lattice and divide the lattice points into two sublattices, whose lattice points are denoted by j and k respectively, so that each j lattice point is surrounded only by k 's lattice points and vice versa. Let $\hbar\mathbf{S}_j$ and $\hbar\mathbf{S}_k$ be the spin angular momentum vectors of the atoms on these lattice points respectively and $J/2$ the exchange integral between the nearest neighbouring pairs.

Then we have for the exchange Hamiltonian, \mathcal{H}_{ex} ,

$$\mathcal{H}_{\text{ex}} = |J| \sum_{\langle j, k \rangle} \mathbf{S}_j \cdot \mathbf{S}_k, \quad (1)$$

where $\langle j, k \rangle$ means the nearest neighbouring pair. When the spin system is subjected to a static magnetic field $\mathbf{H}(H_x, H_y, H_z)$, the field Hamiltonian, $\mathcal{H}_{\text{field}}$,

$$\mathcal{H}_{\text{field}} = g\beta \sum_{j,k} (\mathbf{S}_j + \mathbf{S}_k) \cdot \mathbf{H} \quad (2)$$

should be added to \mathcal{H}_{ex} , where g is the Lande g -factor and β the Bohr magneton. The total Hamiltonian \mathcal{H} is

$$\mathcal{H} = \mathcal{H}_{\text{ex}} + \mathcal{H}_{\text{field}}. \quad (3)$$

If we introduce the creation and annihilation operators a^* and a defined by

$$(n-1|a|n') = (n'|a^*|n-1) = \sqrt{n} \delta_{nn'}, \quad (4)$$

the spin operators in the representation diagonalizing S_z are written after Kubo as

$$\begin{aligned} S_{xj} + iS_{yj} &= (2S)^{1/2} (1 - n_j/2S)^{1/2} a_j, \\ S_{xj} - iS_{yj} &= (2S)^{1/2} a_j^* (1 - n_j/2S)^{1/2}, \end{aligned} \quad (5)$$

$$\begin{aligned} S_{zj} &= S - n_j; \\ S_{xk} + iS_{yk} &= (2S)^{1/2} a_k^* (1 - n_k/2S)^{1/2}, \\ S_{xk} - iS_{yk} &= (2S)^{1/2} (1 - n_k/2S)^{1/2} a_k, \\ S_{zk} &= -S + n_k, \end{aligned} \quad (6)$$

where S is the spin quantum number and where we assume that the j th spin is almost

parallel to the z -direction and the k th spin almost antiparallel. The a 's satisfy the commutation relation

$$aa^* - a^*a = 1. \quad (7)$$

Further, from (5), we observe that

$$n = a^*a, \quad (8)$$

which will be called the spin deviation.

Substituting (5), (6) into (1), (2) and introducing Holstein-Primakoff's approximations

$$(1 - n/2S)^{1/2} \simeq 1, \quad nn' \simeq 0, \quad (9)$$

(3) may be written as

$$\begin{aligned} \mathcal{H} = & -3N|J|S^2 + \sum_{j,k} \{ (6|J|S - g\beta H_z) n_j + (6|J|S + g\beta H_z) n_k \} \\ & + |J|S \sum_{j,k} (a_j a_k + a_j^* a_k^*) + g\beta (S/2)^{1/2} H_x \sum_{j,k} (a_j + a_j^* \\ & + a_k + a_k^*) - i g\beta (S/2)^{1/2} H_y \sum_{j,k} (a_j - a_j^* + a_k - a_k^*), \end{aligned} \quad (10)$$

where N is the total number of atoms. By successive uses of the canonical transformations:

$$\begin{aligned} Q_j &= (a_j + a_j^*)/\sqrt{2}, \quad P_j = (a_j - a_j^*)/\sqrt{2}i, \\ R_k &= (a_k + a_k^*)/\sqrt{2}, \quad S_k = (a_k - a_k^*)/\sqrt{2}i, \end{aligned} \quad (11)$$

and

$$\begin{aligned} Q_\lambda &= (2/N)^{1/2} \sum_j e^{i\lambda j} Q_j, \quad P_\lambda = (2/N)^{1/2} \sum_j e^{-i\lambda j} P_j, \\ R_\lambda &= (2/N)^{1/2} \sum_k e^{i\lambda k} R_k, \quad S_\lambda = (2/N)^{1/2} \sum_k e^{-i\lambda k} S_k, \end{aligned} \quad (12)$$

(10) is transformed into

$$\mathcal{H} = -3N|J|S(S+1) + \sum_\lambda \mathcal{H}_\lambda, \quad (13)$$

$$\mathcal{H}_\lambda = 3|J|S \{ (1 + \epsilon) (Q_\lambda^2 + P_\lambda^2) + (1 - \epsilon) (R_\lambda^2 + S_\lambda^2) + 2\gamma_\lambda (Q_\lambda R_\lambda - P_\lambda S_\lambda) \}, \quad (14)$$

$$\mathcal{H}_0 = (\mathcal{H}_\lambda)_{\lambda=0} + (NS/2)^{1/2} g\beta \{ H_x (Q_0 + R_0) + H_y (P_0 - S_0) \}, \quad (15)$$

where

$$\gamma_\lambda = (\cos \lambda_1 + \cos \lambda_2 + \cos \lambda_3)/3, \quad (16)$$

$$\epsilon = g\beta H_z / 6|J|S. \quad (17)$$

The new set of operators Q_λ , P_λ , R_λ and S_λ prove to satisfy, in virtue of (7), (11) and (12), the commutation relations

$$\begin{aligned} [Q_\lambda, P_{\lambda'}] &= i\delta_{\lambda\lambda'}, \quad [R_\lambda, S_{\lambda'}] = i\delta_{\lambda\lambda'}, \\ [Q_\lambda, R_{\lambda'}] &= 0, \quad [Q_\lambda, S_{\lambda'}] = 0, \\ [P_\lambda, R_{\lambda'}] &= 0, \quad [P_\lambda, S_{\lambda'}] = 0. \end{aligned} \quad (18)$$

*) By λ , j and k we mean the vectors whose components are given by $(\lambda_1, \lambda_2, \lambda_3)$ etc., and by λ, j et we denote scalar products.

Now that \mathcal{H} has been divided into commutative parts, we have only to diagonalize each part \mathcal{H}_λ separately. We first use the orthogonal transformation

$$Q_\lambda = \cos \theta_\lambda Q_{1\lambda} - \sin \theta_\lambda Q_{2\lambda}, \quad P_\lambda = \cos \theta_\lambda P_{1\lambda} - \sin \theta_\lambda P_{2\lambda}, \quad (19)$$

$$R_\lambda = \sin \theta_\lambda Q_{1\lambda} + \cos \theta_\lambda Q_{2\lambda}, \quad S_\lambda = \sin \theta_\lambda P_{1\lambda} + \cos \theta_\lambda P_{2\lambda},$$

$$\tan 2\theta_\lambda = -\gamma_\lambda/\epsilon \quad (20)$$

and obtain

$$\begin{aligned} [3|J|S]^{-1} \mathcal{H}_\lambda &= (1 + \epsilon \cos 2\theta_\lambda + \gamma_\lambda \sin 2\theta_\lambda) Q_{1\lambda}^2 + (1 + \epsilon \cos 2\theta_\lambda - \gamma_\lambda \sin 2\theta_\lambda) P_{1\lambda}^2 \\ &+ (1 - \epsilon \cos 2\theta_\lambda - \gamma_\lambda \sin 2\theta_\lambda) Q_{2\lambda}^2 + (1 - \epsilon \cos 2\theta_\lambda + \gamma_\lambda \sin 2\theta_\lambda) P_{2\lambda}^2 - 4\epsilon \sin 2\theta_\lambda Q_{1\lambda} Q_{2\lambda}. \end{aligned} \quad (21)$$

Then we use the canonical transformation:

$$Q_{1\lambda} = \sigma (\cos \varphi_\lambda q_{1\lambda} - \sin \varphi_\lambda q_{2\lambda}), \quad P_{1\lambda} = \sigma^{-1} (\cos \varphi_\lambda p_{1\lambda} - \sin \varphi_\lambda p_{2\lambda}), \quad (22)$$

$$Q_{2\lambda} = \tau (\sin \varphi_\lambda q_{1\lambda} + \cos \varphi_\lambda q_{2\lambda}), \quad P_{2\lambda} = \tau^{-1} (\sin \varphi_\lambda p_{1\lambda} + \cos \varphi_\lambda p_{2\lambda}),$$

$$\tan 2\varphi_\lambda = \gamma_\lambda \sigma \tau / \epsilon, \quad (23)$$

$$\sigma = (1 + \epsilon \cos 2\theta_\lambda - \gamma_\lambda \sin 2\theta_\lambda)^{1/2}, \quad (24)$$

$$\tau = (1 - \epsilon \cos 2\theta_\lambda + \gamma_\lambda \sin 2\theta_\lambda)^{1/2}$$

and get the diagonalized Hamiltonian

$$\mathcal{H}_\lambda = \mathcal{H}_{1\lambda} + \mathcal{H}_{2\lambda} \quad (25)$$

with

$$[3|J|S]^{-1} \mathcal{H}_{r\lambda} = (n_{r\lambda} + 1/2) \omega_{r\lambda}, \quad (25a)$$

$$\omega_{1\lambda} = 2(\sqrt{1 - \gamma_\lambda^2} + \epsilon), \quad (26)$$

$$\omega_{2\lambda} = 2(\sqrt{1 - \gamma_\lambda^2} - \epsilon)$$

with the help of the commutation relations

$$\begin{aligned} [q_{r\lambda}, p_{r'\lambda}] &= i\delta_{rr'}\delta_{\lambda\lambda'}, \\ [q_{r\lambda}, q_{r'\lambda}] &= [p_{r\lambda}, p_{r'\lambda}] = 0, \end{aligned} \quad (27)$$

which follow from (18), (19), (22) and with the help of (20), (23) and (24).

It may be noticed that the systems $(2, \lambda)$'s are not oscillatory when $\sqrt{1 - \gamma_\lambda^2} < \epsilon$. The system which corresponds to the extreme case of this is $(2, 0)$. Let us consider the behaviour of the system $(r, 0)$ when $\epsilon = 0$. In this case, \mathcal{H}_0 is divided into two commutative parts by the transformation (19) with $\theta_0 = \pi/4$ and with the commutation relations (27). Thus

$$\begin{aligned} \mathcal{H}_0 &= \mathcal{H}_{10} + \mathcal{H}_{20}, \\ \mathcal{H}_{10} &= 6|J|Sq_{10}^2 + \sqrt{NS}g\beta H_x q_{10}, \\ \mathcal{H}_{20} &= 6|J|Sp_{20}^2 + \sqrt{NS}g\beta H_y p_{20}. \end{aligned} \quad (28)$$

In the representation diagonalizing q_{10} and p_{20} , \mathcal{H}_0 is diagonal and hence it reduces to the *energy value*. In this representation, all the knowledge about the quantities p_{10} and q_{20} , which are conjugate to q_{10} and p_{20} respectively, proves to be lost by the uncertainty principle; this circumstance seems to give an interpretation of the divergence of $\langle p_{10}^2 \rangle$ and $\langle q_{20}^2 \rangle$.

As was pointed out by Anderson, the introduction of the anisotropy Hamiltonian of the form:

$$\mathcal{H}_{\text{anis}} = K \left[\sum_j (S_{xj}^2 + S_{yj}^2) + \sum_k (S_{xk}^2 + S_{yk}^2) \right] \quad (29)$$

which corresponds to the anisotropy energy of uniaxial symmetry, makes the motion of the system (2, 0) oscillatory. That the anisotropy energy plays an essential role in antiferromagnetic resonance absorption has already been recognized in the classical theory of it. We shall also see below that this circumstance is related to the oscillatory nature of the system ($r, 0$).

We now denote the new Hamiltonian obtained by adding (29) to (3) as \mathcal{H} . Its sub-Hamiltonian \mathcal{H}_λ is written

$$\begin{aligned} \mathcal{H}_\lambda = & 3(1+\alpha) |J| S \left[(1+\epsilon/(1+\alpha)) (Q_\lambda^2 + P_\lambda^2) + (1-\epsilon/(1+\alpha)) (R_\lambda^2 + S_\lambda^2) \right. \\ & \left. + 2 \gamma_\lambda / (1+\alpha) (Q_\lambda R_\lambda - P_\lambda S_\lambda) \right], \end{aligned} \quad (30)$$

where

$$\alpha = K/3|J|. \quad (31)$$

The process of the diagonalization of this sub-Hamiltonian can be carried out completely parallel with that of the previous Hamiltonian if only the substitution

$$\epsilon \rightarrow \epsilon/(1+\alpha), \quad \gamma_\lambda \rightarrow \gamma_\lambda/(1+\alpha) \quad (32)$$

is made. Hence we have in place of (26)

$$\begin{aligned} [3(1+\alpha) |J| S]^{-1} \mathcal{H}_{r\lambda} &= (n_{r\lambda} + 1/2) \omega_{r\lambda}, \\ \omega_{1\lambda} &= 2/(1+\alpha) \cdot (\sqrt{(1+\alpha)^2 - \gamma_\lambda^2} + \epsilon), \\ \omega_{2\lambda} &= 2/(1+\alpha) \cdot (\sqrt{(1+\alpha)^2 - \gamma_\lambda^2} - \epsilon). \end{aligned} \quad (33)$$

As one sees from (26) or (33), the frequencies of the spin waves fall into two branches, which become identical when $\epsilon=0$. With increasing ϵ , they separate from each other, while their centre of gravity remains unchanged.

§ 3. Magnetic susceptibility

Let M_\parallel , χ_\parallel be the magnetic moment and susceptibility of the system when the applied field is parallel to the magnetic axis (S -axis) and M_\perp , χ_\perp be those when the field is perpendicular.

(1) Magnetic susceptibility at zero temperature

As was described in the preceding section, the centre of gravity of the energy levels

does not move in the presence of the parallel field, so that the gain of the energy due to the parallel field is zero. Hence

$$\chi_{\parallel 0} = 0. \quad (34)$$

Let us next consider $\chi_{\perp 0}$. In case if $u, \epsilon = 0$, we see from (27) that the lowest value of \mathcal{H}_{r0} 's are determined by

$$\partial \mathcal{H}_{10} / \partial q_{10} = 0, \quad \partial \mathcal{H}_{20} / \partial p_{20} = 0. \quad (35)$$

These lead to

$$(\mathcal{H}_0)_{\min} = -N g^2 \beta^2 H_{\perp}^2 / 24 |J|, \\ H_{\perp}^2 = H_x^2 + H_y^2, \quad (36)$$

we thus have

$$\chi_{\perp 0} = (1/H_{\perp}) (-\partial / \partial H_{\perp}) (\mathcal{H}_0)_{\min} = N g^2 \beta^2 / 12 |J| \quad (37)$$

in agreement with the Van Vleck formula.⁸⁾

(2) Magnetic susceptibility at an arbitrary temperature

We calculate χ_{\parallel} , assuming the energy levels of the system to be given by (26). Since the partition function of the oscillator (r, λ) is given by

$$1/2 \sinh \xi \omega_{r\lambda}, \quad \xi = 3|J|S/2kT, \quad (38)$$

the partition function of the spin system, $Z(T, H_{\parallel})$, is written

$$Z(T, H_{\parallel}) = \prod_{\lambda} 1/2 \sinh \xi \omega_{1\lambda} \cdot \prod_{\lambda} 1/2 \sinh \xi \omega_{2\lambda}, \quad (39)$$

which gives

$$M_{\parallel} = kT (\partial / \partial H_{\parallel})_T \cdot \log Z(T, H_{\parallel}) \\ = 8kT \xi^2 \epsilon (\partial \epsilon / \partial H_{\parallel}) \sum_{\lambda} 1 / \sinh^2 2\xi \sqrt{1 - \gamma_{\lambda}^2} + 0(\epsilon^3). \quad (40)$$

To obtain an approximate estimation of the integral in (40) at low temperatures ($\xi \rightarrow 0$), we expand $\sqrt{1 - \gamma_{\lambda}^2}$ into powers of λ by use of (16). Retaining only the linear term we have

$$\sqrt{1 - \gamma_{\lambda}^2} \simeq \lambda / \sqrt{3}.$$

This approximation is based on the observation that, when $\xi \rightarrow 0$, the long wave region gives the dominant contribution to the integral. Then

$$\sum_{\lambda} \frac{1}{\sinh^2 2\xi \sqrt{1 - \gamma_{\lambda}^2}} = \frac{N}{2} \frac{1}{(2\pi)^3} \iiint_{-\pi}^{\pi} \frac{1}{\sinh^2 2\xi \sqrt{1 - \gamma_{\lambda}^2}} d\lambda_1 d\lambda_2 d\lambda_3 \\ \simeq \frac{N}{(2\pi)^2} \left(\frac{\sqrt{3}}{2\xi} \right)^3 \int_0^{\infty} \frac{x^2}{\sinh^2 x} dx \\ \simeq \frac{\sqrt{3}}{32} \frac{N}{\xi^3}, \quad (41)$$

where we used

$$\sum_{n=1}^{\infty} 1/n^2 = \zeta(2) = \pi^2/6.$$

Substituting (41) in (40) and using (17) and (38), we get the T^2 -dependency of χ_{\parallel} , as was found by Kubo, namely

$$\chi_{\parallel} = \frac{\sqrt{3}}{2} \frac{Ng^2\beta^2}{6|J|S} \left(\frac{kT}{6|J|S} \right)^2. \quad (42)$$

It should be noticed that the dominant contribution to χ_{\parallel} comes from the long wave region where there exist non-oscillatory systems, and that we treated all systems as if they were oscillatory.

Finally, we calculate the classical partition function $Z'(T, H_{\perp})$ from (28) and obtain

$$Z'(T, H_{\perp}) = (kT/3|J|S)^{1/2} \exp(Ng^2\beta^2 H_{\perp}^2/24|J|kT) \quad (43)$$

to find

$$\chi_{\perp} = \chi_{\perp 0}. \quad (44)$$

§ 4. Resonance condition

Let us now inquire into the interaction between the spin wave system and the oscillating field with angular frequency ω and amplitude $\mathbf{H}'(H'_x, H'_y, H'_z)$. The interaction Hamiltonian is given by

$$\mathcal{H}'_x = g\beta \left(\sum_j S_{xj} + \sum_k S_{xk} \right) H'_x \cos \omega t, \quad x = x, y, z. \quad (45)$$

As was noted in § 2, the anisotropy energy plays an essential rôle in the antiferromagnetic resonance absorption. Hence we have to take (30) as the Hamiltonian of the spin system rather than (3).

Now, with the help of (19) and (22), we rewrite (45) as

$$\begin{aligned} \mathcal{H}'_x = (NS/2)^{1/2} g\beta \{ & \{\sigma(\sin \theta_0 + \cos \theta_0) \cos \varphi_0 + \tau(-\sin \theta_0 + \cos \theta_0) \sin \varphi_0\} q_{10} \\ & - \{\sigma(\sin \theta_0 + \cos \theta_0) \sin \varphi_0 + \tau(\sin \theta_0 - \cos \theta_0) \cos \varphi_0\} q_{20} \} H'_x \cos \omega t, \end{aligned} \quad (46)$$

$$\begin{aligned} \mathcal{H}'_y = (NS/2)^{1/2} g\beta \{ & \{\sigma^{-1}(-\sin \theta_0 + \cos \theta_0) \cos \varphi_0 - \tau^{-1}(\sin \theta_0 + \cos \theta_0) \sin \varphi_0\} p_{10} \\ & - \{\sigma^{-1}(-\sin \theta_0 + \cos \theta_0) \sin \varphi_0 + \tau^{-1}(\sin \theta_0 + \cos \theta_0) \cos \varphi_0\} p_{20} \} H'_y \cos \omega t, \end{aligned} \quad (47)$$

$$\mathcal{H}'_z = \sum_{\lambda} \mathcal{H}'_{z\lambda} + \sum_{\lambda} \mathcal{H}'_{z2\lambda}, \quad (48)$$

with

$$\mathcal{H}'_{z1\lambda} = -\frac{1}{2} g\beta \left\{ \frac{\sqrt{(1+u)^2 - \gamma_{\lambda}^2} + \epsilon}{1+u} q_{1\lambda}^2 + \frac{1+u}{\sqrt{(1+u)^2 - \gamma_{\lambda}^2} + \epsilon} p_{1\lambda}^2 \right\} H'_z \cos \omega t,$$

$$\mathcal{H}'_{z2\lambda} = \frac{1}{2} g\beta \left\{ \frac{\sqrt{(1+u)^2 - \gamma_{\lambda}^2} - \epsilon}{1+u} q_{2\lambda}^2 + \frac{1+u}{\sqrt{(1+u)^2 - \gamma_{\lambda}^2} - \epsilon} p_{2\lambda}^2 \right\} H'_z \cos \omega t,$$

where we simplified \mathcal{H}'_z by substitution of (20), (23) and (24).

The non-zero matrix elements of $q_{r\lambda}$ and $p_{r\lambda}$ are confined to the following ones:

$$\begin{aligned}(n_{r\lambda}|q_{r\lambda}|n_{r\lambda}-1) &= (n_{r\lambda}-1|q_{r\lambda}|n_{r\lambda}) = (\omega_{r\lambda}/2)^{-1/2} (n_{r\lambda}/2)^{1/2}, \\ (n_{r\lambda}|p_{r\lambda}|n_{r\lambda}-1) &= -(n_{r\lambda}-1|p_{r\lambda}|n_{r\lambda}) = i(\omega_{r\lambda}/2)^{1/2} (n_{r\lambda}/2)^{1/2}\end{aligned}\quad (49)$$

in the representation diagonalizing $\mathcal{H}_{r\lambda}$. Then, to obtain the allowed transitions and resonance frequencies, we consider two cases separately, one in which $H'_z=0$ and $H'_{x,y}\neq 0$, and the other in which $H'_z\neq 0$ and $H'_{x,y}=0$.

$$(1) \quad H'_z=0, \quad H'_{x,y}\neq 0.$$

Considering the case $H_\perp=0$ for the sake of simplicity, the systems $(r, 0)$ absorb or emit the magnetic energy of the oscillating field according to the selection rule

$$\Delta n_{r\lambda} = \pm 1 \quad (50)$$

in virtue of (46), (47) and (49). The frequencies which correspond to this transition are obtained as

$$\begin{aligned}\hbar\omega_1 &= 2\sqrt{K(6|J|+K)}S + g\beta H_z, \\ \hbar\omega_2 &= 2\sqrt{K(6|J|+K)}S - g\beta H_z,\end{aligned}\quad (51)$$

using (33) with (17) and (31). We shall discuss the case $H_\perp\neq 0$ in the last section.

$$(2) \quad H'_{x,y}=0, \quad H'_z\neq 0.$$

As can be seen from (48), only the transitions between the levels of the same oscillator are allowed. In the present case, the selection rule is

$$\Delta n_{r\lambda} = \pm 2, \quad (52)$$

which follows from (48) and (49), and hence the resonance frequencies are obtained as

$$\hbar\omega = 2[2\sqrt{(3|J|+K)^2 - (3J_{r\lambda})^2}S \pm g\beta H_z] \quad (53)$$

in a similar way as we obtained (51).

The resonance frequencies given by (53) are distributed broadly between the low frequency limit $2\omega_r$ and a very high frequency $12|J|S/\hbar$. However, the transition probability of the transition now under consideration is very small compared with that of the transition due to a perpendicular oscillating field, because the former has the order of magnitude equal to $1/N$ times as is easily seen from (46)–(48). Therefore, we leave this case out of consideration.

§ 5. Absorption coefficient

The absorption coefficient, with which the oscillator $(r, 0)$ absorbs the magnetic energy of oscillating field, is obtained in complete parallelism with Karplus and Schwinger's theory, which is based on the use of density matrix.

Writing the interaction operator between the oscillator $(r, 0)$ and the oscillating field as

$$V_r \cos \omega t = - (M'_{xz} H'_x + M'_{yz} H'_y) \cos \omega t, \quad (54)$$

we obtain from (46) and (47)

$$|(n-1|M'_{ex}|n)|^2 = \frac{1}{4} N g^2 \beta^2 S \left(\frac{a}{2+a} \right)^{1/2} n, \quad x=x, y, \quad (55)$$

with the help of (20), (23), (24) and (49).

Then, by applying Karplus and Schwinger's formula mutatis mutandis to the present case, and by using (55), the magnetic susceptibility due to the perpendicular oscillating field is written as

$$\chi_w = \frac{1}{4} N g^2 \beta^2 S \left(\frac{a}{2+a} \right)^{1/2} \frac{1}{\hbar} \sum_{r=1,2} \frac{1}{\omega_r} \left[2 - \left\{ \frac{\omega}{\omega - \omega_r + i/\tau} + \frac{\omega}{\omega + \omega_r + i/\tau} \right\} \right], \quad (56)$$

where τ is the relaxation time for the oscillator determined through the frequency of collisions with other subsystems.

As one sees easily, (56) tends to the static susceptibility when $\omega \rightarrow 0$:

$$[\chi_w]_{\omega=0} = \frac{N g^2 \beta^2}{12|J|} \left(\frac{a}{2+a} \right)^{1/2} \left[\frac{1}{\sqrt{(2+a)a+\epsilon}} + \frac{1}{\sqrt{(2+a)a-\epsilon}} \right], \quad (57)$$

which reduces to (37) when $a, \epsilon = 0$.

The absorption coefficient per unit time, A_{sec} , defined by⁹⁾

$$A_{\text{sec}} = 8\pi\omega \Im(\chi_w) \quad (58)$$

is written

$$A_{\text{sec}} = 2 \frac{\pi}{\hbar} N g^2 \beta^2 S \left(\frac{a}{2+a} \right)^{1/2} \left[\left(\frac{\omega^2}{\omega_1} \right) \frac{1/\tau}{(\omega - \omega_1)^2 + 1/\tau^2} + \left(\frac{\omega^2}{\omega_2} \right) \frac{1/\tau}{(\omega - \omega_2)^2 + 1/\tau^2} \right] \quad (59)$$

by use of (56), where we omitted the terms irrelevant to the resonance absorption. In particular, we have in the neighbourhood of the resonance frequency

$$A_{\text{sec}} \simeq 4 \frac{\pi}{\hbar^2} N g^2 \beta^2 K S^2 \tau, \quad \omega \simeq \omega_r,$$

assuming that $\epsilon = 0$.

§ 6. Comparison with the classical theory

According to the results obtained with the classical theory of resonance absorption by Nagamiya, the resonance frequencies are as follows:

$$(1) \quad H_{\parallel} = H, \quad H_{\perp} = 0$$

$$\omega = \gamma [\sqrt{2AK'} \pm H], \quad (T \simeq 0^\circ K), \quad (61)$$

$$\omega = (\gamma/2) [\sqrt{8AK' + H^2} \pm H], \quad (T \simeq \text{Curie point}). \quad (62)$$

$$(2) \quad H_{\parallel} = 0, \quad H_{\perp} = H$$

$$\omega_1 = \gamma \sqrt{2AK'}, \quad (63)$$

$$\omega_2 = \gamma \sqrt{2AK' + H^2}, \quad (64)$$

where

$$\begin{aligned} A &= 12|J|g^{-2}\beta^{-2}/N, \quad K' \cong NKS^2, \\ \gamma &= ge/2mc. \end{aligned} \quad (65)$$

Assuming that $|J| \propto K$, the resonance frequencies given by (51) are in agreement with (61) on substituting (65). However, the former does not show any temperature dependence, while the latter varies from (61) to (62) when the temperature is increased.

In the case of $H_{\perp} = 0$, the resonance frequencies corresponding to the case (2) are obtained from (51) by putting $H_z = 0$. They degenerate into (63). This degeneracy cannot be removed even in the presence of H_{\perp} , because the last term in (15) contributes to \mathcal{H}_0 through the second order perturbation, which causes merely a uniform shift of the energy levels of the both systems of $(r, 0)$'s equal to

$$-\frac{1}{12} \frac{Ng^2\beta^2}{(2+\alpha)|J|} H_{\perp}^2 \quad (66)$$

by use of (55).

This qualitative discrepancy between these two different theories might arise from the approximations made in the present calculations of the spin wave theory. If the perturbational calculation of the terms of cubic, quartic and higher orders of the products of operators could be included, it would make the present theory approach the classical theory better. In actuality, however, this program is very difficult to perform because of its complicity.

§ 7. Resonance frequencies in the orthorhombic system

The anisotropy energy introduced in (30) has a uniaxial symmetry. It was in virtue of this high symmetry that the diagonalization of \mathcal{H}_{λ} was straight-forward. We see, however, that the case of the anisotropy energy with orthorhombic symmetry can be treated very similarly. In this case we can write

$$\mathcal{H}_{\text{anis}} = K_1 \left(\sum_j S_{xj}^2 + \sum_k S_{xk}^2 \right) + K_2 \left(\sum_j S_{yj}^2 + \sum_k S_{yk}^2 \right) \quad (67)$$

and when this is introduced into \mathcal{H}_{λ} , the diagonalization of \mathcal{H}_{λ} can still be performed in complete parallelism with the previous case.

The present \mathcal{H}_{λ} can be written, in place of (14), as

$$\begin{aligned} [3|J|S]^{-1} \mathcal{H}_{\lambda} &= (1 + \epsilon + a_1) Q_{\lambda}^2 + (1 + \epsilon + a_2) P_{\lambda}^2 + (1 - \epsilon + a_1) R_{\lambda}^2 \\ &\quad + (1 - \epsilon + a_2) S_{\lambda}^2 + 2\gamma_{\lambda} (Q_{\lambda} R_{\lambda} - P_{\lambda} S_{\lambda}), \end{aligned} \quad (68)$$

$$a_1 = K_1/3|J|, \quad a_2 = K_2/3|J|. \quad (69)$$

After successive uses of the transformations given by (19) with (20) and (22) with

$$\tan 2\varphi_\lambda = 2\epsilon\sigma\tau \sin 2\theta_\lambda / \{(2 + \alpha_1 + \alpha_2)\epsilon \cos 2\theta_\lambda - (\alpha_2 - \alpha_1)\gamma_\lambda \sin 2\theta_\lambda\}, \quad (70)$$

$$\sigma = (1 + \alpha_2 + \epsilon \cos 2\theta_\lambda - \gamma_\lambda \sin 2\theta_\lambda)^{1/2}, \quad (71)$$

$$\tau = (1 + \alpha_2 - \epsilon \cos 2\theta_\lambda + \gamma_\lambda \sin 2\theta_\lambda)^{1/2}$$

instead of (23) and (24), we arrive at the diagonalized Hamiltonian (25) with the frequencies given by

$$\omega_{r\lambda} = 2[(1 + \alpha_1)(1 + \alpha_2) + \epsilon^2 - \gamma_\lambda^2 \pm \{(2 + \alpha_1 + \alpha_2)^2 \epsilon^2 + (\alpha_2 - \alpha_1)^2 \gamma_\lambda^2 - 4\epsilon^2 \gamma_\lambda^2\}^{1/2}]^{1/2}, \quad (72)$$

which replaces (26). The + and - of the double sign in the right-hand side of (72) correspond to $r=1$ and $r=2$ respectively.

Putting $\lambda=0$ in (72), we obtain the resonance frequencies in the presence of a parallel field as

$$\begin{aligned} \hbar\omega_r &= 3|J|S\omega_{r0} \\ &\cong 6|J|S[\alpha_1 + \alpha_2 + \epsilon^2 \pm \{4(\alpha_1 + \alpha_2)\epsilon^2 + (\alpha_2 - \alpha_1)^2\}^{1/2}]^{1/2}. \end{aligned} \quad (73)$$

In the absence of a parallel field (73) reduces to

$$\hbar\omega_1 = 6|J|S\sqrt{2\alpha_2}, \quad \hbar\omega_2 = 6|J|S\sqrt{2\alpha_1}, \quad (74)$$

assuming that $\alpha_2 > \alpha_1$. Thus the degeneracy of resonance frequencies in the case of uniaxial symmetry does not appear in the case of orthorhombic symmetry. By the same considerations as in the case of uniaxial symmetry, however, we see that both frequencies are independent of the perpendicular field in the present case at the stage of the approximation of spin wave theory.

In conclusion, the writer wishes to express his cordial thanks to Prof. T. Nagamiya for his continual interest and discussions and to Dr. K. Yosida for his valuable discussions during the stay at Osaka University.

Note added in proof

1. The anisotropy Hamiltonian adopted in this paper are somewhat phenomenological in nature. According to the recent investigation by F. Keffer (to be published in the Phys. Rev.) the anisotropy energy of MnF_2 arises mainly from the dipole interaction among spins. Then, it will be desirable to treat the Hamiltonian inclusive of the dipole interaction. The main results will, however, remain unchanged even if it was done. We shall report them later.

2. The resonance frequency dependent on the perpendicular field, (64), may formally be derived from the method of using the spin wave approximation (9) in the representations which diagonalize the spin operators in the directions of the mean magnetization of two sublattices, which are determined so that the perpendicular susceptibility may agree with the Van Vleck formula (37). But, we fail to determine the directions of the mean magnetization of two sublattices by such spin wave theory itself consistently, and then the above method is unsatisfactory with regard to this point. We are grateful to Mr. Yosida for awakening our attention to the above method.

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On the Structure of the Interactions of the Elementary Particles, II

—Does the Interaction of the Second Kind exist in the Nature?—

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As was shown in the previous paper^{**}), it would have a very important meaning for the formation of the future theory of the elementary particles to decide whether there are interactions of the second kind in the nature or not. In this paper, we shall first look for generally those types of fields which may have the interactions of the first kind, and then discuss whether the model of the mesons can be explained by means of these fields with interaction of the first kind only.

§ 1. Introduction

In the foregoing paper^{**}), we have investigated the applicability of the renormalization theory and classified interactions into the first and the second kinds. It was shown there that the renormalization theory would fail at once, as soon as even one interaction of the elementary particles was found to be that of the second kind.

It has been generally believed that the renormalization theory had achieved a great success in the spinor quantum electrodynamics, though it seemed to fail in the theory of mesons. But, in our opinion, such a mode of expression is certainly wrong, because all the elementary particles existing in the nature, as we have stressed repeatedly¹⁾, are so related each other, that they can not treat separately. In fact, if the existence of the interaction of the second kind were required in the theory of mesons, all of the S-matrix elements in the spinor quantum electrodynamics would also be divergent. Every Feynman diagram in the spinor quantum electrodynamics will always contain in itself meson lines as the internal ones, if the higher order approximations are taken into account correctly.

Thus we are now standing on a turning point where the existence or non-existence of the interactions of the second kind play a decisive role. If their existence were proved, we should be confronted at once with the problem of the structure of the elementary particles. So that investigations of the non-local interactions²⁾ as well as the non-local field³⁾ would be required strongly. On the contrary, if their non-existence were clarified, a consistent non-singular formalism would be obtained in accordance with the program of the renormalization theory⁴⁾. Of course, even if it were so, we should still have to

* Read at the Tokyo meeting of the physical society of Japan held on October 5, 1951.

** S. Sakata, H. Umezawa and S. Kamefuchi, Prog. Theor. Phys. 7 (1952), 377, which will be cited as I. Throughout this paper we use the same notation as I.

investigate the structure of the elementary particles hidden behind the renormalization theory. If such a hidden parameter were not found, not only the renormalization procedure would not been justified, but also we could not understand the reason why the interactions of the first kind were merely realized in the nature.

In order to investigate the existence or non-existence of the interactions of the second kind in detail, we shall discuss in this paper the following two questions:

- i) What types of fields may realize the interactions of the first kind? (§ 3)
- ii) Whether the various properties of the mesons can be described merely in terms of the interactions of the first kind? (§ 4)

Before discussing these questions, we shall first classify the well known interactions in the theory of usual fields having spin ≤ 1 (§ 2).

§ 2. Interactions between the usual fields

In this section, we shall discuss the interactions between the usual fields, which are denoted as follows:

A_μ : the electromagnetic field,

U : the scalar or pseudoscalar field with $\kappa \approx 0$,

ψ : the spinor field with spin=1/2 and $\kappa \approx 0$,

U_μ : the vector or pseudovector field with $\kappa \approx 0$.

In Table I, the respective $\Gamma_{ii'}$'s and $\beta^{\alpha\beta}$'s for these fields are shown.

Table I

	A_μ	U	ψ	U_μ
$\Gamma_{ii'}$	$\delta_{\mu\nu}$	1	$\{(\gamma_\mu \partial_\mu) + \kappa\}$	$\delta_{\mu\nu} - \frac{1}{\kappa^2} \partial_\mu \partial_\nu$
β^α	0	0	1	2

i) *The spinor electrodynamics* The vector coupling $\bar{\psi} \gamma_\mu \psi A_\mu$ between A_μ and ψ belongs to the interaction of the first kind with $\eta_i = 0$. It is well known that all the divergencies

appeared in this case can be removed by the renormalization of the mass and the charge⁵⁾.

The tensor coupling $\bar{\psi} \gamma_\mu \gamma_\nu \psi F_{\mu\nu}$ ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) belongs, however, to the interaction of the second kind with $\eta_i = 1$. In fact, the renormalization of the divergent term $A_\mu \square \int_\mu^{(6)}$, induced by this interaction in the problem of the vacuum polarization, gives rise to new diverging terms, which require the further introduction of infinitely many interaction terms as the counter terms. Therefore, if the tensor coupling or the so called Pauli term were included in the spinor quantum electrodynamics, we could not obtain a closed theory by the renormalization procedure.

ii) *Interactions of the scalar or pseudoscalar field* The scalar (pseudoscalar) coupling between U and the spinor field ψ , and the vector coupling between the charged U and A_μ belong to the interactions of the first kind. As we can easily see from (2.21) in I, those graphs which consist of four external U -lines are primitive divergent ones. The corresponding counter term may be written as the form λU^4 which belongs to the interaction of the first kind with $\eta_i = 0$. In the case when U is neutral, the introduction

of the counter term having the form $\lambda' U^3$ ($\eta_l = -1$) is also required. It is well known that in the theory of the scalar or pseudoscalar mesons, which interact with the nucleons and the electromagnetic field by the above mentioned couplings, all the divergencies can be removed by the renormalizations of masses, charges, λU^4 - and $\lambda' U^3$ -terms⁷⁾.

iii) *Interactions of the vector or pseudovector field* The vector coupling with the electromagnetic field has the form $U_\mu G_{\mu\nu} A_\nu$ where $G_{\mu\nu}$ is defined by

$$G_{\mu\nu} \equiv \partial_\mu U_\nu - \partial_\nu U_\mu = \partial_{[\mu\nu]} U_{\mu'}. \quad (2.1)$$

Taking into account the discussions made in the Appendix of I, the true degree a^* for the derivation operator $\partial_{\mu\nu\mu'}$ in (2.1) is 0. Therefore this coupling belongs to the interaction of the second kind with $\eta_l = 1$.**)

Moreover, the tensor coupling $U_\mu U_\nu F_{\mu\nu}$ with the electromagnetic field belongs also to the interaction of the second kind with $\eta_l = 2$.

In fact, these couplings give rise, for example, to the diverging term $A_\mu [\square] J_\mu^{(8)}$ in the problem of the vacuum polarization, so that the renormalization procedure cannot give a closed theory.

iv) *The direct interaction between spinor particles* The direct interaction $\bar{\psi}^{(a)} \psi^{(b)} \bar{\psi}^{(c)} \psi^{(d)}$, which appears, for example, in Fermi's theory of the beta decay, belongs to the interaction of the second kind with $\eta_l = 2$. As Kamefuchi¹⁰⁾ has shown, the introduction of infinitely many counter terms having successively higher derivatives of field quantities is required in this case, so that we cannot obtain a closed theory.

In Table II we have shown the properties of important interactions between the usual fields indicated in the row and the column.

Table II

	U		U_μ		ϕ	
	Charged	Neutral	Charged	Neutral	Charged	Neutral
A_μ	$V(0)$ 1st [$U^* U U^* U(0)$]		$V(1)$ 2nd $T(2)$ 2nd		$V(0)$ 1st $T(1)$ 2nd	
ϕ	$S(0)$ 1st [$U^* U U^* U(0)$]	$S(0)$ 1st [$U^4(0), U^3(0)$]	$V(1)$ 2nd	$V(0)$ for v^* 1st $V(1)$ for $\not{p}v$ 2nd	Direct Interaction (2) 2nd	
	$V(1)$ 2nd	V^* for s $V(1)$ 2nd for $\not{p}s$	$T(1)$ 2nd	$T(1)$ 2nd		

S , V and T denote the scalar, vector and tensor couplings respectively. The values in () show η_l for respective interactions. The counter interactions required for the closure of the respective interactions are written in []. s , $\not{p}s$, v and $\not{p}v$ denote the scalar, pseudoscalar, vector and pseudovector fields respectively.

* The vector coupling of the neutral vector field or neutral scalar field will be discussed in detail in the Appendix 1.

** Of course, we have also $e^2 U_\mu U_\nu A_\mu A_\nu$ -term ($\eta_l = 2$) for the electromagnetic interaction of the charged vector field because of the gauge invariance. In general we can easily prove that the e^2 -interaction term required by gauge invariance has the same η_l as e -interaction term or the larger η_l than e -interaction for the cases in which the latter interaction belongs to the first or the second kind respectively.

§ 3. The interactions of the fields with arbitrary spins

Now we shall investigate generally the fields with arbitrary spins which may interact with each other through the interactions of the first kind. As we have shown in I, the condition that an arbitrary interaction term II_i is the interaction of the first kind can be written in the following form :

$$\eta_i = -K_i = A_i + B_i/2 + C_i - 4 \leq 0. \quad (3.1)$$

Taking account of the fact that A_i and all b^a 's are not negative, we obtain the condition

$$b^a \leq 4 \quad (3.2)$$

for an ω -field in order that it may have the interaction of the first kind.

On the other hand, there are simple relations between b^a and the spin s of the ω -field, as we may see from the commutation relations. The canonical commutation relations for the free fields with arbitrary spins will be discussed in detail in Appendix II. From the relations (A-2-10) and (A-2-12) given there, we find generally the following relation for the ω -field with a spin value s and a non vanishing mass κ :

$$b^a = 2s \quad \text{for} \quad \kappa \neq 0. \quad (3.3)$$

In cases when the mass of the field vanishes identically, we find instead of (3.3)

$$b^a = \begin{cases} 0 & \text{for } \kappa=0 \text{ and } s=\text{integer,} \\ 1 & \text{for } \kappa=0 \text{ and } s=\text{half integer.} \end{cases} \quad (3.4)$$

From (3.2) and (3.3), it becomes clear that *the fields with $s > 2$ and $\kappa \neq 0$ can never have the interactions of the first kind.* It should be noted that this conclusion will not be altered, even if we take into account the circumstances that the vector couplings of the neutral vector fields may happen to have smaller values of b^a than those given by (3.3). (See Appendix I)

For the field $U_{\mu\nu}$ with $s=2$ and $\kappa \neq 0$, the following interaction has the smallest η_i ($=0$) :

$$H_i = U_{\mu\nu} B_{\mu\nu} \quad (3.5)$$

where $B_{\mu\nu}$ is the field with $s=2$ and $\kappa=0$. All other interactions belong to those of the second kind.

For the field χ_μ with $s=3/2$ and $\kappa \neq 0$, the following interaction has the smallest η_i ($=0$) :

$$H_i = \chi_\mu \omega_\mu \quad (3.6)$$

where ω_μ is the field with $s=3/2$ and $\kappa=0$. All other interactions belong to those of the second kind.

For the field U_μ with $s=1$ and $\kappa \neq 0$, there are four types of interactions which belong to those of the first kind ($\eta_i=0$) :

$$H_i = F_{\mu\nu} F_{\mu\nu}', \quad H_i = U_\mu U_\mu', \quad (3.7)$$

$$H_i = U_\mu \partial_\mu U, \quad (3.8)$$

$$H_i = U_\mu C_{\mu\nu}^1 \dots C_{\nu\mu}^2, \quad (3.9)$$

where U'_μ and U are fields with $s=1$ and $s=0$ respectively. $C_{\mu\nu\dots}^1$ and $C_{\nu\dots}^2$ are fields with $\kappa=0$ having $s=m+1$ and m respectively.

In the case of the neutral vector field V_μ the vector coupling with the spinor field ($s=1/2$)

$$H_i = V_\mu \bar{\psi} \gamma_\mu \psi \quad (3.10)$$

and the vector coupling with the scalar field U

$$H_i = V_\mu U^* \partial_\mu U \quad (3.11)$$

belong also to the interactions of the first kind. (See Appendix I) All interactions of the fields U_μ and V_μ , excepting the above mentioned ones, belong to those of the second kind.

As we have seen from the above investigations, there are remarkable distinctions between the interactions of the fields with $s > 1$ and $s \leq 1$, as long as the masses of the fields do not vanish. In fact, the fields with $s > 1$ and $\kappa \neq 0$ can not have the interactions of the first kind composed of more than two lines, that is, $C_i \geq 3$. The interactions (3.5) and (3.6) are composed of two lines. Moreover, in such cases the mass terms $\kappa^2 U_{\mu\nu\dots\rho} U_{\mu\nu\dots\rho}$ in the Hamiltonian belong to the interactions of the second kind, so that we could not apply the usual procedure of the mass renormalization. It would be very probable that these circumstances should restrict the spins of the elementary particles existing in the nature.

For the fields with vanishing masses, there are many types of interactions belonging to those of the first kind.

§ 4. The models in the meson theory

In this section, we shall investigate briefly the question whether we may obtain a consistent model in the meson theory by taking into account merely interactions of the first kind or not. As we have seen from the discussions in § 3, the spin value s of the charged π -meson must be zero in order to interact with nucleon via the interaction of the first kind. This conclusion is consistent with the phenomenological investigations on the behaviours of the negative π -mesons in the light elements¹⁰⁾.

As is well known, the fact, that the π -meson decays not into the electron, but into the μ -meson, requires the existence of the *direct interaction* between the nucleon and the lepton, though there is an exceptional case¹¹⁾ that the π -meson is described by the pseudoscalar field interacting with the lepton via the pseudovector coupling which belong to the interaction of the second kind. Similarly, it is also required the existence of the direct interaction between the μ -meson and the lepton.

On the other hand, we have shown in § 2 that the direct interaction between the spinor fields belongs to the interaction of the second kind*. So that, if we assume that

* Even if the spin of the μ meson is 0, its direct interaction with the lepton belongs also to that of the second kind ($\eta_l=1$).

there exist only the interactions of the first kind in the nature, the interactions between the nucleons, the leptons and μ -mesons will be considered to be intermediated by another kind of Bose field which is not identical with the π -meson.

It is required for this new field that it is a charged field having a spin value $s=0$ and gives rise to the coupling of the tensor type in the theory of the nuclear beta-decay¹²⁾. However, it is the well known fact that the beta-coupling of the tensor type can not be obtained from the primary interactions of the forms

$$(\Psi\gamma\Psi')U \quad \text{and} \quad (\bar{\psi}\gamma\psi')U \quad (4.1)$$

where Ψ , Ψ' , ψ and ψ' denote the field quantities of the proton, the neutron, the electron and the neutrino respectively, and γ is equal to 1 or γ_5 , according as the U -field is scalar or pseudoscalar. In order to obtain the beta-coupling of the tensor type, we must take the following interactions instead of (4.1):

$$(\bar{\psi}\gamma\Psi')U \quad \text{and} \quad (\bar{\Psi}\gamma\psi')U. \quad (4.2)$$

In fact, by assuming the interaction (4.2) with $\gamma=1$, we get the following beta-coupling in the second approximation of the perturbation calculation:

$$(\bar{\Psi}\psi')(\bar{\psi}\Psi') \quad (4.2)^*$$

which may be rewritten in the form, as was shown by Fierz,¹³⁾

$$(\bar{\Psi}\psi')(\bar{\psi}\Psi') = 1/4 \cdot \sum_{k=1}^5 J_k \quad (4.4)$$

where $J_k (k=1, \dots, 5)$ represent respectively the well known five sorts of the beta-couplings appeared in Fermi's theory. Hence it is clear that (4.2) contains the beta-coupling of the tensor type.

In this way it would be possible to establish a consistent model in the meson theory, involving only interactions of the first kind, if the existence of the new Bose field having the interactions of the form (4.2) were really confirmed.

Recently, Brueckner¹⁴⁾ has, however, shown by the perturbation calculation that the production of the meson by the nucleon-nucleon collisions could not be accounted for by the pseudoscalar coupling**.

Moreover, the fact, that the absorption cross section of the negative π -meson by the deuteron depends strongly on the momentum of the π -meson, can not explained by the pseudoscalar coupling in the perturbation theory¹⁵⁾. These results suggest that the interaction between the π -meson and the nucleon does not belong to the first kind, though it is not definite proof.

* For the high energy region the beta-coupling obtained from (4.2) in the second order approximation of the perturbation calculation will have a different form. But we may confine ourselves to the low energy region as long as the nuclear beta-decays are concerned.

** In these connections, it should be noted that the pseudovector coupling of the meson would become to be renormalizable if we took a new mixed theory of pseudoscalar and pseudovector mesons which was suggested recently by Beard & Bethe¹⁵⁾.

Furthermore, we have some experimental evidences for the multiple production of π -mesons by the nucleon-nucleon collisions, which seems to suggest the existence of the interaction of the second kind between the π -meson and the nucleon. The fact that the dependence of the most probable number of the produced mesons on the incident energy is stronger than logarithmic, could not be accounted for by the interaction of the first kind.*

In conclusion, the author wishes to express his sincere gratitude to Prof. S. Sakata for his continuous guidance throughout this work. and to Messers S. Kamefuchi and R. Kawabe for their valuable discussions.

Appendix I

The neutral vector field with the vector coupling

(The generalized Glauber's theorem)

In this Appendix two theorems will be proved: (i) the vector coupling between a neutral vector field U_μ and a source field, whose current satisfies an equation of continuity, belongs to the interaction of the first kind or to the second kind according to the spin s of the source field $s < 1$ or $s \geq 1$, and (ii) the usual theory of the electromagnetic field may be obtained in the limit $\kappa \rightarrow 0$ from the theory of the neutral vector field with the mass κ . The latter theorem was firstly established by Glauber** for the special case in which the spin of the source field was $\frac{1}{2}$.

First we shall *generally* establish the theorem (ii). In this section the bold letters and the usual letters are used for the quantities in the Heisenberg- and interaction- representations respectively. The interaction Hamiltonian $H'(x)$ in the interaction representation for the neutral vector field with the vector coupling may be written as follows by the well-known unitary transformation:

$$H' = j_\mu U_\mu + 1/2 \cdot j_{\mu\nu} U_\mu U_\nu + 1/2\kappa^2 \cdot (j_\mu n_\mu)^2 \quad (\text{A.I.1})$$

where j_μ and $j_{\mu\nu}$ do not contain the quantities of the U_μ -field.***

On the other hand the current \mathbf{J}_μ in the Heisenberg representation takes the following form:

$$\mathbf{J}_\mu = \mathbf{j}_\mu + \mathbf{j}_{\mu\nu} \mathbf{U}_\nu \quad (\text{A.I.2})$$

where $\mathbf{j}_{\mu\nu}$ does not have time components:

$$j_{\mu\nu} n_\nu = j_{\mu\nu} n_\nu = 0. \quad (\text{A.I.3})^{****}$$

Moreover, the current \mathbf{J}_μ satisfies the continuity-equation:

$$\partial_\mu \mathbf{J}_\mu = 0. \quad (\text{A.I.4})$$

* The beta disintegration based on the interaction (4.1) was previously discussed by Y. Tanikawa (Prog. Theor. Phys. **3** (1948), 338).

** R. J. Glauber, "On the Gauge-Invariance of the Neutral Vector Meson Theory", (unpublished paper).

*** For example, for the source field of the spinor type,

$$j_\mu = g \bar{\psi} \gamma_\mu \psi \quad \text{and} \quad j_{\mu\nu} = 0.$$

See note added in proof (I).

**** See note added in proof (II).

In the interaction representation, the continuity-equation will be transformed into the following form :

$$\partial_\mu J_\mu(x) = i \int [J_\mu(x), H'(x')] d\sigma'_\mu = i \int [j_\mu(x), H'(x')] d\sigma'_\mu, \quad (\text{A.I.5})$$

$$J_\mu \equiv j_\mu + j_{\mu\nu} U_\nu. \quad (\text{A.I.6})$$

Comparing the terms with the same power of U_μ in the both sides of (A.I.5), we obtain following relations :

$$\begin{aligned} \partial_\mu j_\mu &= 0, \\ n_\mu [j_\mu(x), j_\nu(x')] &= i n_\mu \partial_\rho \partial_\mu \mathcal{A}(x-x') j_{\rho\nu}(x'), \\ n_\mu n_\nu [j_\mu(x), j_\nu(x')] &= 0, \\ n_\rho [j_\mu(x), j_{\rho\nu}(x')] &= 0. \end{aligned} \quad (\text{A.I.7})$$

In the Stueckelberg's formalism for the neutral vector field there are following relations :

$$\begin{aligned} U_\mu &= A_\mu + (1/\kappa) \partial_\mu B, \\ [A_\mu(x), A_\nu(x')] &= i \delta_{\mu\nu} \mathcal{A}(x-x'), \\ [B(x), B(x')] &= i \mathcal{A}(x-x'), \\ [B(x), A_\mu(x')] &= 0, \quad (\partial_\mu A_\mu + \kappa B) \Psi[\sigma] = 0, \end{aligned} \quad (\text{A.I.8})$$

where A_μ and B denote a vector field and a scalar field respectively.

Introducing the following unitary transformation :

$$\begin{aligned} \Psi[\sigma] &\rightarrow \Psi'[\sigma] = U^{-1}[\sigma] \Psi[\sigma], \\ U[\sigma] &\equiv \exp \left[-i \int_\sigma j_\mu(x') B(x') d\sigma'_\mu \right] \end{aligned} \quad (\text{A.I.9})$$

and using (A.I.7), we obtain the following Schrödinger equation and supplementary condition :

$$\begin{aligned} \left[i \frac{\partial}{\partial \sigma(x)} - j_\mu A_\mu - \frac{1}{2} j_{\mu\nu} A_\mu A_\nu \right] \Psi'[\sigma] &= 0, \\ \left[\partial_\mu A_\mu + \kappa B - \int_\sigma j_\mu(x') \mathcal{A}(x-x') d\sigma'_\mu \right] \Psi'[\sigma] &= 0. \end{aligned} \quad (\text{A.I.10})$$

As (A.I.10) gives in the limit $\kappa \rightarrow 0$ the Schrödinger equation and the Lorentz condition in the quantum electrodynamics, we have proved the theorem (ii).

Moreover we obtain the theorem (i) as follows. From the commutation relation (A.I.8), we see that b^α for A_μ is 0. As the Schrödinger equation in (A.I.10) does not contain B -field it is proved that the vector coupling between the neutral vector field U_μ and the source field belongs to the first kind only for the source with the spin 0 or 1/2, and to the second kind for the sources of another types.

Similarly we can easily show that the vector coupling $H'(x')$ of the neutral scalar field $U(x)$, whose source satisfies the continuity-equation of the current, has no physical effect.

$$H'(x) = j_\mu \partial_\mu U + 1/2 \cdot j_{\mu\nu} \partial_\mu U \cdot \partial_\nu U + 1/2 x^2 (j_\mu n_\mu)^2. \quad (\text{A.I.11})$$

There arises also similar circumstances for the neutral fields $U_{\mu\nu\dots}$ with the spin $s > 1$ and the mass $x \neq 0$. For the interaction of the certain types, δ^α has the same value as if in the case of $x=0$ (and so $\delta^\alpha=0$ or 1) owing to the existence of a kind of continuity equation. However these interactions* belong to the second kind, because they must contain higher derivatives in order to contract the tensor-suffices of $U_{\mu\nu\dots}$.

Appendix II

The commutation relations of fields with the arbitrary spin

In this Appendix we shall determine the commutation relations of fields with the arbitrary spin, which are compatible with the canonical formalism. Such commutation relation has already been given by M. Fierz¹⁷⁾, but as his results contain some errors in the case of Bose fields, we shall improve them here.

The field $U_{\mu_1\mu_2\dots\mu_s}$ (μ_1, μ_2, \dots being the tensor suffices) with the integer spin s and the mass $x \neq 0$ satisfies following equations:

$$(\square - x^2) U_{\mu_1\dots\mu_s} = 0, \quad (\text{A.II.1})$$

$$\partial_\nu U_{\nu\mu_2\dots\mu_s} = 0, \quad (\text{A.II.2})$$

$$U_{\nu\nu\mu_3\dots\mu_s} = 0. \quad (\text{A.II.3})$$

$U_{\mu_1\dots\mu_s}$ is the symmetrical tensor of rank s .

The commutation relations must be compatible with the following canonical equations:

$$i \cdot \partial f / \partial t = [H, f] \quad (\text{A.II.4})$$

where H is the Hamiltonian density in the absence of interactions.

The commutation relation for $U_{\mu_1\dots\mu_s}(x)$ and $U_{\mu'_1\dots\mu'_s}(x')$ will have to satisfy the following requirements: (i) It must contain the delta function differentiated by the tensor operator of rank $2s$ composed of ∂_ν and $\partial_{\mu\nu}$. (ii) It must be symmetrical with respect to the tensor suffices $(\mu_1\dots\mu_s)$ (and also $(\mu'_1\dots\mu'_s)$). (iii) It must be compatible with (A.II.2). Thus, we must determine the following expression so as to satisfy these requirements:

$$[U_{\mu_1\dots\mu_s}(x), U_{\mu'_1\dots\mu'_s}(x')] = [\alpha_0 + \alpha_1 \sum^{(1)} + \dots + \alpha_r \sum^{(r)}] \\ P(\mu'_i) \prod_{i=1}^s R(\mu_i, \mu'_i) \Delta(x - x') \quad (\text{A.II.5})$$

where $r=s/2$ for even s and $r=(s-1)/2$ for odd s . $P(\mu'_i)$ means the summation over all possible permutations of suffices (μ'_i) . $R(\mu, \nu)$ is the derivation operator defined as follows:

* The only possibility of the interactions of such fields, which does not belong to the second kind, is such a form as $\psi \gamma_\mu \gamma_\nu \dots \gamma_\rho \psi \ell_{\mu\nu\dots\rho}$ but it vanishes identically in virtue of the symmetrical property of $U_{\mu\nu\dots\rho}$ as well as the following subsidiary condition $U_{\mu\mu, \sigma\dots\rho} = 0$.

$$R(\mu, \nu) = \delta_{\mu\nu} - \frac{1}{x^2} \partial_\mu \partial_\nu. \quad (\text{A.II.6})$$

In (A.II.5) the notation $\sum^{(j)}$ means the following procedure: First, pick up arbitrary j pairs of two elements $R(\mu_i, \mu'_i)$ and $R(\mu_k, \mu'_k)$ from $\prod_{i=1}^s R(\mu_i, \mu'_i)$ and then in each of these pairs exchange their arguments μ'_i and μ_k (for example, $R(\mu_1, \mu'_1)R(\mu_2, \mu'_2) \rightarrow R(\mu'_1, \mu'_2)R(\mu_1, \mu_2)$). Second, sum up over all the possible ways of picking out j pairs. $\alpha_0, \alpha_1, \dots, \alpha_r$ are the constants which are to be determined by the requirement that the commutation relation (A.II.5) must be compatible with the subsidiary condition (A.II.3).

Using the following relations for $R(\mu, \nu)$:

$$\begin{aligned} R(\mu, \nu)R(\nu, \sigma)J(x) &= R(\mu, \sigma)J(x), \\ R(\mu, \mu)J(x) &= 3J(x), \end{aligned} \quad (\text{A.II.7})$$

we can obtain the recurrence formula:

$$\begin{aligned} \text{Trace } (\mu_1, \mu_2) \sum^{(j)} P(\mu'_i) \prod_{i=1}^s R(\mu_i, \mu'_i) &= (2s - 2j + 1) P(\mu'_1) R(\mu'_1, \mu'_2) \\ &\times \sum_{i=3}^{(j-1)} R(\mu_i, \mu'_i) + P(\mu'_i) R(\mu'_i, \mu'_2) \sum_{i=3}^{(j)} \prod_{i=1}^s R(\mu_i, \mu'_i) \end{aligned} \quad (\text{A.II.8})$$

where $\text{Trace } (\mu_1, \mu_2)$ means the trace with respect to μ_1 and μ_2 .

By means of (A.II.7), (A.II.8) and the subsidiary condition (A.II.3) we can obtain the relation between α_i and α_{i-1} :

$$\alpha_i = -(1/2s - 2j + 1)\alpha_{i-1}$$

and so:

$$\alpha_j = (-1)^j \left\{ \prod_{m=0}^{(j-1)} \frac{1}{(2s - 2j + 1 + 2m)} \right\} \alpha_0. \quad (\text{A.II.9})$$

Hence, we obtain the commutation relation as follows:

$$\begin{aligned} [U_{\mu_1, \dots, \mu_g}(x), U_{\mu'_1, \dots, \mu'_s}(x')] &= \alpha_0 \sum_{j=0}^r P(\mu'_i) \\ &\times \left[(-1)^j \left\{ \prod_{m=0}^{(j-1)} \frac{1}{(2s - 2j + 1 + 2m)} \right\} \sum_{i=0}^{(j)} \prod_{i=1}^s R(\mu_i, \mu'_i) \right] J(x - x'). \end{aligned} \quad (\text{A.II.10})$$

The constant α_0 is to be determined by the requirement that the commutation relation (A.II.10) must be compatible with the canonical equation (A.II.4). The actual form of the energy tensor $T_{44} = -H$ was concretely given by M. Fierz¹⁷⁾. By use of this T_{44} , α_0 is to be determined by the equation:

$$\begin{aligned} \alpha_0 \left(\prod_{i=1}^s \sum_{\mu_i} \delta_{\mu_i \mu'_i} \right) \sum_{j=0}^r P(\mu'_i) &\left[(-1)^j \left\{ \prod_{m=0}^{(j-1)} \frac{1}{(2s - 2j + 1 + 2m)} \right\} \right. \\ &\times \left. \sum_{i=1}^{(j)} \prod_{i=1}^s R(\mu_i, \mu'_i) \right] J(x) = (2S + 1) J(x). \end{aligned} \quad (\text{A.II.11})$$

In contrast with (A.II.9) Fierz has obtained $\alpha_1/\alpha_0 = -1/\{2 + s(s-1)/2\}$ and $\alpha_2 = \alpha_3 = \dots = \alpha_r = 0$.¹⁷⁾ This is correct only for the case of the spin $s < 4$, but incorrect for the case of $s \geq 4$, because of the non-vanishing α_i 's ($i \geq 2$).

For the case of the half-integer spin $s = m + \frac{1}{2}$ Fierz obtained the following commutation relation in the spinor representation :

$$\left[\varphi_{u_1 \dots u_m}^{\dot{t}_1 \dots \dot{t}_{m+1}}(x), \varphi_{u_1 \dots u_m}^{\dot{t}_1 \dots \dot{t}_{m+1}}(x') \right]_+ = \alpha \sum_{l=0}^m (-x^2)^{-P} P(u_l) P(t_k) \\ \times \left\{ \prod_{l=1}^P \partial_{u_l u_l} \right\} \left\{ \prod_{j=1}^{m-P} \partial_{u_{P+j} \dot{t}_j} \partial_{u_{P+j} \dot{t}_j} \right\} \left\{ \prod_{i=m-P+1}^{m+1} \partial_{\dot{t}_i \dot{t}_i} \right\} \Delta(x-x') \quad (\text{A.II.12})$$

where α is a constant.

In the case of the mass $x=0$ the terms with negative powers of the mass x must not be allowed in (A.II.10) and (A.II.12), and so the terms with higher derivatives disappear. Then $R(\mu, \nu)$ must be replaced by $\delta_{\mu\nu}$ and a_j by the following a'_j :

$$a'_j = (-1)^j \prod_{m=0}^{(j-1)} \frac{1}{(2s-2j+2+2m)}. \quad (\text{A.II.13})$$

The latter replacement comes from the difference between $\delta_{\mu\mu} (=4)$ and $R(\mu, \mu) \Delta(x) (=3\Delta(x))$. While such a commutation relation of fields ($x=0$) is compatible with (A.II.3), it is not compatible with (A.II.2). Therefore in the case $x=0$ we must regard (A.II.2) as the condition imposed on the state vector. As an example of such situations we know the Lorentz condition for the electromagnetic field.*)

Note added in proof: After the author had finished the Appendix II, he could have chance to see the Fierz's paper (Helv. Phys. Acta. 22 (1950), 111), in which Fierz improved his previous results for the commutation relation. Although his discussions are very brief, we can see that Fierz's new result is equivalent to (A.II.9) taking into account the fact that Fierz used $c_l P(\mu_l) P(\mu_l')$ instead of $a_j P(\mu_l')$ in (A.II.5).

Note added in proof:

(I) We shall discuss now the several important points in the discussions in Appendix I.

We shall give firstly a short review of the derivation of (A.I.6), (A.I.1) and (A.I.3), the detailed discussions of which will be developed in the forthcoming paper⁽⁹⁾. When U_μ is the electromagnetic field A_μ ($x=0$), we can prove the following relation between the interaction Hamiltonian \mathcal{H}' and the current J_μ in the interaction representation :

$$J_\mu (= S J S^{-1}) = \frac{\partial}{\partial A_\mu} \mathcal{H}' \quad (1)$$

and so

$$J_\mu = j_\mu + j_{\mu\nu} A_\nu, \\ \mathcal{H}' = j_\mu A_\mu + \frac{1}{2} j_{\mu\nu} A_\mu A_\nu, \quad (2)$$

where S is the unitary transformation connecting the Heisenberg representation with the interaction representation. In the derivation of (1) and (2) we use the form of J_μ and \mathcal{H}' denoted by L and L (Lagrangian functions in both representations) and the following relation for the independent canonical variables :

$$\frac{\partial L}{\partial Q_{\alpha;A}} = S \left(\frac{\partial L}{\partial Q_{\alpha;A}} \right) S^{-1}. \quad (3)$$

* The simple and general rule for determining the commutation relations will be discussed in the forthcoming paper (Y. Takahashi and H. Umezawa, Prog. Theor. Phys. in press.).

Moreover, using (3) we can derive the important relation (A.I.3)¹⁷⁾.

When U_μ is not the electromagnetic field but the neutral vector field with the mass κ , we can obtain the generalized relation (A.I.1) from (2).

Now, we can prove the following important theorem according to the analogous method to that in Appendix I: The theory for the system of the arbitrary charged fields and the electromagnetic field is invariant for the following gauge-transformation:

$$\Psi[\sigma] \rightarrow \Psi'[\sigma] = \exp \left[-i \int \sigma j_\mu(x') A(x') d\sigma \mu' \right] \Psi[\sigma],$$

$$A_\mu \rightarrow A_\mu + \partial_\mu A = 0, \quad \square A = 0.$$

This is the general theorem of the gauge-invariance which has been concretely discussed for the particular cases of the charged fields with spin 1, 1/2 and 0.

(II) We can easily see the important feature of the energy spectrum of the multiple meson production derived from the nuclear interaction of the first kind, when we apply the dimensional analysis in § 3 of I to the "meson cloud spectrum $u(x, \sigma; x') d\lambda'$," by which the S-matrix for the multiple meson production was described in our previous paper (H. Umezawa, Y. Takahashi and S. Kamefuchi, *Mesonic Proyer Field*, Phys. Rev. **85** (1952), 505). In fact, taking into account the fact, for the nuclear interaction of the first kind, the quantities of the dimension of length affecting the high energy process is only the wave length λ and that the cloud spectrum has the same dimension as the meson wave function $U(\lambda)$, we can easily show that the highest power of the momentum in the cloud spectrum including the higher order effects in perturbation calculation is same as that of the lowest order effect, which gives the logarithmic energy spectrum for the produced mesons, for the nuclear interaction of the first kind. The detailed analysis for the energy spectrum of the produced mesons in the cosmic ray underground is developed by S. Ogawa¹⁸⁾.

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The Mass Variation with Velocity in the Bopp's Unitary Field Theory, II

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In the Bopp type non-local field theory the total energy and momentum of the field are calculated, starting from the expression for the energy and momentum tensor recently derived by Y. Ono and it is shown that the results is the same as that obtained in the previous paper on the same subject, starting from the energy and momentum tensor derived by Bopp. Therefore nothing is modified in the condition to ensure the four vector variation of the total energy and momentum of the field, namely the condition for the vanishment of the self-stress. Further it is shown that the two expressions for the energy and momentum tensor are essentially equivalent to each other. As an interesting example we examine a non-local field theory proposed by Pais and Uhlenbeck and show that the condition for the vanishment of the self-stress and the regulator condition for the finiteness of the self-energy are satisfied simultaneously in this case.

§ 1. Introduction

In the unitary field theory, the inertia, which we observe as mass of a particle, should be entirely due to the inertia of the field which, produced by the particle around itself, has a certain amount of energy and momentum and, therefore, also a certain inertia. The variation of mass with respect to the velocity of the particle is described by the well-known relativistic mechanical formulae. The reason why unitary field theoretical standpoint cannot be applicable to the current field theory is that the energy and momentum of the field produced by a certain source do not form a four vector in general. In our previous investigations on the same subject^{1)*} we remarked that in the usual linear field theory the energy and momentum of the field produced by a certain source do not in general form a four vector, it is, however, permitted to form a four vector if a certain condition is satisfied. To this particular example belongs the old Bopp's proposal²⁾, where he tried to introduce the next higher derivatives into the Lagrangian. In I we obtained the condition which assures the correct behavior of the energy and momentum of the field in the more general non-local proposal by Bopp³⁾. These calculations were based entirely upon the expression derived by Bopp⁴⁾ and Heisenberg⁵⁾ for the energy-momentum tensor of the Bopp-type non-local field. However, as was discussed in I, the above expression has many unsatisfactory points and recently Y. Ōno⁶⁾ has obtained the most perfect expression for the energy and momentum tensor of the Bopp-type non local field.

We examine in this paper how the above condition is modified by the Ōno's revision

* This preceding paper is the first part of the present one and will be referred to as I.

of the energy-momentum tensor and the differences of the two expressions. As a result of the present investigation it is shown that both of them give just the same value for the total energy and momentum of the field and the required condition therefore is not modified at all. Furthermore it is able to bring the expression given by Bopp into a more convenient form which makes it easy to compare the above two expressions to reveal the difficult points of the Bopp's expression. From these comparisons it can be said that the Bopp's expression is right in essence, but it must be improved in fine respects as is discussed in detail below.

In Section 4 we consider the non-local field theory proposed by Pais and Uhlenbeck⁷⁾, which is the most general type of mixed field theory on the realistic standpoint. In this case the so called "regulator condition" is automatically satisfied, so that the finiteness of the self-energy is ensured. It was interesting for us to examine whether the required condition is satisfied in this case. As is shown in the Appendix, the condition for the correct variation of the total energy and momentum of the field with velocity of the particle is identically satisfied in the Pais and Uhlenbeck case. This is one of the most important conclusions obtained in this paper. Therefore we can say that there remains some hope to generalize the present mixed field theory without violating the finiteness of the self energy and the correct variation of the energy and momentum of the field.

In the final section we remark the same condition is directly derived from the usual condition for the vanishment of the self-stress.

§ 2. Calculation of the total energy and momentum of the field and the derivation of the requisite condition

We use the same notations as was used in I. The modified Lagrangian in the Bopp's non-local field theory is given by

$$\bar{L} = -1/16\pi \int f_{\alpha\beta}(x) \varepsilon(x-x') f_{\alpha\beta}(x') dx dx' + 1/c \int \varphi_\alpha(x) S_\alpha(x) dx, \quad (1)$$

where $\varepsilon(x)$ is the characteristic Lorentz invariant "*fernwirkungsfunktion*." The Bopp's expression for the energy and momentum tensor of the above Bopp-type non-local field is

$$T_{\alpha\beta}(x) = 1/(2\pi)^4 \int \tilde{T}_{\alpha\beta}(k) e^{-ik_\gamma x_\gamma} dk, \quad (2)$$

where

$$\begin{aligned} \tilde{T}_{\alpha\beta}(k) = & \frac{1}{256\pi^5} \int (k_\alpha - 2k'_\alpha) (k_\beta - 2k'_\beta) \frac{(k-k')^2 \tilde{\varepsilon}(k-k') - k'^2 \tilde{\varepsilon}(k')}{(k-k')^2 - k'^2} \tilde{\varphi}_\gamma(k') \tilde{\varphi}_\gamma(k-k') dk' \\ & - \frac{1}{32\pi^4 c} \partial_{\alpha\beta} \int \tilde{\varphi}_\gamma(k') \tilde{S}_\gamma(k-k') dk' + \frac{1}{16\pi^4 c} \int \tilde{\varphi}_\alpha(k') \tilde{S}_\beta(k-k') dk' \end{aligned} \quad (3)$$

and all quantities like $\tilde{A}(k)$ are the Fourier coefficients of the corresponding quantities $A(x)$, $\varphi_\alpha(x)$ field quantities and $S_\alpha(x)$ source functions of the field. Ono's revision of the above expression is

$$\tilde{T}_{\alpha\beta}(k) = \frac{1}{16\pi} \frac{1}{(2\pi)^4} \int dk' \left[-\tilde{f}_{\mu\nu}(k-k') \tilde{\varepsilon}(k') \tilde{f}_{\mu\nu}(k') \partial_{\alpha\beta} \right]$$

$$\begin{aligned}
& + 2 \tilde{\varepsilon}(k') \left\{ \tilde{f}_{\alpha\mu}(k-k') \tilde{f}_{\beta\mu}(k') + \tilde{f}_{\beta\mu}(k-k') \tilde{f}_{\alpha\mu}(k') \right\} \\
& + 2 \left\{ \tilde{f}_{\mu\nu}(k-k') \tilde{f}_{\mu\nu}(k') \left(-(k-k')_{\alpha} k'_{\beta} + \frac{1}{2} \delta_{\alpha\beta} k_{\gamma} (k-k')_{\gamma} \right) \right. \\
& + \tilde{f}_{\mu\nu}(k-k') \tilde{f}_{\beta\nu}(k') k_{\mu} (k-2k')_{\alpha} \\
& \left. + \tilde{f}_{\mu\nu}(k-k') \tilde{f}_{\alpha\nu}(k') k_{\mu} (k-2k')_{\beta} \right\} \frac{\tilde{\varepsilon}(k-k') - \tilde{\varepsilon}(k')}{(k-k')^2 - k^2} \Big]
\end{aligned} \quad (4)$$

with the same notations as above. The total energy E and the total momentum \mathbf{P} of the field are given by the equations:

$$E = - \int T_{44}(x) dV, \quad (5)$$

$$\mathbf{P} = - \frac{1}{ic} \int \vec{T}_{k4}(x) dV. \quad (k=1,2,3) \quad (6)$$

We now proceed the calculations of E and \mathbf{P} from the latter expression of the energy and momentum tensor.

First of all the equation of motion derived from the Lagrangian (1) is

$$\frac{\partial}{\partial x_{\beta}} \int \varepsilon(x-x') f_{\alpha\gamma}(x') dx' = \frac{4\pi}{c} S_{\alpha}(x') \quad (7)$$

or using the Lorentz condition $\partial\varphi_{\alpha}(x)/\partial x_{\alpha}=0$,

$$\square \int \varepsilon(x-x') \varphi_{\alpha}(x') dx' = - \frac{4\pi}{c} S_{\alpha}(x). \quad (8)$$

Transforming into momentum representation, one obtains from Eq. (8)

$$k_{\alpha}^2 \tilde{\varepsilon}(k) \tilde{\varphi}_{\alpha}(k) = \frac{4\pi}{c} \tilde{S}_{\alpha}(k), \quad (9)$$

from which it follows

$$\tilde{\varphi}_{\alpha}(k) = \frac{4\pi}{c} \frac{\tilde{S}_{\alpha}(k)}{k_{\beta}^2 \cdot \tilde{\varepsilon}(k)}. \quad (10)$$

For a point electron which has a constant velocity \mathbf{v} and was situated at the origin at time $t=0$, the four current $S_{\alpha}(x)$ takes the form

$$\tilde{S}_{\alpha}(x) = (i, ic\rho), \quad (11)$$

where

$$\rho(x, t) = e\delta(x-vt)$$

and

$$i(x, t) = ev\delta(x-vt),$$

or in momentum representation

$$\tilde{S}_{\alpha}(k) = 2\pi en_g \delta(kv - \omega), \quad (12)$$

where

$$u_\alpha = (v, ic), \quad k_\alpha = (k, i\omega/c).$$

If we integrate the both sides of the conservation law of energy and momentum of the field $\partial T_{\alpha\beta}/\partial x_\beta = -(1/c)f_{\alpha\beta}S_\beta$ over the whole field, we obtain the total Lorentz force acting on the point electron

$$-\frac{1}{c} \int f_{\alpha\beta} S_\beta dV, \quad (13)$$

whose vanishment is required by the conservation of the total energy and momentum of the field for this case and can be proven quite in the same way as in I. The symmetry requirement is now automatically satisfied in this case.

In calculating the total energy E and total momentum \mathbf{P} of the field from equations (2), (4), (5) and (6), it is most important to notice that the contributions to the total energy and momentum in the integrands of Eq. (2) come only from the Fourier coefficients at $k=0$. This can easily be seen from the definition of E and \mathbf{P} by the equations (5), (6) and the factor $\delta(\mathbf{k}\mathbf{v}-\omega)$ contained in $\tilde{S}_\alpha(k)$, as is discussed in detail in I. In order not only to perform the calculations unambiguously but also to clarify the differences of the two expressions for $T_{\alpha\beta}(k)$, it is indispensable to carry out the calculations always bearing this point in mind. Otherwise it is impossible to deduce a unique conclusion because the products of various delta-functions appear and they give ambiguous results depending upon the order of integration. Further it is also important to notice that the expression like $k_\alpha \tilde{\varphi}_\alpha(k)$ can be put equal to zero owing to the Lorentz condition $\partial\varphi_\alpha(x)/\partial x_\alpha = 0$, which was introduced before. Considering these points, the following calculations are permissible concerning the first term of Eq. (4):

$$\begin{aligned} & -\tilde{f}_{\mu\nu}(k-k')\tilde{\varepsilon}(k')\tilde{f}_{\mu\nu}(k') \\ & = ((k-k')_\mu\tilde{\varphi}_\nu(k-k') - (k-k')_\nu\tilde{\varphi}_\mu(k-k'))\tilde{\varepsilon}(k')(k'_\mu\tilde{\varphi}_\nu(k') - k'_\nu\tilde{\varphi}_\mu(k')) \\ & = (k-k')_\mu k'_\mu \tilde{\varphi}_\nu(k-k')\tilde{\varepsilon}(k')\tilde{\varphi}_\nu(k') + (k-k')_\nu k'_\nu \tilde{\varphi}_\mu(k-k')\tilde{\varepsilon}(k')\tilde{\varphi}_\mu(k') \quad (14) \\ & = -2(k-k')_\mu^2 \tilde{\varphi}_\nu(k-k')\tilde{\varepsilon}(k-k')\tilde{\varphi}_\nu(k') = -\frac{8\pi}{c}\tilde{\varphi}_\nu(k')\tilde{S}_\nu(k-k'). \end{aligned}$$

Above calculations are rigorous when the above expression is multiplied by the four dimensional delta function $\delta(k_\alpha)$, which is certainly realized in this case, as is discussed in detail in I. Therefore the first term of the integrand of Eq. (4) becomes

$$-\frac{1}{32\pi^4 c} \partial_{\alpha\beta} \tilde{\varphi}_\tau(k') \tilde{S}_\tau(k-k'), \quad (15)$$

which is just the same as the second term of the old expression (3). The contributions to the total energy and momentum of the field arising from the first term turn out to be the same as the corresponding parts in I and are given by Eqs. (36) and (37) of I.

Concerning the second form of Eq. (4), it can first be seen that both terms give the same contributions because of the factor $\delta(k_\alpha)$. Thus the second term of the integrand of Eq. (4) becomes

$$\begin{aligned}
& 4\tilde{\varepsilon}(k') \tilde{f}_{\alpha\mu}(k-k') \tilde{f}_{\nu\mu}(k') / 16\pi(2\pi)^4 \\
&= -4\tilde{\varepsilon}(k') \left\{ (k-k')_{\alpha} \tilde{\varphi}_{\mu}(k-k') - (k-k')_{\mu} \tilde{\varphi}_{\alpha}(k-k') \right\} \times \\
&\quad \times \left\{ k_{\beta}' \tilde{\varphi}_{\mu}(k') - k_{\mu}' \tilde{\varphi}_{\beta}(k') \right\} / 16\pi(2\pi)^4 \\
&= -4\tilde{\varepsilon}(k') \left\{ (k-k')_{\alpha} k_{\beta}' \tilde{\varphi}_{\mu}(k-k') \tilde{\varphi}_{\mu}(k') + \right. \\
&\quad \left. + (k-k')_{\mu} k_{\mu}' \tilde{\varphi}_{\alpha}(k-k') \tilde{\varphi}_{\beta}(k') \right\} / 16\pi(2\pi)^4 \quad (16) \\
&= \frac{1}{256\pi^5} 4k_{\alpha}' k_{\beta}' \tilde{\varepsilon}(k') \tilde{\varphi}_{\mu}(k') \tilde{\varphi}_{\mu}(k-k') + \frac{1}{16\pi^4 c} \tilde{\varphi}_{\alpha}(k') \tilde{S}_{\beta}(k-k'),
\end{aligned}$$

when in addition to the Lorentz condition use is made of Eq. (10) as well as the even nature of $\tilde{\varepsilon}(k)$, $\tilde{S}(k)$ and $\tilde{\varphi}(k)$, and the delta function $\delta(k_{\alpha})$ contained in the integrand of Eqs. (5) and (6) is again taken into account. The latter term of Eq. (16) is quite the same as the third term of Eq. (3) and the contributions to the total energy and momentum arising from this term are given by Eqs. (33) and (34) in I. The first term in Eq. (16) will be considered in the following paragraph together with the discussions of the remaining part in the integrand of Eq. (4).

This remaining term contains the characteristic factor

$$\{\tilde{\varepsilon}(k-k') - \tilde{\varepsilon}(k')\} / \{(k-k')^2 - k'^2\},$$

which must be considered first. When we put $k_{\alpha}=0$ owing to the factor $\delta(k_{\alpha})$, both the numerator and denominator of the above factor vanish simultaneously. We must, therefore evaluate the limiting value of this factor by expanding $\tilde{\varepsilon}(k-k')$ in Taylor series as in I:

$$\begin{aligned}
\tilde{\varepsilon}(k-k') &= \tilde{\varepsilon}(k' - k) = f(k_{\alpha}'^2 - 2k_{\alpha}' k_{\alpha} + k_{\alpha}^2) \\
&= f(k_{\alpha}'^2) - 2k_{\alpha}' k_{\alpha} f'(k_{\alpha}'^2) + \dots \\
&= \tilde{\varepsilon}(k') - 2k_{\alpha}' k_{\alpha} f'(k_{\alpha}'^2) + \dots,
\end{aligned} \quad (17)$$

where $f(k_{\alpha}'^2)$ is $\tilde{\varepsilon}(k')$ itself considered as a function of $k_{\alpha}'^2$ instead of $\sqrt{k_{\alpha}'^2}$ and $f'(k_{\alpha}'^2)$ is the derivative of $f(k_{\alpha}'^2)$ with respect to its argument $k_{\alpha}'^2$. Thus we obtain

$$\lim_{k_{\alpha} \rightarrow 0} \frac{\tilde{\varepsilon}(k-k') - \tilde{\varepsilon}(k')}{(k-k')^2 - k'^2} = f'(k_{\alpha}'^2). \quad (18)$$

Owing to the delta function factor $\delta(k_{\alpha})$, those parts which are multiplied by k_{α} do not give any contributions to E and \mathbf{P} . Thus, dropping out these unimportant parts, the remaining term of the integrand of Eq. (4) becomes

$$\begin{aligned}
& \frac{1}{128\pi^5} k_{\alpha}' k_{\beta}' \tilde{f}_{\mu\nu}(k-k') \tilde{f}_{\mu\nu}(k') f'(k_{\alpha}'^2) \\
&= \frac{1}{256\pi^5} 4k_{\alpha}' k_{\beta}' k_{\mu}'^2 \tilde{\varphi}_{\nu}(k-k') \tilde{\varphi}_{\nu}(k') f'(k_{\alpha}'^2),
\end{aligned} \quad (19)$$

using Eq. (14). Adding the first term of Eq. (16) to the above expression, we finally obtain

$$-\frac{1}{256\pi^5} 4k'_\alpha k'_\beta \tilde{\varphi}_\nu(k-k') \tilde{\varphi}_\nu(k') \left\{ \tilde{\varepsilon}(k') + k'^{\prime 2}_\mu f'(k'^{\prime 2}_\alpha) \right\}, \quad (20)$$

which is just the first term of the integrand of Eq. (3), if we drop out the terms multiplied by k_α and use the following limiting value

$$\lim_{k_\alpha \rightarrow 0} \frac{(k-k')^2 \tilde{\varepsilon}(k-k') - k'^2 \tilde{\varepsilon}(k)}{(k-k')^2 - k'^2} = \tilde{\varepsilon}(k') + k'^{\prime 2}_\mu f'(k'^{\prime 2}_\alpha), \quad (21)$$

as was calculated by Eqs. (38) and (39) in I.

Thus we can see that the values of the total energy and momentum of the field calculated from the energy and momentum tensor (4) is quite the same as that calculated from Eq. (3). Therefore the requisite condition to assure the four vector behavior of the total energy and momentum of the field is the same as that obtained in I and is given according to Eq. (51) in I by

$$2 \int_0^\infty \frac{\tilde{\varepsilon}(K) + K^2 f'(K^2)}{(\tilde{\varepsilon}(K))^2} dK = 3 \int_0^\infty \frac{dK}{\tilde{\varepsilon}(K)}. \quad (22)$$

If the above condition holds, the total energy E and total momentum \mathbf{P} are given according to Eqs. (52) and (53) in I by

$$E = \frac{1}{2} e \varphi(o) / \sqrt{1 - \beta^2} \quad (23)$$

and

$$\mathbf{P} = \frac{1}{2} e \mathbf{v} \varphi(o) / c^2 \sqrt{1 - \beta^2}, \quad (24)$$

where

$$\varphi(o) = \frac{2e}{\pi} \int_0^\infty \frac{dK}{\tilde{\varepsilon}(K)} \quad \text{and} \quad \beta^2 = v^2/c^2. \quad (25)$$

§ 3. The comparison between the two expressions for $\tilde{T}_{\alpha\beta}(k)$

From the calculations in § 2, it is shown that the two expressions for $\tilde{T}_{\alpha\beta}(k)$ or Eqs. (3) and (4) give the same total energy and momentum of the field. Therefore it is verified that the Bopp's expression (3) for the energy-momentum tensor gives the exact total energy and momentum of the field. In this meaning we can say that the Bopp's expression is correct in essence. In the following we discuss the three difficult points concerning the Bopp's expression (3):

- (i) It depends explicitly upon source functions $\tilde{S}_\alpha(k)$.
- (ii) It does not tend to the usual one in the limit of $\tilde{\varepsilon}(k)=1$ or $\varepsilon(x)=\delta(x)$.
- (iii) It is neither symmetric nor gauge invariant.

Concerning the first point we must put $\tilde{S}_\alpha(k) = c k_\alpha^2 \tilde{\varepsilon}(k) \tilde{\varphi}_\alpha(k) / 4\pi$ according to Eq. (10) in the integrand of Eq. (3) and eliminate $\tilde{S}_\alpha(k)$ entirely. The resulting expression must be considered as the Fourier coefficient of the energy and momentum tensor of the field. Then the second term of the integrand of Eq. (3) becomes

$$-\frac{1}{8\pi} \frac{1}{(2\pi)^4} \partial_{\alpha\beta} (k-k')_{\mu}^2 \tilde{\varepsilon}(k-k') \tilde{\varphi}_{\nu}(k') \tilde{\varphi}_{\nu}(k-k'), \quad (26)$$

which must be replaced according to Eq. (14) by

$$-\frac{1}{16\pi} \frac{1}{(2\pi)^4} \partial_{\alpha\beta} \tilde{f}_{\mu\nu}(k-k') \tilde{\varepsilon}(k') \tilde{f}_{\mu\nu}(k'), \quad (27)$$

which is the first term of Eq. (4) and is free from the defects (ii) and (iii). Indeed it was shown in § 2 that Eqs. (26) and (27) give the same contributions to E and \mathbf{P} .

In order to see the behavior in the limiting case of $\tilde{\varepsilon}(k)=1$ or $\varepsilon(x)=\delta(x)$, we must modify the characteristic factor in the following way:

$$\frac{(k-k')^2 \tilde{\varepsilon}(k-k') - k'^2 \tilde{\varepsilon}(k')}{(k-k')^2 - k'^2} = \tilde{\varepsilon}(k') + (k-k')^2 \frac{\tilde{\varepsilon}(k-k') - \tilde{\varepsilon}(k')}{(k-k')^2 - k'^2}, \quad (28)$$

where the first term remains in the Maxwell limit and the second term, which vanishes in the Maxwell limit, has the same characteristic factor as appears in the integrand of Eq. (4). Thus the sum of the first term, in which the characteristic factor is simply replaced by the first term of Eq. (28), and the third term, in which $\tilde{S}_{\alpha}(k)$ is replaced by $\tilde{\varphi}_{\alpha}(k)$, of the integrand of Eq. (3) becomes

$$\frac{1}{16\pi} \frac{1}{(2\pi)^4} \left\{ (k_{\alpha} - 2k'_{\alpha})(k_{\beta} - 2k'_{\beta}) \tilde{\varepsilon}(k') \tilde{\varphi}_{\tau}(k') \tilde{\varphi}_{\tau}(k-k') + (k-k')_{\tau}^2 (\tilde{\varepsilon}(k-k') \tilde{\varphi}_{\alpha}(k') \tilde{\varphi}_{\beta}(k-k')) \right\}, \quad (29)$$

which must be replaced according to Eq. (16) by

$$\frac{1}{16\pi} \frac{1}{(2\pi)^4} 2\tilde{\varepsilon}(k') \left\{ \tilde{f}_{\alpha\mu}(k-k') \tilde{f}_{\beta\mu}(k') + \tilde{f}_{\beta\mu}(k-k') \tilde{f}_{\alpha\mu}(k') \right\}, \quad (30)$$

which is the second term of Eq. (4) and free from other above mentioned defects. It is shown in § 2 that above two expressions (29) and (30) do give the same contributions to E and \mathbf{P} .

The remaining term of the integrand of Eq. (3)

$$\frac{1}{16\pi} \frac{1}{(2\pi)^4} (k_{\alpha} - 2k'_{\alpha})(k_{\beta} - 2k'_{\beta})(k-k')^2 \tilde{\varphi}_{\tau}(k') \tilde{\varphi}_{\tau}(k-k') \times \\ \left\{ \tilde{\varepsilon}(k-k') - \tilde{\varepsilon}(k') \right\} / \left\{ (k-k')^2 - k'^2 \right\},$$

must correspond to the last term of Eq. (4) and both of them do really give the same contributions to E and \mathbf{P} as was shown in § 2.

Thus the correspondence of the two expressions (3) and (4) is clarified and it is clear how the Bopp's expression must be modified and supplemented.

§ 4. Application to Pais-Uhlenbeck type non-local field

In order to see the significance of the condition (22), we consider a case in which according to Pais and Uhlenbeck⁷⁾ we put

$$\varepsilon(x) = \prod_{i=1}^N (1 - \square/\mu_i^2) \delta(x), \quad (31)$$

where μ_i^2 's are constants and we assume they are all real and distinct. From Eq. (31) it follows that

$$\tilde{\varepsilon}(k) = \int \varepsilon(x) e^{ikx} dx = \prod_{i=0}^N (1 + k^2/\mu_i^2). \quad (32)$$

Therefore,

$$\begin{aligned} f'(K^2) &= (\partial/\partial K^2) \prod_{i=1}^N (1 + K^2/\mu_i^2) \\ K^2 f'(K^2) &= \sum_{k=1}^N (K^2/\mu_k^2) \prod_{j \neq k} (1 + K^2/\mu_j^2) = \prod_{i=1}^N (1 + K^2/\mu_i^2) \sum_{k=1}^N \frac{(K^2/\mu_k^2)}{(1 + K^2/\mu_k^2)}. \end{aligned} \quad (33)$$

The condition (22) can be rewritten as follows:

$$2 \int_0^\infty \frac{K^2 f'(K^2)}{(\tilde{\varepsilon}(K))^2} dK = \int_0^\infty \frac{dK}{\tilde{\varepsilon}(K)}. \quad (34)$$

Using the following partial fraction decomposition of $1/\tilde{\varepsilon}(K)$,

$$\frac{1}{\tilde{\varepsilon}(K)} = \frac{1}{f(K^2)} = \frac{1}{\prod_{i=1}^N (1 + K^2/\mu_i^2)} = \sum_k \frac{\eta_k \mu_k^2}{1 + K^2/\mu_k^2}, \quad (35)$$

where

$$\eta_k = 1/\{\mu_k^4 f'(-\mu_k^2)\}, \quad (36)$$

the integral of the right hand side of Eq. (34) becomes

$$\int_0^\infty \frac{dK}{\tilde{\varepsilon}(K)} = \sum_{k=1}^N \int_0^\infty \frac{\eta_k \mu_k^2}{1 + K^2/\mu_k^2} dK = \frac{\pi}{2} \sum_{k=1}^N \eta_k \mu_k^3. \quad (37)$$

On the other hand the integral on the left hand side of Eq. (34) can be calculated as follows:

$$\begin{aligned} \int_0^\infty \frac{K^2 f'(K^2)}{(\tilde{\varepsilon}(K))^2} dK &= \int_0^\infty \frac{1}{\tilde{\varepsilon}(K)} \sum_{j=1}^N \frac{K^2/\mu_j^2}{1 + K^2/\mu_j^2} dK \\ &= \sum_{k=1}^N \int_0^\infty \frac{\eta_k K^2}{(1 + K^2/\mu_k^2)^2} dK + \sum_{k=1}^N \sum_{j \neq k} \int_0^\infty \frac{\eta_k \mu_k^2 K^2/\mu_j^2}{(1 + K^2/\mu_k^2)(1 + K^2/\mu_j^2)} dK \\ &= \frac{\pi}{4} \sum_{k=1}^N \eta_k \mu_k^3 + \frac{\pi}{2} \sum_{k=1}^N \sum_{j \neq k} \eta_k \frac{\mu_k^4}{\mu_k + \mu_j}. \end{aligned} \quad (38)$$

The second term of the right hand side of Eq. (38) can be proven to vanish identically, as is shown in detail in the Appendix. Thus the condition (22) is satisfied in this case. In the case of Pais-Uhlenbeck type non-local field, it is well known that the so-called "regulator condition" is automatically satisfied; thus theories of this type automatically lead to the finiteness of the self-energy. In the above calculations this corresponds to the finiteness of the integral appearing on the right hand side of Eq. (25), which is calculated and is proven to be in fact finite according to Eq. (37).

Thus it is shown that the condition ensuring the four vector character of the total energy and momentum of the field and the regulator condition ensuring the finiteness of the self-energy are both satisfied at the same time in the Pais-Uhlenbeck type non-local field theory. In this respect the mixed field theory can be considered to be satisfactory. Also it can be said that the Bopp-type non-local field theory is quite general, so that there remains some possibility to find out functional forms for the characteristic non-local function $\varepsilon(x)$ which satisfies the above mentioned two conditions and is in addition free from the difficult points of the present mixed field theory.

§ 5. Concluding remarks

According to the general transformation properties of energy and momentum tensor, the total energy and momentum of the field can be expressed in terms of the corresponding quantities in the rest system of the particle :

$$E = - \int T_{44} dV = - \frac{\int \dot{T}_{44} d\dot{V} - \beta^2 \int \dot{T}_{11} d\dot{V}}{\sqrt{1 - \beta^2}}$$

$$\mathbf{P} = \frac{1}{ic} \int \vec{T}_{k4} dV = - \frac{\mathbf{v}}{c^2} \frac{\int \dot{T}_{44} d\dot{V} - \int \dot{T}_{11} d\dot{V}}{\sqrt{1 - \beta^2}}, \quad (39)$$

where \circ refers to the rest system of the particle. From Eq. (39) one can readily see that in order to have the four vector variation of the total energy and momentum of the field the quantity $\int \dot{T}_{11} d\dot{V}$, which is called "self-stress", must vanish, viz.,

$$\int \dot{T}_{11} d\dot{V} = 0. \quad (40)$$

In the rest system of the particle the calculations are much simplified, so that it is easy to deduce the condition (22) directly from Eq. (40). Therefore the condition obtained before is nothing but the condition for the vanishment of the self-stress.

As a conclusion, in the Pais-Uhlenbeck type non-local field theory, the condition for the vanishment of the self-stress and the condition for the finiteness of the self-energy are satisfied at the same time. As the Bopp-type non-local field theory is more general, it may be permitted to generalize the mixed field theory without violating the fulfilment of the above two requirements.

Appendix

It is shown here that the second term on the right hand side of Eq. (38) vanishes. Through Eqs. (36) and (33) what is to be demonstrated leads to

$$\sum_{k=1}^N \sum_{j \neq k} \frac{1}{\mu_k + \mu_j} \frac{1}{\prod_{i \neq k} (\mu_i^2 - \mu_k^2)} = 0. \quad (\text{A. 1})$$

For this verification we prefer to employ the following identity :

$$\sum_{l=1}^N \frac{1}{\prod_{i \neq l} (\mu_i^2 - \mu_l^2)} = 0, \quad (\text{A. 2})$$

the establishment of which we shall defer until the close of this appendix. On differentiating Eq. (A. 2) with respect to μ_k we get

$$\sum_{j \neq k} \frac{\mu_k}{\mu_j^2 - \mu_k^2} \left[-\frac{1}{\prod_{i \neq j} (\mu_i^2 - \mu_j^2)} + \frac{1}{\prod_{i \neq k} (\mu_i^2 - \mu_k^2)} \right] = 0, \quad (\text{A. 3})$$

and likewise the differentiation with respect to μ_j ($j \neq k$) gives

$$\sum_{k \neq j} \frac{\mu_j}{\mu_k^2 - \mu_j^2} \left[-\frac{1}{\prod_{i \neq k} (\mu_i^2 - \mu_j^2)} + \frac{1}{\prod_{i \neq j} (\mu_i^2 - \mu_k^2)} \right] = 0. \quad (\text{A. 4})$$

From Eqs. (A. 3) and (A. 4) we get

$$\sum_{k=1}^N \sum_{j \neq k} \frac{1}{\mu_k + \mu_j} \left[\frac{1}{\prod_{i \neq k} (\mu_i^2 - \mu_k^2)} + \frac{1}{\prod_{i \neq j} (\mu_i^2 - \mu_j^2)} \right] = 0. \quad (\text{A. 5})$$

The left hand side of Eq. (A. 5) is the same as the addition of the left hand side of Eq. (A. 1) to its equivalent expression:

$$\sum_{k=1}^N \sum_{j \neq k} 1 / \{ (\mu_k + \mu_j) \prod_{i \neq j} (\mu_i^2 - \mu_j^2) \} = 0. \quad (\text{A. 6})$$

Thus Eq. (A. 1) is verified.

Finally we shall briefly refer to the verification of Eq. (A. 2). This can easily be carried out by performing formally a partial fraction decomposition of $1 / \prod_{i \neq l} (\mu_i^2 - \mu_l^2)$ as follows:

$$1 / \prod_{i \neq l} (\mu_i^2 - \mu_l^2) = \sum_{j \neq l} a_j / (\mu_j^2 - \mu_l^2), \quad (\text{A. 7})$$

where a_j 's ($j \neq l$) are constants independent of μ_l^2 , and it follows that

$$a_j = 1 / \prod_{i \neq l, j} (\mu_i^2 - \mu_j^2) \quad (\text{A. 8})$$

Inserting a_j 's into Eq. (A. 7), Eq. (A. 2) is reproduced after transposition of the right hand side to the left.

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The Capture Process of π Mesons by Deuterons

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Capture process of positive π mesons by deuterons are theoretically treated under the assumption that mesons are of pseudoscalar type and have pseudovector coupling. We have also taken into consideration the forces in the final triplet P -state of p - p system and investigated their influence to the final results. The calculation shows that if one assumes strong repulsion in P -state, better agreement with experiments is to be obtained.

§ 1. Introduction

The experiment on the capture processes of π mesons by various nuclei have widely been made by many authors. Among those the capture of negative π mesons by deuterons, investigated early by Panofsky et al.,¹⁾ have revealed that there occur two competing processes, $\pi^- + d \rightarrow n + n$ and $\pi^- + d \rightarrow 2n + \gamma$ by the ratio of 2 : 1, while no evidence of $\pi^- + d \rightarrow 2n + \pi^0$ has been observed. This indicates that the possibility that π mesons are scalar particles must be excluded by the selection rules.¹⁾²⁾

Recently, R. Darbin et al.³⁾ have made the alternative experiments, $\pi + d \rightarrow p + p$, with more than 20 Mev mesons and investigated the angular distribution of emitted protons and the energy dependence of the process. The reaction and its inverse, $p + p \rightarrow \pi^+ + d$, are correlated by detailed balancing. In this balancing it has been shown that there exists the difference by the factor 3, whether one assumes π mesons to have spin 0 or spin 1. From this fact one could determine the spin of π mesons. According to the experiments on $p + p \rightarrow \pi^+ + d$ made by Cartwright et al.⁵⁾ it seems better to assume that π mesons have spin 0.

Thus we may presumably assume the spin of π mesons to be zero, then they must be pseudoscalar particles because the case to be scalar particles has been excluded by Panofsky's experiments. There remains now to determine whether they have pseudoscalar or pseudovector coupling with nucleons. The problem will be decided by scattering of π mesons by protons. The experiments show the strong increase of scattering probabilities with meson energies, which seems to prefer pseudoscalar coupling.

In this paper we are dealing with the capture process of π mesons by deuterons under the assumption that they are pseudoscalar and have pseudovector coupling with nucleons. The theoretical treatments of the problem have already been made by S. Tamor, W. Cheston, J. Gunn et al.⁶⁾ They assumed, however, that there exist no force in the final $3p$ state of p - p (or n - n) system, according to Serber's consideration of high energy n - p scattering. But in p - p scattering the situation will be somewhat different and whether

there exist forces or not are still undetermined. The purpose of this paper is to investigate the influence of the force on the probability.

§ 2. The process of $\pi^+ + d \rightarrow p + p$

The matrix element for the capture process by two-nucleon system is given in the first order approximation by

$$M = -\frac{1}{\sqrt{2}\omega} \frac{f}{\mu} \int \psi_f^* \left[\left\{ (\sigma^{(1)} \mathbf{k}) - \omega \rho_1^{(1)} \right\} \tau_+^{(1)} \exp(i\mathbf{k} \mathbf{r}_1) + \left\{ (\sigma^{(2)} \mathbf{k}) - \omega \rho_1^{(2)} \right\} \tau_+^{(2)} \exp(i\mathbf{k} \mathbf{r}_2) \right] \psi_i d\mathbf{r}$$

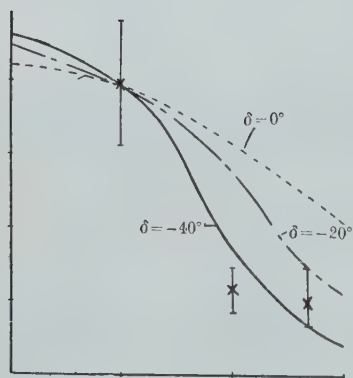


Fig. 1. Angular distribution of protons for $E_m = 53 \text{ Mev}$ in C.M.

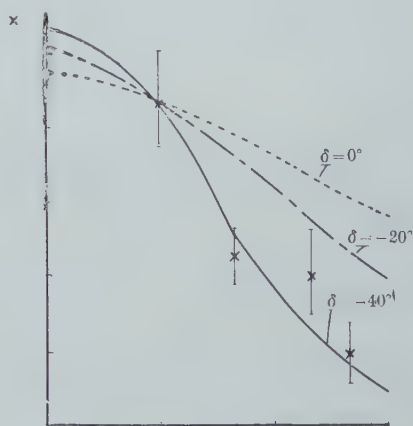


Fig. 2. Angular distribution of protons for $E_m = 40 \text{ Mev}$.

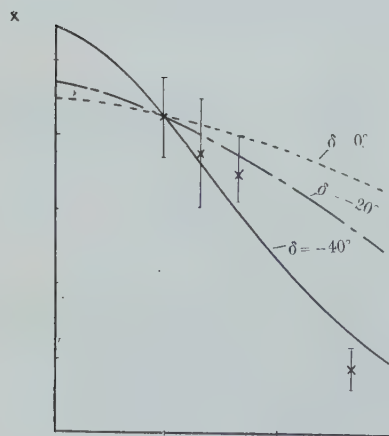


Fig. 3. Angular distribution of protons for $E_m = 25 \text{ Mev}$.



Fig. 4. Energy dependence of differential cross section at 30° .

where ψ_i denote the initial and final nucleon wave function, μ the meson mass, $\omega = \sqrt{\mu^2 + k^2}$ and τ_+ is the operator which changes the nucleon from neutron state to proton state. To obtain the values due to the matrix $\sigma\tau_+$, we take for the charge and spin functions the forms $\psi_i = \frac{1}{\sqrt{2}} \{a(1)b(2) - a(2)b(1)\} {}^3\chi_1^m(1,2) U_d(r)$, for the initial deuteron state; $\psi_f = a(1)a(2) {}^3\chi_1^m(1,2) U_e(r)$, for the final ${}^3(\text{odd})$ state; or $a(1)a(2) {}^1\chi_1^0(1,2) U_o(r)$, for the final ${}^1(\text{even})$ state. Here a, b denote the isotopic spin function, χ the spin functions. The transitions to the other states are forbidden by the matrix $\sigma\tau_+$. Substitution of the above expressions yields

$$M = -\frac{1}{\sqrt{2\omega}} \frac{f}{\mu} \frac{1}{\sqrt{2}} \sum_{f,m} \left[\chi_f^m U_f^*(r) \left[-\mathbf{k} \cdot \left((\sigma^{(1)} \exp(i\mathbf{k}r_1) - \sigma^{(2)} \exp(i\mathbf{k}r_2)) \right) \right. \right. \\ \left. \left. + \omega (\rho_1^{(1)} \exp(i\mathbf{k}r_1) - \rho_1^{(2)} \exp(i\mathbf{k}r_2)) \right] {}^3\chi_1^m U_d(r) \right] dv,$$

where the suffix f refers to the ${}^3(\text{odd})$ or ${}^1(\text{even})$ state.

We shall here deal with the mesons of 20~50 Mev. In these meson energies the recoil of the nucleon system after the meson capture can be neglected because of the small meson mass and we may simplify the calculation by assuming that the centre of gravity of the nucleon system is fixed throughout the process. Thus in the relative coordinates we can put $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{r}_1 = -\mathbf{r}_2 = \frac{\mathbf{r}}{2}$. In the non-relativistic approximation ρ_1 can be substituted by $i(\sigma\Delta)/M$, and then we have

$$M = \left(\frac{1}{4\omega} \right)^{1/2} \frac{f}{\mu} \sum_{f,m} \left[\chi_f^m U_f^*(r) \left[\mathbf{k} \cdot \left(\sigma^{(1)} \exp(i\mathbf{k}r/2) - \sigma^{(2)} \exp(-i\mathbf{k}r/2) \right) \right. \right. \\ \left. \left. + \frac{i\omega}{M} \left(\sigma^{(1)} \exp(i\mathbf{k}r/2) + \sigma^{(2)} \exp(-i\mathbf{k}r/2) \right) \cdot \Delta \right] {}^3\chi_1^m U_d(r) \right] dv.$$

The summation over m , gives

$$M = \left(\frac{1}{4\omega} \right)^{1/2} \frac{2if}{\mu} \int u_f^*(r) \left[k(\mathbf{k}r/2) - \frac{\omega}{M} \cos(\mathbf{k}r/2) \cdot \Delta_z \right] u_d(r) dv, \text{ for } {}^3S \rightarrow {}^3(\text{odd}), \\ = \left(\frac{1}{4\omega} \right)^{1/2} \frac{2f}{\mu} \int u_f^*(r) \left[k \cos(\mathbf{k}r/2) + \frac{\omega}{M} \sin(\mathbf{k}r/2) \cdot \Delta_z \right] u_d(r) dv, \\ \text{for } {}^3S \rightarrow {}^1(\text{even}),$$

as only the transitions $m=1 \rightarrow 1$ and $m=-1 \rightarrow -1$ for ${}^3S \rightarrow {}^3(\text{odd})$ and $m=0 \rightarrow 0$ for ${}^3S \rightarrow {}^1(\text{even})$ are allowed. Here we have taken k to coincide with z-axis of the coordinate system.

In the case where kinetic energies of mesons are less than 50 Mev, only S, P and D waves of the nucleon are effective to the process. Therefore we may confine our consideration to the following transitions, ${}^3S \rightarrow {}^3P$ and ${}^3S \rightarrow {}^1S, {}^1D$. Then the matrix element can be written as

$$M_1 = \left(\frac{1}{4\omega}\right)^{1/2} \frac{f}{\mu} (I_1 + I_2), \quad M_2 = \left(\frac{1}{4\omega}\right)^{1/2} \frac{f}{\mu} (I_3 + I_4 + I_5),$$

with

$$I_1 = ik^2 \frac{4\pi}{3} \cos \theta \int u_P(r) u_d(r) r^3 dr,$$

$$I_2 = \frac{2i\omega}{M} \cdot \frac{4\pi}{3} \int u_P(r) \frac{\partial}{\partial r} u_d(r) r^2 dr,$$

$$I_3 = 8\pi k \int u_S(r) u_d(r) r^2 dr,$$

$$I_4 = \frac{2\omega}{M} k \frac{4\pi}{3} \int u_S(r) \frac{\partial}{\partial r} u_d(r) r^3 dr,$$

$$I_5 = \frac{2\omega}{M} k \frac{\pi}{2} \cos \theta \int u_D(r) \frac{\partial}{\partial r} u_d(r) r^3 dr,$$

where the suffices represent the nucleon radial functions for S , P , D , states. Further we have taken $\cos(kr/2) \sim 1$, $\sin(kr/2) \sim kr/2$, which introduces about $2 \sim 3\%$ errors in the final results for meson energy of 50 Mev. In the above integrations I_1 is of the order of 0.001, independent of incident meson energy, and we may neglect it. For other quantities we perform the integration with the wave functions

$$u_d(r) = \frac{N}{\sqrt{4\pi}} \frac{e^{-\alpha r} - e^{-\beta r}}{r}, \quad u_S(r) = (1 - e^{-\eta r}) \sin pr/pr, \quad u_D(r) = 5 \cdot j_{5/2}(pr)/pr,$$

and

$$U_P(r) = 3 \cdot j_{5/2+\Delta}(pr)/pr, \quad \text{with} \quad \Delta = -\frac{2}{\pi} \delta,$$

δ being P -phase shift. When $\delta = 0$, $I_1 < I_2$; for $\delta > 0$, I_1 decreases with increasing δ , while I_2 increases; and for $\delta < 0$, I_1 increases and I_2 decreases as δ decreases. The contributions of I_3 , I_5 are small, compared to I_1 , I_2 . Therefore, the cross sections, angular distributions, energy dependence vary with the value of δ . For negative δ , I_1 becomes large and the angular distribution will come to like the shape of $\cos^2 \theta$. Further, as I_1 is proportional to k^2 , the cross section increases with meson energies.

The differential cross section is antisymmetrized with respect to the final protons and we have

$$\frac{d\sigma}{d\Omega} = \frac{f^2}{4\pi} \cdot \frac{\rho M}{\mu^2 k} \cdot N^2 \left(\frac{4\pi}{3}\right)^2 F(\theta).$$

The calculated $F(\theta)$ for various values of δ and incident meson energies E_m are listed in the Table. In Fig. 1, 2 and 3 are plotted the angular distributions of emitted

Table. Values of $F(\theta)$ for various values of phase shifts δ and meson energies E_m , in unit of 10^{-26} cm².

δ (in degree)	$E_m=25$ Mev	$E_m=40$ Mev	$E_m=53$ Mev
20°	0.0058 ($\text{Cos}^2\theta+5.4$)	0.0094 ($\text{Cos}^2\theta+2.5$)	0.0162 ($\text{Cos}^2\theta+1.8$)
0°	0.0072 ($\text{Cos}^2\theta+4.1$)	0.0121 ($\text{Cos}^2\theta+1.8$)	0.0198 ($\text{Cos}^2\theta+1.0$)
-20°	0.0167 ($\text{Cos}^2\theta+1.2$)	0.0306 ($\text{Cos}^2\theta+0.51$)	0.0432 ($\text{Cos}^2\theta+0.31$)
-40°	0.0306 ($\text{Cos}^2\theta+0.31$)	0.0594 ($\text{Cos}^2\theta+0.16$)	0.0918 ($\text{Cos}^2\theta+0.11$)

protons, which are normalized at 30°. The justification of this normalization is checked by the energy dependence in Fig. 4. where the values are normalized to $E_m=25$ Mev.

§ 3 Conclusion

From the figures we see that for zero or positive phase shifts the angular distributions will be too flat and the differential cross section does not increase sufficiently with incident meson energies, while for negative phase the agreement with experiments becomes better. This seems to contradict to the concept of charge independence of the nuclear forces. In the high energy $n\text{-}p$ scattering, the angular distribution of the scattered nucleon is symmetric around 90°. To explain this experimental facts, Serber has introduced the nuclear potential, which does not exert force in the P -state, because P -wave gives asymmetry around 90° by interfering with S -wave. However, there is another possible explanation. If the nuclear forces are negative and suitably strong for P -state, the interference will again vanish, because interference term contains the factor which is the cosine of the difference of S and P phase shifts. Whether we may extend the above results to the $n\text{-}p$ system, or not, can not be decided at present. However, if applied to the photodisintegration of the deuteron, we can have large cross section at high energy region, which agrees well with the recent experiments.

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On the Well-ordered S-matrix

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With the help of the relation between T-product and S-product given by Wick, a new formula for the S-matrix has been obtained, which expresses the S-matrix directly as S-product and from which several already known results can be derived much easily.

§ 1. Introduction

The evaluation of the S-matrix should be performed for the present in the form of power series expansion of the coupling constant, each term of which becomes finite, as proved by Dyson¹⁾, after the renormalizations of mass and charge are carried out. The convergence of the series, however, remains now to be proved. For this purpose, more detailed knowledge about the structure of the S-matrix—finite part of the S-matrix will be required. Therefore it is desirable to express the S-matrix in a simpler (closed) form and it is also desirable in order to make it possible to evaluate energy eigenvalues of the bound state from the S-matrix by means of analytic continuation and to find a clue to the future theory as well.

Recently Feynman²⁾ succeeded to write the S-matrix in a compact form, using his operational calculus. For calculating its matrix element, however, we must again transform it into a sum of the normal products, namely, into the usual power series. We will try from the beginning to express it as an S-product—a sum of ordered products.

§ 2. Well-ordering of the S-matrix

Recently Wick³⁾ has introduced algebraic operations named T-product and S-product which make the treatment of the S-matrix much easier. Using his T-product symbol, one can write the S-matrix as,

$$S = \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n T(H(x_1)H(x_2)\cdots H(x_n)), \quad (1)$$

where $H(x)$ is the Hamiltonian density of the system in consideration.

Since T-product is a linear operation, we can change the order of T-product operation and since $H(x)$ contains even numbers (including zero) of Fermi field variables in the case of physical interest, $H(x)$'s in the T-product commute each other. Therefore we obtain

$$S = T \left[\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left(\int_{-\infty}^{\infty} H(x) dx \right)^n \right]$$

or

$$S = T \left[e^{-i \int_{-\infty}^{\infty} H(x) dx} \right].$$

Considering the meaning of the T-product, this expression is in complete accordance with Feynman's result regarding the time as a disentangling parameter.

To fix the idea, we will confine ourselves to the quantum electrodynamics. Further we will consider the transformation function $S[\sigma]$ instead of the S-matrix. The S-matrix can be obtained immediately by letting $\sigma \rightarrow \infty$; $S = S[\infty]$.

The relation between T-product and S-product was given by Wick⁽³⁾:

$$\begin{aligned} T(UVW \cdots XYZ) = & UVW \cdots XYZ + :U \cdot V \cdot W \cdots XYZ : + \cdots \\ & + : UVW \cdots XY \cdot Z : + \cdots + : U \cdot V \cdots W \cdots X \cdots Y \cdot Z : \end{aligned} \quad (3)$$

where on the right hand side the summation is over all possible contractions with respect to $UVW \cdots XYZ$.

In the case of the quantum electrodynamics the Hamiltonian density is given by,

$$H_i(x) = -j_\mu(x) A_\mu(x), \quad (4)$$

where

$$j_\mu(x) = ie : \bar{\psi}(x) \gamma_\mu \psi(x) : \quad (5)$$

and then we can readily see that

$$\begin{aligned} S[\sigma] = & \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\sigma} dx_1 \cdots \int_{-\infty}^{\sigma} dx_n T(j_{\mu 1}(x_1) \cdots j_{\mu n}(x_n)) \times \\ & T(A_{\mu 1}(x_1) \cdots A_{\mu n}(x_n)). \end{aligned} \quad (6)$$

When the second T-product (with respect to A_μ) in the Eq. (6) is transformed into S-product, the terms which contain no contraction can be written as,

$$\sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\sigma} dx_1 \cdots \int_{-\infty}^{\sigma} dx_n T(j_{\mu 1}(x_1) \cdots j_{\mu n}(x_n)) : A_{\mu 1}(x_1) \cdots A_{\mu n}(x_n) :$$

or

$$S_A \left[\sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\sigma} dx_1 \cdots \int_{-\infty}^{\sigma} dx_n T(j_{\mu 1}(x_1) \cdots j_{\mu n}(x_n)) A_{\mu 1}(x_1) \cdots A_{\mu n}(x_n) \right]. \quad (7)$$

Here we have used a notation S_A instead of colons to express S-product only with respect to A_μ .

Because

$$A_\mu(x) A_\nu(y) = 1/2 \delta_{\mu\nu} D_F(x-y), \quad (8)$$

introducing the operator

$$\mathcal{A}[\sigma] = \frac{1}{2} \int_{-\infty}^{\sigma} dx \int_{-\infty}^{\sigma} dy D_F(x-y) \frac{\delta^2}{\delta A_{\mu}(x) \delta A_{\mu}(y)}, \quad (9)$$

we can write the terms which contain a contraction,

$$S_A \left[\sum_{n=0}^{\infty} \mathcal{A}[\sigma] \frac{i^n}{n!} \int_{-\infty}^{\sigma} dx_1 \cdots \int_{-\infty}^{\sigma} dx_n T(j_{\mu 1}(x_1) \cdots j_{\mu n}(x_n)) A_{\mu 1}(x_1) \cdots A_{\mu n}(x_n) \right], \quad (10)$$

and the terms containing two contractions,

$$S_A \left[\sum_{n=0}^{\infty} \frac{1}{2!} \mathcal{A}^2[\sigma] \frac{i^n}{n!} \int_{-\infty}^{\sigma} dx_1 \cdots \int_{-\infty}^{\sigma} dx_n T(j_{\mu 1}(x_1) \cdots j_{\mu n}(x_n)) A_{\mu 1}(x_1) \cdots A_{\mu n}(x_n) \right].$$

The factor $1/2!$ is necessary because the same terms with two contractions arise in two ways. In general the terms with r contractions can be written as,

$$S_A \left[\sum_{n=0}^{\infty} \frac{1}{r!} \mathcal{A}^r[\sigma] \frac{i^n}{n!} \int_{-\infty}^{\sigma} dx_1 \cdots \int_{-\infty}^{\sigma} dx_n T(j_{\mu 1}(x_1) \cdots j_{\mu n}(x_n)) A_{\mu 1}(x_1) \cdots A_{\mu n}(x_n) \right].$$

Finally they are collected to be

$$S_A \left[\sum_{n=0}^{\infty} \left(\sum_{r=0}^{\left[\frac{n}{2} \right]} \frac{1}{r!} \mathcal{A}^r[\sigma] \right) \frac{i^n}{n!} \int_{-\infty}^{\sigma} dx_1 \cdots \int_{-\infty}^{\sigma} dx_n \times \right. \\ \left. T(j_{\mu 1}(x_1) \cdots j_{\mu n}(x_n)) A_{\mu 1}(x_1) \cdots A_{\mu n}(x_n) \right],$$

where $\left[\frac{n}{2} \right] = \frac{n}{2}$ for even n and $\frac{n-1}{2}$ for odd n . We may replace this upper limit of the summation by infinity, because the alteration merely give rise to a vanishing effect. Therefore we obtain

$$S[\sigma] = S_A \left[e^{\mathcal{A}[\sigma]} \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\sigma} dx_1 \cdots \int_{-\infty}^{\sigma} dx_n T(j_{\mu 1}(x_1) \cdots j_{\mu n}(x_n)) \times \right. \\ \left. A_{\mu 1}(x_1) \cdots A_{\mu n}(x_n) \right]. \quad (11)$$

In this case the definition of the functional derivatives with respect to the operator A_{μ} exhibits no serious difficulty, for the functional derivatives always appear in the S-symbol.

Next, noting that

$$\psi_{\alpha}(x) \bar{\psi}_{\beta}(y) = \frac{1}{2} S_{F\alpha\beta}(x-y) \quad (12)$$

and defining the operator

$$\Sigma[\sigma] = \frac{1}{2} \int_{-\infty}^{\sigma} dx \int_{-\infty}^{\sigma} dy S_{F\alpha\beta}(x-y) \frac{\delta^2}{\delta \psi_{\alpha}(x) \delta \bar{\psi}_{\beta}(y)}, \quad (13)$$

we can treat the first T-product (with respect to the electron field) in the entirely similar manner with only exception of precaution to the sign. Combinations such as $\bar{\psi}(x)\psi(x)$ or $j_{\mu}(x)$ commute each other in the S-symbol not changing its sign. Hence we can

always arrange $\psi(x)$ and $\bar{\psi}(y)$ in such order as they stand; $\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y)$. After $\Sigma[\sigma]$ operates, this factor changes into $\bar{\psi}(x)S_F\psi(y)$ which again contains a $\bar{\psi}$ and a ψ and commute with j_μ or each other in the S-symbol, and so on. Thus we need only define that the factor arising by the differentiation with respect to $\psi(x)$ must be written always on the left side of the factor arising by the differentiation of $\bar{\psi}(y)$. The situation may be more explicitly manifested* by writing (13) as

$$\Sigma[\sigma] = \frac{1}{2} \int_{-\infty}^{\sigma} dx \int_{-\infty}^{\sigma} dy \frac{\delta}{\delta \psi(x)} S_F(x-y) \frac{\delta}{\delta \bar{\psi}(y)}. \quad (14)$$

Eventually $S[\sigma]$ can be expressed as an S-product**:

$$\begin{aligned} S[\sigma] &= : e^{\Sigma[\sigma]} e^{\Delta[\sigma]} \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[\int_{-\infty}^{\sigma} j_\mu(x) A_\mu(x) dx \right]^n : \\ &= : e^{\Sigma[\sigma]} e^{\Delta[\sigma]} e^{i \int_{-\infty}^{\sigma} j_\mu(x) A_\mu(x) dx} :. \end{aligned} \quad (15)$$

It is the main feature of $S[\sigma]$ written in the form (15) that every $A_\mu(x)$ in the Eq. (15) are commutable and $\bar{\psi}(x)$, $\psi(x)$ are anticommutable each other, respectively and commutation relations which should be appear are all involved in the operator $e^{\Sigma[\sigma]}$ and $e^{\Delta[\sigma]}$. Therefore it will be very opportune for the discussion of many problems about S-

matrix. The operator $e^{\Sigma[\sigma]}$ and $e^{\Delta[\sigma]}$ play the rôle to combine the fundamental graph (Fig. 1) representing $H_i(x) = -ij_\mu(x)A_\mu(x)$ in every possible ways.

We can easily shown that even if Hamiltonian density in consideration contains not only $H_i(x)$ (4) but the mass renormalization term,

$$H_s(x) = -\delta mc^2 : \bar{\psi}(x)\psi(x) :.$$



Fig. 1.

or the Gupta's term⁴⁾ or both, we can write similarly

$$S[\sigma] = : e^{\Sigma[\sigma]} e^{\Delta[\sigma]} e^{-i \int_{-\infty}^{\sigma} H(x) dx} : \quad (16)$$

But the latter cases need some remarks.

§ 3. Remarks on the Gupta's term

We can automatically take into account the effect of the charge renormalization by adding the Gupta's term¹⁾ to the Hamiltonian density $H(x)$. Since this term contains the time derivatives of the potential $A_\mu(x)$, a normal dependent term necessarily appears in the Hamiltonian. Further we would have to modify $\Delta[\sigma]$ because of the presence of $F_{\mu\nu}(x)$ in $H(x)$. If we use nevertheless the same expression for $\Delta[\sigma]$, the result obtained by

* The definition $\delta^2/\delta \psi(x)\delta \bar{\psi}(y) = -\delta^2/\delta \bar{\psi}(y)\delta \psi(x)$, may be also available.

** In the Eq. (15) the Eqs. of motion for the field variables should be taken in consideration after $e^{\Sigma[\sigma]}$ and $e^{\Delta[\sigma]}$ have already operated.

applying $\mathcal{A}[\sigma]$ can be summarised as follows : First we rewrite $F_{\rho\sigma}(x) = (\partial_{\mu\rho} \frac{\partial}{\partial x_\sigma} - \partial_{\mu\sigma} \frac{\partial}{\partial x_\rho}) A_\mu(x)$. Second when $\mathcal{A}[\sigma]$ is applied, $F_{\rho\sigma}(x)$ differentiated functionally with respect to $A_\mu(x)$ bequeathes $(\partial_{\mu\rho} \frac{\partial}{\partial x_\sigma} - \partial_{\mu\sigma} \frac{\partial}{\partial x_\rho})$ and this always operate to $D_F(x-y)$ contained in $\mathcal{A}[\sigma]$. For example,

$$\begin{aligned}
 \mathcal{A}[\sigma] \exp & \left[i \int_{-\infty}^{\infty} j_\mu(x) A_\mu(x) dx + \frac{if}{4} \int_{-\infty}^{\infty} F_{\rho\sigma}(x) F_{\rho\sigma}(x) dx \right] \\
 &= \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left\{ ij_\mu(x) + \frac{if}{2} F_{\rho\sigma}(x) \left(\partial_{\mu\rho} \frac{\partial}{\partial x_\sigma} - \partial_{\mu\sigma} \frac{\partial}{\partial x_\rho} \right) \right\} D_F(x-y) \times \\
 &\times \left\{ ij_\mu(y) + \frac{if}{2} \left(\partial_{\mu\lambda} \frac{\partial}{\partial y_\kappa} - \partial_{\mu\kappa} \frac{\partial}{\partial y_\lambda} \right) F_{\lambda\kappa}(y) \right\} \times \\
 &\times \exp \left[i \int_{-\infty}^{\infty} j_\mu(x) A_\mu(x) dx + \frac{if}{4} \int_{-\infty}^{\infty} F_{\rho\sigma}(x) F_{\rho\sigma}(x) dx \right] \\
 &= \left[-\frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy j_\mu(x) D_F(x-y) j_\mu(y) + \right. \\
 &\quad + f \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy j_\mu(x) \frac{\partial}{\partial y_\rho} D_F(x-y) F_{\rho\mu}(y) \\
 &\quad \left. - \frac{f^2}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy F_{\rho\mu}(x) \frac{\partial^2}{\partial x_\rho \partial y_\lambda} D_F(x-y) F_{\lambda\mu}(y) \right] \times \\
 &\times \exp \left[-i \int_{-\infty}^{\infty} j_\mu(x) A_\mu(x) dx + \frac{if}{4} \int_{-\infty}^{\infty} F_{\rho\sigma} F_{\rho\sigma} dx \right]. \tag{17}
 \end{aligned}$$

If \mathcal{A} 's no more operate from the left, the second and the third term in the first factor vanishes due to the Eq. of motion (cf. the foot note on p. 804). The result above mentioned amounts to the definitions,

$$A_\mu(x) F_{\rho\sigma}(y) = \left(\partial_{\mu\rho} \frac{\partial}{\partial y_\sigma} - \partial_{\mu\sigma} \frac{\partial}{\partial y_\rho} \right) D_F(x-y), \tag{18}$$

$$F_{\mu\nu}(x) F_{\rho\sigma}(y) = \left(\partial_{\alpha\mu} \frac{\partial}{\partial x_\nu} - \partial_{\alpha\nu} \frac{\partial}{\partial x_\mu} \right) \left(\partial_{\alpha\rho} \frac{\partial}{\partial y_\sigma} - \partial_{\alpha\sigma} \frac{\partial}{\partial y_\rho} \right) D_F(x-y). \tag{19}$$

The Eqs. (18) and (19) are correct except that to the Eq. (19) should be added a singular term containing a δ -function. The normal dependent term plays the part of compensating the singular expression. Eventually we need only retain the first term of the Gupta's term,

$$H_p = -\frac{1}{2} f F_{\mu\nu}(x) F_{\mu\nu}(x)$$

with $\mathcal{A}[\sigma]$ not modified.

§ 4. The diffusion equations

Now we replace the charge e by $\varepsilon = a b^{\frac{1}{2}} e$ and rewrite

$$\sqrt{b} A_{\mu}(x) \rightarrow A_{\mu}(x), \quad \sqrt{a} \bar{\psi}(x) \rightarrow \bar{\psi}(x), \quad \sqrt{a} \psi(x) \rightarrow \psi(x)$$

Then we have instead of (8) and (12)

$$A_{\mu}(x) A_{\nu}(y) = \frac{1}{2} b \delta_{\mu\nu} D_F(x-y) \quad (20)$$

$$\psi_{\alpha}(x) \bar{\psi}_{\beta}(y) = \frac{1}{2} a S_{F\alpha\beta}(x-y) \quad (21)$$

and instead of $S[\sigma]$,

$$S[a, b; \sigma] = : e^{a\bar{\Sigma}[\sigma]} e^{b\bar{\Delta}[\sigma]} e^{-i \int_{-\infty}^{\sigma} H(x) dx} : \quad (22)$$

Let $A_{\mu}(x)$ be an arbitrary c-number function and $\bar{\psi}(x)$, $\psi(x)$ be a set of an arbitrary c-number spinor functions and $\bar{\Delta}[\sigma]$ and $\bar{\Sigma}[\sigma]$ be the same operators as $\Delta[\sigma]$ and $\Sigma[\sigma]$ replaced A_{μ} and $\bar{\psi}$, ψ by A_{μ} and $\bar{\psi}$, ψ , and

$$\mathcal{S}[a, b; \sigma] = e^{a\bar{\Sigma}[\sigma]} e^{b\bar{\Delta}[\sigma]} e^{-i \int_{-\infty}^{\sigma} H(x) dx} \quad (23)$$

$S[a, b; \sigma]$ can be obtained from $\mathcal{S}[a, b; \sigma]$ by replacing A_{μ} , $\bar{\psi}$, ψ by A_{μ} , $\bar{\psi}$, ψ and taking its S-product. One can readily see that

$$\frac{\partial S[a, b; \sigma]}{\partial a} = \bar{\Sigma}[\sigma] S[a, b; \sigma], \quad (24)$$

$$\frac{\partial S[a, b; \sigma]}{\partial b} = \bar{\Delta}[\sigma] S[a, b; \sigma]. \quad (25)$$

It might seem paradoxical at first sight the S-matrix involving divergences (15) and the renormalized (16) satisfy the same diffusion equations. This is because the initial conditions at $a=0$ and $b=0$ differ in the both cases. The Eqs. (24) and (25) correspond to the diffusion equations obtained by Tomonaga and Fukuda²⁾, but in their work above consideration about the difference $S[a, b; \sigma]$ and between $\mathcal{S}[a, b; \sigma]$ was made only implicitly.

Next, since

$$\begin{aligned} \Delta[\sigma] e^{i \int_{-\infty}^{\sigma} j_{\mu}(x) A_{\mu}(x) dx} &= \\ &= -\frac{1}{2} \int_{-\infty}^{\sigma} dx \int_{-\infty}^{\sigma} dy j_{\mu}(x) D_F(x-y) j_{\mu}(y) e^{i \int_{-\infty}^{\sigma} j_{\mu}(x) A_{\mu}(x) dx}, \end{aligned} \quad (26)$$

operating $\Delta[\sigma]$ again, we obtain

$$\begin{aligned} \Delta^2[\sigma] e^{i \int_{-\infty}^{\sigma} j_{\mu}(x) A_{\mu}(x) dx} &= \\ &= \left\{ -\frac{1}{2} \int_{-\infty}^{\sigma} dx \int_{-\infty}^{\sigma} dy j_{\mu}(x) D_F(x-y) j_{\mu}(y) \right\}^2 e^{i \int_{-\infty}^{\sigma} j_{\mu}(x) A_{\mu}(x) dx}, \end{aligned}$$

and similarly in general,

$$\begin{aligned} \mathcal{A}^n[\sigma] e^{i \int_{-\infty}^{\sigma} j_{\mu}(x) A_{\mu}(x) dx} &= \\ &= \left\{ -\frac{1}{2} \int_{-\infty}^{\sigma} dx \int_{-\infty}^{\sigma} dy j_{\mu}(x) D_F(x-y) j_{\mu}(y) \right\}^n e^{i \int_{-\infty}^{\sigma} j_{\mu}(x) A_{\mu}(x) dx}. \end{aligned}$$

Thus we have

$$\begin{aligned} S[a, b; \sigma] &= : e^{a \Sigma[\sigma]} e^{-\frac{1}{2} \int_{-\infty}^{\sigma} dx \int_{-\infty}^{\sigma} dy j_{\mu}(x) D_F(x-y) j_{\mu}(y)} \times \\ &\quad e^{i \int_{-\infty}^{\sigma} j_{\mu}(x) A_{\mu}(x) dx} : \end{aligned} \quad (27)$$

In the Eq. (27) there is no more $\mathcal{A}[\sigma]$ contained, so that we may replace $A_{\mu}(x)$ by $\mathfrak{A}_{\mu}(x)$ which denotes only real field (and external field which we have hitherto omitted out of our consideration) and may be treated as if it was c-number function*. Returning back S-Product into T-Product we obtain

$$S[a, b; \sigma] = T(e^{-\frac{1}{2} b \int_{-\infty}^{\sigma} dx \int_{-\infty}^{\sigma} dy j_{\mu}(x) D_F(x-y) j_{\mu}(y)} e^{i \int_{-\infty}^{\sigma} j_{\mu}(x) \mathfrak{A}_{\mu}(x) dx}). \quad (28)$$

Regarding the time as a disentangling parameter and remembering that $D_F(x)$ is equal to $-\frac{i}{2\pi} \partial_+ (s_{1/2}^{\circ})$, we find that the Eq. (28) is identical to that obtained by Feynman ($\sigma \rightarrow \infty$). The same result was obtained by Glauber⁶⁾ but in the case of c-number $j_{\mu}(x)$. The Eq. (25) can easily afford to derive the Feynman's diffusion equation**

$$\frac{\partial S}{\partial b} = -\frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy D_F(x-y) \frac{\partial^2 S}{\partial \mathfrak{A}_{\mu}(x) \partial \mathfrak{A}_{\mu}(y)},$$

which is essentially the same as the Eq. (25).

In conclusion the author wishes to express his hearty thanks to Prof. S. Ozaki for his helpful discussions. He is also very grateful to the Press the Chubu Nippon for the financial aid.

* It operates to the state vector and creates or annihilates a photon.

** In the Eq. (26) S should be replaced by \mathcal{S} in the case of not using the Feynman's operator calculus

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Letters to the Editor

Note on the Non-Relativistic Limit of the Compton Scattering

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Thirring¹⁾ has shown the disappearance of the Thomson limit of the Compton scattering by the Fermi particles. In this note it will be shown that we can obtain the same conclusion from only the Lorentz covariance and the gauge invariance.

The matrix element of the Compton scattering (Fig. 1 (a)) in momentum representation may be expressed in virtue of the covariance as follows:

$$S_a = A_v^*(k_2) A_u(k_1) \bar{\psi}(p_2) \Gamma_{\mu\nu}^a \psi(p_1),$$

where

$$I_{\mu\nu}^a = U_{\mu\nu} + V_{\mu\nu} + V'_{\nu\mu} + W_{\mu\nu},$$

$$U_{\mu\nu} = C_1 \delta_{\mu\nu} + C_2 \gamma_\mu \gamma_\nu,$$

$$V_{\mu\nu} = D_1 p_{1\mu} \gamma_\nu + (p_1 \rightarrow p_2, k_1, k_2),$$

$$V'_{\nu\mu} = D'_1 p_{1\nu} \gamma_\mu + (p_1 \rightarrow p_2, k_1, k_2),$$

$$W_{\mu\nu} = E_1 p_{1\mu} p_{2\nu} + (p_1, p_2 \rightarrow p_1, p_2, k_1, k_2).$$

In the above expressions, the coefficients C_1 , D_1 , D'_1 and E_1 etc. are, respectively, the functions of the covariant expressions of p_1 ,

p_2 , k_1 , k_2 and the masses of the related particles. The symbol $(p_1 \rightarrow p_2, k_1, k_2)$ indicates the terms, exchanged the letter " p_1 " of the very previous term for the letter p_2 , k_1 or k_2 and changed the coefficient by the appropriate ones. The meanings of all the other symbols are as usual.

Now we consider the non-relativistic limit. Since $p_1 = -k_1$, $p_2 = -k_2$ in the centre of mass system, $p_{1\mu}$ etc. tend to zero in the limit of $k_1 \rightarrow 0$. (Since p_1 and p_2 are the energy-momentums of the Fermi particle, their time components do not vanish, if $x \approx 0$ (x : mass of the Fermi particle), in the limit of $k_1 \rightarrow 0$. But we now consider the scattering of the *photon*, hence it is not necessary to take into account of the components of μ , $\nu=4$.) Thus if the coefficients C_1 etc. were not singular in the limit of $k_1 \rightarrow 0$ ²⁾, then

$$I_{\mu\nu}, I'_{\nu\mu}, W_{\mu\nu} \rightarrow 0,$$

$$U_{\mu\nu} \rightarrow C_1^0 \delta_{\mu\nu} + C_2^0 \gamma_\mu \gamma_\nu \equiv U_{\mu\nu}^0,$$

where C_1^0 and C_2^0 are the limits of $k_1 \rightarrow 0$ and the finite c -numbers respectively.³⁾ Hence we obtain:

$$S_a \rightarrow A_v^*(k_2) A_u(k_1) \bar{\psi}(p_2) U_{\mu\nu}^0 \psi(p_1).$$

However, there is certainly a diagram (Fig. 1 (b)) in momentum representation

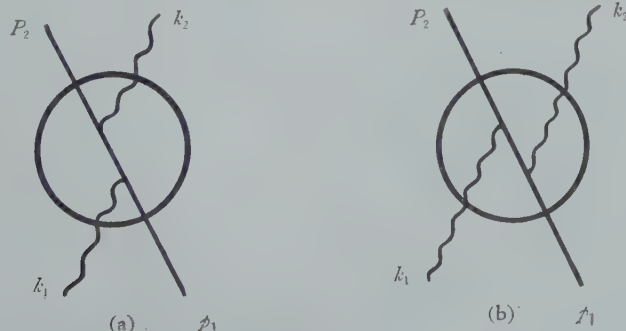


Fig. 1.

which has exchanged mutually for absorption and emission with respect to only the real photon in the diagram (a). Since for internal lines the real momentum transfer is zero in the case of $k_1 \rightarrow 0$,

$$S_b \rightarrow A_\nu^*(k_2) A_\mu(k_1) \bar{\psi}(p_2) U_{\nu\mu}^0 \psi(p_1),$$
$$U_{\nu\mu}^0 = C_1^0 \delta_{\mu\nu} + C_2^0 \gamma_\nu \gamma_\mu.$$

Thus we obtain the total matrix element in the limit of $k_1 \rightarrow 0$ as follows :

$$S = S_a + S_b \rightarrow A_\nu^*(k_2) A_\mu(k_1) \bar{\psi}(p_2) \\ \times (U_{\mu\nu}^0 + U_{\nu\mu}^0) \psi(p_1) \\ = 2(C_1^0 + C_2^0) A_\mu^*(k_2) A_\mu(k_1) \bar{\psi}(p_2) \psi(p_1),$$

which *must* vanish, considering with all the other diagrams,⁴⁾ because of its non-gauge invariance. Hence the Thomson limit becomes zero.³⁾

Actually we have ascertained the above conclusion for the Compton scattering by the nucleons, which interacts with the scalar (or pseudo scalar) meson field,⁵⁾ in the approximation $e^2 f^2$. (f ; coupling constant of the nucleon and the meson field.)

In conclusion I should like to express my cordial thanks to Prof. S. Sakata for his kind guidance and to Mr. H. Umezawa for his valuable discussions.

- 1) W. Thirring, *Phil. Mag.* **41** (1950), 1193.
- 2) If they were singular, the Pauli term, for instance, would remain and then we could not establish the below conclusion. In fact this circumstance happens in the case of the approximation e^2 , but generally does not for the higher order approximation if $\alpha \neq 0$.
- 3) It can be easily seen that the "Compton scattering" does not include any new divergences after performing the mass and the charge renormalizations. Moreover applying the equation of motion thoroughly C_1^0 and C_2^0 become the finite c -numbers in the limit of $k_1 \rightarrow 0$. See also reference 5.
- 4) Z. Koba, N. Mugibayashi and S. Nakai, *Prog. Theor. Phys.* **6** (1951), 322.
- 5) The validity of reference 3 is not changed also for these meson fields by performing the appropriate renormalizations. Cf., P. T. Mathews, *Phys. Rev.* **81** (1951), 936.

Seasonal Variation of Large Cosmic-Ray Bursts

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May 1, 1952

In compiling the data of the large cosmic-ray bursts at sea level and at basement, described in the recent paper¹⁾, we found a marked seasonal variation in burst-frequencies. To ascertain it, the bursts on the records were counted as much as possible, and the preliminary report on the study of seasonal variation is presented here.

Burst-frequencies at sea level during 1941-1943 were obtained by five SRI ion chambers of the same type installed at the same place, details of which were shown in (1). The burst sizes measured by them were normalized to those at 40 atm. and 30°C, each chamber and recording apparatus being thermo-regulated at 30°C. Burst-frequencies at basement* were obtained by one of them (SRI ion chamber No. 1) during 1939-1941.

Fig. 1. Seasonal variation of burst-frequencies

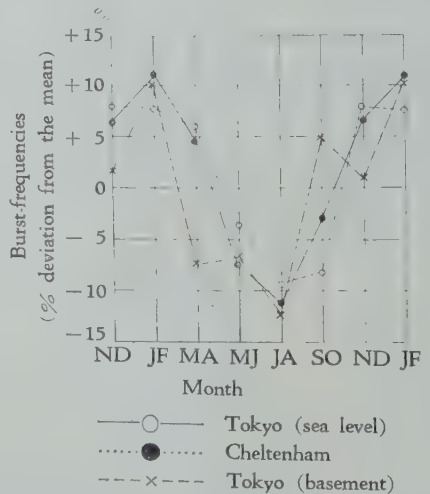


Fig. 1 shows the seasonal variation of the frequencies of bursts of more than size

4 (13.4×10^6 ion pairs), and each point represents the percentage deviation of the frequencies in successive two months from the mean frequencies. In the Fig., bursts of more than 350 particles measured at Cheltenham are also plotted.²⁾

As shown in the Fig., seasonal variations of the burst-frequencies are clearly seen at each station. Moreover, by application of χ^2 -test on the data the existence of seasonal variation is justified in each case. Assuming that these seasonal variations are attributed to the variations of atmospheric temperature*, i.e., the variations of isobar heights, the relations between burst-frequencies and atmospheric temperature are graphed in Figs. 2, 3 and 4. Here, the atmospheric temperature is that at sea level, and that at Cheltenham is replaced by that at Washington, because we have not the available data of outdoor temperature at Cheltenham. Close correlation between burst-frequencies and atmospheric temperature is seen in the Figs., and the temperature coefficients obtained from the Figs. are as follows;

at sea level

$$\text{Tokyo } a_s = -(0.95 \pm 0.1) \% / ^\circ\text{C}^{(3)}$$

$$\text{Cheltenham } a'_s = -(0.8 \pm 0.2) \% / ^\circ\text{C}$$

at basement

$$\text{Tokyo } a_B = -(1.1 \pm 0.3) \% / ^\circ\text{C}^{(3)}$$

the errors are estimated considering the total number, errors in reading, and the electrical leakage at the insulators of the electrode system (on the bursts at basement).

In connection with the temperature effect of the burst-frequencies, latitude effect of large cosmic-ray bursts reported by several authors⁴⁾ may be attributed to the variation of isobar heights with latitude, namely, the temperature effect.

Admitting that the seasonal variation is caused by the variation of atmospheric temperature, it is necessary to know the temperature effect of the burst-frequencies at mountain altitude of medial or high latitude, in order to find the reason why such large temperature effect as these occurs. Though we have not any available data for it, the paper reported by Stinchcomb⁵⁾ may throw some light on the subject, and following it the temperature effect at mountain altitude appears to be smaller than $-0.5 \% / ^\circ\text{C}$. Therefore, if the frequency ratios of μ -bursts to N -bursts at mountain altitude, at sea level and at basement are taken into consideration¹⁾, bursts which

Fig. 2 Burst-frequencies vs. atmospheric temperature (Tokyo; sea level)

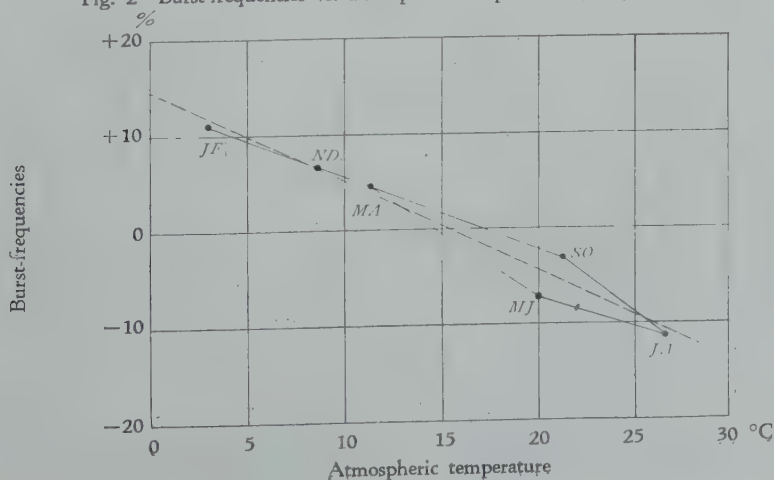


Fig. 3 Burst-frequencies vs. atmospheric temperature (Cheltenham)

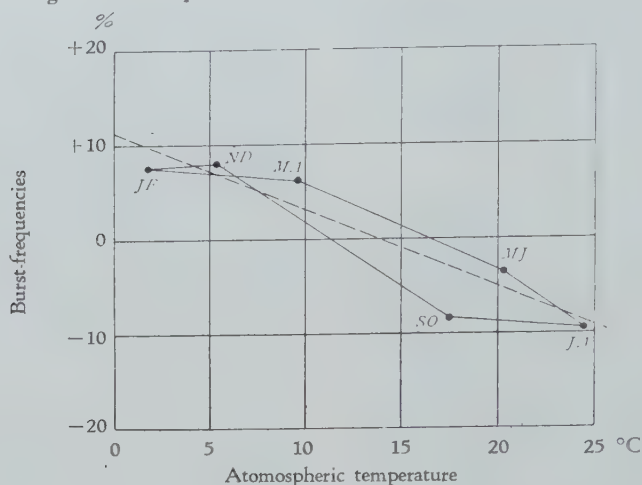
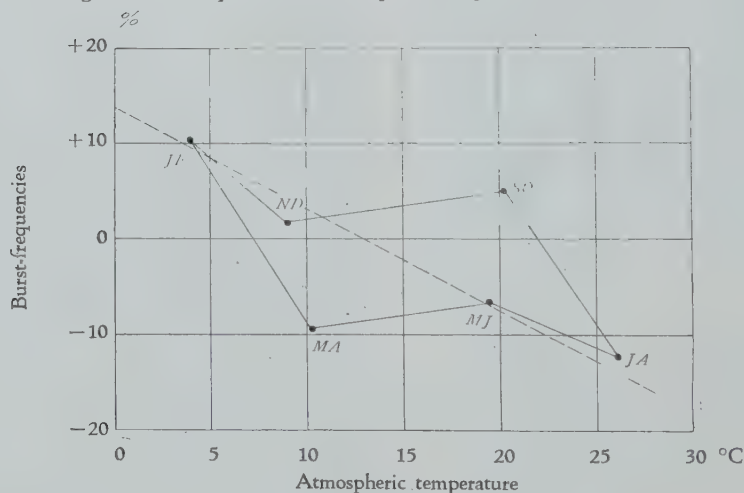


Fig. 4 Burst-frequencies vs. atmospheric temperature (Tokyo; basement)



change the frequencies with temperature seem to be contained in μ -bursts, and the following possibility may not be ruled out; that is, some of the so-called μ -bursts are not initiated by the bremsstrahlung of μ -mesons, but are produced by some particles which have short life time ($\lesssim 10^{-9}$ sec) and absorption mean free path several times longer than that of N -components.

Statistical treatment of the data and details of the subject will be soon published in Journal of the Scientific Research Institute.

* Under 170 g/cm² concrete.

** This assumption is justified by the fact that the seasonal variation lessens as increasing burst size.

- 1) T. Kameda and M. Wada, Prog. Theor. Phys. **7** (1952), 1.
- 2) Cosmic Ray Results from Huancayo Observatory, presented to Dr. Sekido by Dr. Forbush. We thank Dr. Sekido who lent it to us.
- 3) Correcting for the barometric effects, $a_S = -1.0\%/^{\circ}\text{C}$, $a_R = -1.1\%/^{\circ}\text{C}$.
- 4) W. P. Jesse and P. S. Gill, Phys. Rev. **55** (1939), 414.
M. Schein and P. S. Gill, Rev. Mod. Phys. **11** (1939), 267.
- 5) T. G. Stinchcomb, Phys. Rev. **83** (1951), 422.

On The Equivalence Principle*

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May 2, 1952

The equivalence of the scalar (pseudoscalar) and vector (pseudovector) couplings of the meson and nucleon fields is shown by means of a change of variables in the Lagrangian. It is also shown that this modification of the Lagrangian is equivalent to a unitary transformation of the Hamiltonian.

Let us take a typical example of the neutral pseudoscalar meson field:

$$L_{\text{total}} = L + L_m,$$

$$L_m = -\frac{1}{2} \{ (\partial_\mu \phi)^2 + x^2 \phi^2 \}, \quad (1)$$

$$-L = \bar{\psi} \gamma_\mu \partial_\mu \psi + x_0 \bar{\psi} \psi + if \bar{\psi} \gamma_5 \psi \phi$$

$$+ \frac{ig}{x} \bar{\psi} \gamma_\mu \gamma_5 \psi \partial_\mu \phi, \quad \bar{\psi} = \psi^* \gamma_4. \quad (2)$$

Making the transformation

$$\psi = U \psi', \quad U = \exp \left(-i \frac{g}{x} \gamma_5 \phi \right),$$

$$\psi = (U \psi')^* \gamma_4 = \psi'^* U^{-1} \gamma_4 = \bar{\psi}' U, \quad (3)$$

the original Lagrangian becomes

$$-L(\psi) = \bar{\psi}' U \gamma_\mu \partial_\mu (U \psi') + x_0 \bar{\psi}' U^2 \psi'$$

$$+ if \bar{\psi}' U \gamma_5 U \psi' \phi + \frac{ig}{x} \bar{\psi}' U \gamma_\mu \gamma_5 U \psi' \partial_\mu \phi$$

$$= \bar{\psi}' \gamma_\mu \partial_\mu \psi' + x_0 \bar{\psi}' U^2 \psi' + if \bar{\psi}' \gamma_5 U^2 \psi' \phi$$

$$\equiv -L'(\psi') \quad (4)$$

by virtue of the relations

$$U \gamma_\mu = \gamma_\mu U^{-1},$$

$$U^{-1} \partial_\mu U = U^{-1} \left(-i \frac{g}{x} \gamma_5 \partial_\mu \phi \right) U$$

$$= -\frac{ig}{x} \gamma_5 \partial_\mu \phi.$$

The latter relation is guaranteed because we here treat the field variables as *c*-numbers. The new expression (4) of the Lagrangian is obviously void of the pseudo-vector coupling term.

Now, we shall show this modification of the Lagrangian is equivalent to some unitary transformation of the Hamiltonian. The line of reasoning runs along the following scheme:

$$\left. \begin{array}{l} L(\psi) \xrightarrow{\text{change of variables}} L'(\psi') \xrightarrow{\text{quantization}} H'(\psi') \\ L(\psi) \xrightarrow{\text{quantization}} H(\psi) \xrightarrow{\text{unitary transformation}} H''(\psi') \\ \xrightarrow{\hspace{1cm}} H'(\psi') = H''(\psi'). \end{array} \right\}$$

The Hamiltonian $H'(\psi')$ runs

$$H'(\psi') = \psi'^+ \dot{\psi}' + \dot{\phi}^+ \dot{\phi} - L'_{\text{total}}$$

$$= \bar{\psi}' \gamma_4 \partial_4 \psi' + x_0 \bar{\psi}' U^2 \psi' + if \bar{\psi}' \gamma_5 U^2 \psi' \phi$$

$$+ \frac{1}{2} \{ (\phi'^+)^2 + (\partial_4 \phi')^2 + x^2 \phi'^2 \}, \quad (5)$$

where ψ'^+ and ϕ'^+ are canonically conjugate to ψ' and ϕ' , respectively, i.e.

$$\psi'^+ = \frac{\partial L'(\psi')}{\partial \dot{\psi}'} = i \psi'^*, \quad \phi'^+ = \dot{\phi}'.$$

Whereas we get from (1) and (2) the following expression;

$$H(\psi) = \psi^+ \dot{\psi} + \dot{\phi}^+ \dot{\phi} - L_{\text{total}}$$

$$= \bar{\psi} \gamma_4 \partial_4 \psi + x_0 \bar{\psi} \psi$$

$$+ \frac{1}{2} \{ (\phi^+)^2 + (\partial_4 \phi)^2 + x^2 \phi^2 \} + if \bar{\psi} \gamma_5 \psi \phi$$

$$+ \frac{ig}{x} \bar{\psi} \gamma_\mu \gamma_5 \psi \partial_\mu \phi + \frac{g}{x} \bar{\psi} \gamma_4 \gamma_5 \psi \dot{\phi}$$

$$+ \frac{1}{2} \left(\frac{g}{x} \right)^2 (\bar{\psi} \gamma_4 \gamma_5 \psi)^2, \quad (6)$$

$$\psi^+ = i \psi^*, \quad \phi^+ = \dot{\phi} - \frac{g}{x} \bar{\psi} \gamma_4 \gamma_5 \psi.$$

Now, let us introduce the following unitary transformation :

$$\begin{aligned} T &= \exp \left(-i \frac{g}{x} \int \bar{\psi} \gamma_4 \gamma_5 \psi \phi^3 dx \right), \\ \psi' &= T \psi T^{-1} = \exp \left(+i \frac{g}{x} \gamma_5 \phi \right) \psi = U^{-1} \psi, \\ \phi' &= T \phi T^{-1} = \phi, \\ \phi'^+ &= T \phi^+ T^{-1} = \phi^+ + \frac{g}{x} \bar{\psi} \gamma_4 \gamma_5 \psi = \phi^+. \end{aligned} \quad (7)$$

Note that these new variables ψ' , ϕ' etc. are nothing but the quantities ψ , ϕ etc. already used in exp. (3).

Thus the transformed Hamiltonian becomes, in terms of the new variables

$$\begin{aligned} H''(\psi') &= H(\psi(\psi'\phi'), \phi^+(\psi'\phi')) \\ &= \bar{\psi}' \gamma_4 U^{-1} \partial_t (U \psi') + x_0 \bar{\psi}' U^2 \psi' \\ &\quad + \frac{1}{2} \left\{ \left(\phi'^+ - \frac{g}{x} \psi'^+ \gamma_5 \psi' \right)^2 + (\partial_t \phi')^2 + x^2 \phi'^2 \right\} \\ &\quad + i f \bar{\psi}' U \gamma_5 U \psi' \phi' + \frac{i g}{x} \bar{\psi}' U \gamma_4 \gamma_5 U \psi' \partial_s \phi' \\ &\quad + \frac{g}{x} \bar{\psi}' \gamma_4 \gamma_5 U \psi' \left(\phi'^+ - \frac{g}{x} \psi'^+ \gamma_5 \psi' \right) \\ &\quad + \frac{1}{2} \left(\frac{g}{x} \right)^2 (\bar{\psi}' U \gamma_4 \gamma_5 U \psi')^2 \\ &= \bar{\psi}' \gamma_4 \partial_t \psi' + x_0 \bar{\psi}' U^2 \psi' + \frac{1}{2} \{ (\phi'^+)^2 + (\partial_t \phi')^2 \\ &\quad + x^2 \phi'^2 \} + i f \bar{\psi}' \gamma_5 U^2 \psi' \phi', \end{aligned} \quad (8)$$

which is just the same with the exp. (5). Q.E.D.

If we make use of the interaction representation, the Hamiltonian becomes as follows :

$$\begin{aligned} H_i(\phi) &= x_0 \bar{\psi} (U^2 - 1) \psi + i f \bar{\psi} \gamma_5 U^2 \psi \phi \\ &= i \left(f - \frac{x_0}{x} 2g \right) \bar{\psi} \gamma_5 \psi \phi + \left(\frac{2fg}{x} - \frac{2g^2}{x^2} x_0 \right) \\ &\quad \times \psi \phi \phi^2 + \dots \end{aligned}$$

an expression already given by Case.

Our procedure can be extended easily to other kinds of meson fields.

The results only are listed :

Neutral scalar field: In this case two couplings are completely equivalent, i.e. the vector couplings has no effect.

Charged scalar field: After the transformation $\psi = U \psi'$, $U = \exp \left(-\frac{i g}{x} (\tau_+ \phi^* + \tau_- \phi) \right)$.

$$\begin{aligned} -L'_{\text{total}}(\psi') &= \bar{\psi}' \gamma_\mu \partial_\mu \psi' + x_0 \bar{\psi}' \psi' \\ &\quad + f \bar{\psi}' (\tau_+ \phi^* + \tau_- \phi) \psi' + \bar{\psi}' \gamma_\mu F(\phi, \phi^*, \tau) \psi' \\ &\quad - i c \bar{\psi}' \gamma_\mu \frac{1 - \tau_3 \gamma_5}{2} A_\mu - L'_m, \\ -L'_m &= \{ (\partial_\mu \phi^* + i e A_\mu \phi^*) (\partial_\mu \phi - i e A_\mu \phi) \\ &\quad + x^2 \phi^* \phi \}, \end{aligned}$$

$$\begin{aligned} F(\phi^*, \phi, \tau) &= \sum_{m=0}^{\infty} \frac{1}{2^m} \left(i \frac{g}{x} \right)^{2m+2} \tau_3 (\phi^* \phi)^m \\ &\quad \times \{ \phi^* (\partial_\mu - i e A_\mu) \phi - \phi (\partial_\mu + i e A_\mu) \phi^* \} \\ &\quad \frac{2m+1}{(2m+2)!} + \sum_{m=0}^{\infty} \frac{1}{2^m} \left(i \frac{g}{x} \right)^{2m+3} (-\tau_+ \phi^* + \tau_- \phi) \\ &\quad \times (\phi^* \phi)^m \{ \phi^* (\partial_\mu - i e A_\mu) \phi - \phi (\partial_\mu + i e A_\mu) \phi^* \} \\ &\quad \times \frac{2m+2}{(2m+3)!}, \end{aligned}$$

Charged pseudoscalar field: After the transformation

$$\begin{aligned} \psi' &= U \psi, \quad U = \exp \left\{ -i \frac{g}{x} (\tau_+ \phi^* + \tau_- \phi) \right\}, \\ -L'_{\text{total}}(\psi') &= \bar{\psi}' \gamma_\mu \partial_\mu \psi' + x_0 \bar{\psi}' U^2 \psi' \\ &\quad + i f \bar{\psi}' \gamma_5 (\tau_+ \phi^* + \tau_- \phi) U^2 \psi' \\ &\quad + \bar{\psi}' \gamma_\mu G(\phi, \phi^*, \tau) - i c \bar{\psi}' \gamma_\mu \frac{1 - \tau_3 \gamma_5}{2} A_\mu \\ &\quad - L'_m. \end{aligned}$$

$G(\phi, \phi^*, \tau)$ is obtained by replacing the factor $(-\tau_+ \phi^* + \tau_- \phi)$ in $F(\phi, \phi^*, \tau)$ with $\gamma_5 (-\tau_+ \phi^* + \tau_- \phi)$.

These complicated expressions (charged field) result from the non-commutativity of τ -spin.

* The contents of this letter are read at the annual meeting of the Physical Society of Japan in October 9th in 1951.

K. M. Case, Phys. Rev. **76** (1949), 14.

F. J. Dyson, Phys. Rev. **73** (1948), 929.

Remark on the Self Energy Problem of the Photon

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May 8, 1952

The divergence of the self energy of the photon has been a dead lock in quantum electrodynamics and many attempts¹⁾ have been done to overcome this difficulty. But none of them seems to gain a final success. We point out in this note that the re-examination of Lorentz transformation suggests a way from this situation.

Let us consider, for example, the interaction of the electromagnetic field with the electron-field. The vacuum is defined according to Dirac's positron theory as a state in which all negative energy states are occupied. This state is however nothing but a sort of Fermi gas at the absolute zero of temperature, and it may be expected that the light velocity is altered from that of no interaction. Here we must remark that what Einstein used as a signal in his theory of the relativity is the light in this electron gas, or in other words, photons "clothed" with virtual electron-positron pairs, so that the constant c that appears

in Lorentz transformation must be regarded as the propagation velocity of these clothed photons. This means that c is altered as the result of the interaction, and that it would be necessary to consider the self energy problem of the photon in the relation with this alteration of c .

The rest mass of a particle is determined from its energy and momentum by the equation

$$E^2 = p^2 c^2 + \mu^2 c^4, \quad (1)$$

therefore

$$E = pc \quad (2)$$

is the condition that the particle has a zero rest mass. As to the photons, (2) is of course satisfied if there is no interaction. Upon switching on interaction between the photons and the electrons, the self energy appears accompanied by the alteration of the propagation velocity. The former can be calculated by the ordinary perturbation method, and the latter is obtained from the equation of motion of the clothed photons in the free space.

$$\square \langle A_\mu \rangle_{\text{vac}} = -\frac{1}{c} \delta J_\mu \quad (3)$$

where A_μ represents the electromagnetic four potential in Heisenberg representation, and $\langle A_\mu \rangle_{\text{vac}}$ means the expectation value of A_μ in the vacuum state of the electron, while δJ_μ is the induced current due to the photons, and we can show, at least in the second order approximation, that the following relation is satisfied between $E + \Delta E$ and $c + \Delta c$;

$$E + \Delta E = p(c + \Delta c). \quad (4)$$

(4) shows that the photons are observed to have zero rest mass even when there is interaction if Lorentz transformation is constructed using $c + \Delta c$ instead of unobservable c , and the self energy term which

has been believed to violate the gauge invariance of the theory can be ignored by replacing c by $c + \Delta c$. Thus we can use the self-consistent subtraction technique proposed by Tomonaga²⁾ and Schwinger³⁾ just as in the case of the self energy problem of the electron. Writting the Hamiltonian of the electromagnetic field $H(c)$ in the form

$$H(c) = H(c + \Delta c) - \frac{\partial H}{\partial c} \Delta c, \quad (5)$$

we replace $c + \Delta c$ by the observed value of c , and treat $H(c + \Delta c)$ as the Hamiltonian of the free clothed photon, while $-\frac{\partial H}{\partial c} \Delta c$ serves as a new counter term to cancel divergent effects produced by ordinary calculations. In this way, we can formulate a theory which is Lorentz invariant, gauge invariant, and at least in the second order approximation of perturbation calculation free from the pathological divergence of the photon self energy.

At the present stage of quantum electrodynamics, we can not avoid mathematical ambiguities due to the singular character of D functions and we can not say conclusively whether our renormalization technique can be a final one. But we want to emphasize that the self energy problem of the photon would not be settled by computing the self energy alone, and it would be necessary to take into account the alteration of the light velocity due to the interaction, together with the appearance of the self

energy, since both are the simultaneous effects of the induced current.

We wish to express our sincere gratitude to Prof. S. Sakata for his interest and encouragement. Details will soon appear in this journal.

- 1) J. Schwinger, Phys. Rev. **75** (1949), 651.
H. Umezawa, J. Yukawa and E. Yamada, Prog. Theor. Phys. **3** (1948) 317.
D. Feldman, Phys. Rev. **76** (1949) 1369.
W. Pauli and F. Villars, Rev. Mod. Phys. **21** (1949), 434.
- 2) S. Tomonaga and T. Tati, Prog. Theor. Phys. **3** (1948), 309.
- 3) J. Schwinger, *ibid*.

On the Model of V-Particle and Meson-Nucleon Scattering

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May 8, 1952

Recently, various models of V -particle have been proposed by many authors¹⁾, and essential point of these theories is to separate V -production from its decay. We have estimated the effect of the existence of V' -particle to the meson-nucleon scattering, assuming V' is the mother particle of V and V -production and -decay follows as in Fig. 1.

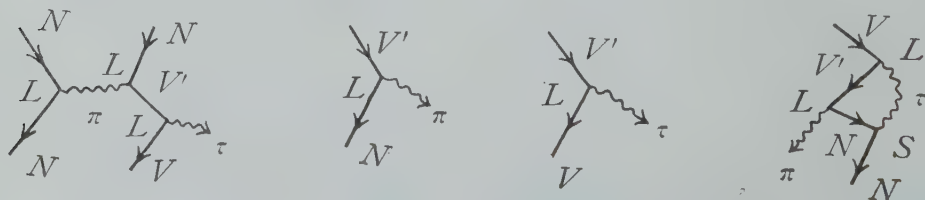


Fig. 1. (L and S means large and small coupling constant, respectively)

In this figure N is the nucleon, and τ is not necessarily identical with the observed τ -meson. We mean by such model only that the V -production is of two steps through strongly interacting V' -particle. This model infers as the mass of V' -particle, $m_{V'} > m_V + m_\tau \approx 2800m_e$.

The existence of such strongly interacting heavy particle affects the scattering of meson with nucleon. If we confine our arguments to mesons of kinetic energy up to 200 Mev, the breadth of resonance and effect of damping may be neglected (resonance takes place at about 600Mev meson in the above model). The effect is shown in Fig. 2. We can offer a simple argument for the effect of the intermediate V' -particle as follows; Denoting the mass of V' -particle by m_1 and coupling constant by g_1 (the coupling constant when they transmutates to nucleon and vice versa), we have as the matrix element in the pseudo-scalar pseudo-vector coupling meson: (in Feynman notation)



Process I (destructive interference)

$$\begin{aligned}
 M_I = & -4\pi \left[\left(g^2 \right. \right. \\
 & + \frac{[(p_0 + q_0)^2 + m^2 + 2mm_1][(p_0 + q_0)^2 - m^2]}{[(p_0 + q_0)^2 + 3m^2][(p_0 + q_0)^2 - m_1^2]} \\
 & \times g_1^2 \left. \right) \frac{(p_0 + q_0)^2 + 3m^2}{(p_0 + q_0)^2 - m^2} (q_0 \gamma) \\
 & - \left(g^2 + \frac{[m + m_1][(p_0 + q_0)^2 - m^2]}{2m[(p_0 + q_0)^2 - m_1^2]} g_1^2 \right) \\
 & \times \frac{2m(\mu^2 + 2(p_0 q_0))}{(p_0 + q_0)^2 - m^2} \left. \right] . \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 M_{II} = & 4\pi \left[\left(g^2 \right. \right. \\
 & + \frac{[(p_0 - q_0)^2 + m^2 + 2mm_1][(p_0 - q_0)^2 - m^2]}{[(p_0 - q_0)^2 + 3m^2][(p_0 - q_0)^2 - m_1^2]} \\
 & \times g_1^2 \left. \right) \frac{(p_0 - q_0)^2 + 3m^2}{(p_0 - q_0)^2 - m^2} (q_0 \gamma) \\
 & + \left(g^2 + \frac{[m + m_1][(p_0 - q_0)^2 - m^2]}{2m[(p_0 - q_0)^2 - m_1^2]} g_1^2 \right) \\
 & \times \frac{2m(\mu^2 - 2(p_0 q_0))}{(p_0 - q_0)^2 - m^2} \left. \right] . \quad (2)
 \end{aligned}$$

Now;

$$\begin{aligned}
 & \left[\frac{m + m_1}{2m} - \frac{(p_0 + q_0)^2 + m^2 + 3mm_1}{(p_0 + q_0)^2 + 3m^2} \right] \\
 & \quad \left[\frac{m + m_1}{2m} \right] \approx 0.04, \\
 & \left[\frac{m + m_1}{2m} - \frac{(p_0 - q_0)^2 + m^2 + 2mm_1}{(p_0 - q_0)^2 + 3m^2} \right] \\
 & \quad \left[\frac{m + m_1}{2m} \right] \approx 0.07, \quad (3)
 \end{aligned}$$



Process II (constructive interference)

Fig. 2.

for $\epsilon_0 \approx 3\mu$, $m_1 = 3000m_e$ and in laboratory system. The values diminish considerably for $\epsilon_0 < 3\mu$ from the above magnitude. So we have approximately,

$$\begin{aligned}
 M_I = & -4\pi \left(g^2 + \frac{[m + m_1][(p_0 + q_0)^2 - m^2]}{2m[(p_0 + q_0)^2 - m_1^2]} g_1^2 \right) \\
 & \frac{[(p_0 + q_0)^2 + 3m^2](q_0 \gamma) - 2m[\mu^2 + 2(p_0 q_0)]}{(p_0 + q_0)^2 - m^2} , \quad (4)
 \end{aligned}$$

$$M_{II} = 4\pi \left(g^2 + \frac{[m + m_1][(\mathbf{p}_0 - \mathbf{q})^2 - m^2]}{2m[(\mathbf{p}_0 - \mathbf{q})^2 - m_1^2]} g_1^2 \right) \frac{[(\mathbf{p}_0 - \mathbf{q})^2 + 3m^2](q\gamma) + 2m[\mu^2 - 2(\mathbf{p}_0 \mathbf{q})]}{(\mathbf{p}_0 - \mathbf{q})^2 - m^2} \quad (5)$$

This means that in such energy region the effect of V' -particle appears as a change in the coupling constant. For the process I, the coupling diminishes considerably when the meson energy is raised, for II they build up. The former effect is larger than the latter (see Table, in comparison).

The exact evaluation of the matrix element and cross-sections shows that the argument given above is approximately correct for energies 40 to 200 Mev. Thus, the scattering cross section, $\pi^- + P \rightarrow \pi^- + P$, is of type I and so they suppressed from the theoretical value obtained by Ashkin, Simon and Marshak²⁾, and for $\pi^+ + P \rightarrow$

$\pi^+ + P$ which is of type II is raised above their results. The behavior of the total cross sections for such processes is plotted in Fig. 3 (exact evaluation).

As to the charge exchange process, the experiments seems to indicate³⁾ the existence of plateau for $\pi^- + P \rightarrow \pi^0 + N$ at about 150 Mev meson energy, and to fit this fact we have assumed pure neutral type interaction for neutral meson interaction $V' \rightleftharpoons N + \pi_0$, with our constants indicated in Fig. 3. (This destroys the symmetry of the process $\pi^+ + N \rightarrow \pi^0 + P$ and above process, the difference becomes notable above 100 Mev energy; the cross section rises along the lower side of $\pi^+ + P \rightarrow \pi^+ + P$ curve. Normal scattering is not affected by this assumption).

Thus, the existence of mother V which is strongly interacting with nucleon modifies the meson nucleon scattering cross section

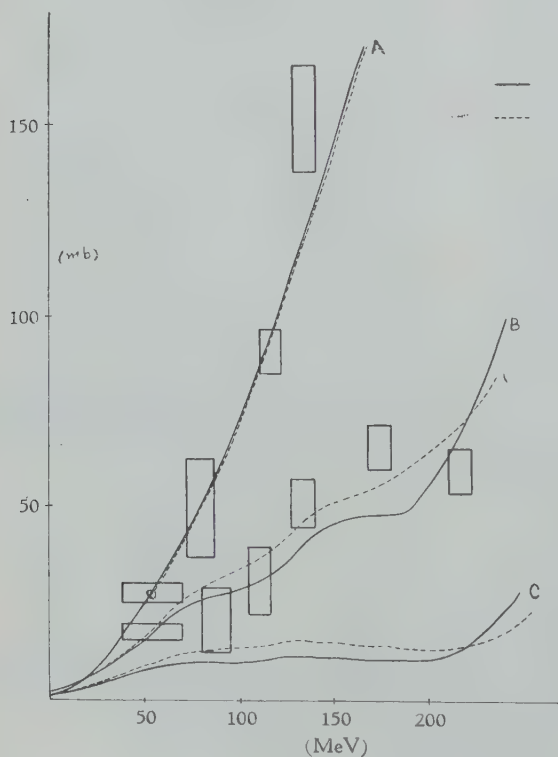


Fig. 3. Total Cross-Sections.

Curve A: $\pi^+ + P \rightarrow \pi^+ + P$,

Curve B: $\left\{ \begin{array}{l} \pi^- + P \rightarrow \pi^- + P \\ \pi^- + P \rightarrow \pi^0 + N \end{array} \right\}$,

Curve C: $\pi^- + P \rightarrow \pi^- + P$.

Coupling constants are such that (g_n, g_p) has opposite sign; (g_{n1}, g_{n2}) , (g_n, g_{n1}) and (g, g_1) has same sign.

And $g^2 = 2g_n^2$, $g_1^2 = 2g_{n1}^2$.

E.g., symmetrical theory for the $N-N-\pi$ interaction.

The mark \circ indicates the normalization point. Experimental data are taken from reference 3.

—: $g_1^2 = 1.0g^2$; $\frac{2\pi g^2}{\mu^2} = 37.3$ mb,

---: $g_1^2 = 0.9g^2$; $\frac{2\pi g^2}{\mu^2} = 38.7$ mb.

Table I.
Change of coupling constant from the simple formula (4) and (5).

Meson Kinetic Energy (Mev)	40	80	120	160	Coupling constant was taken $g_1=0.8g$ and the mass of V' -particle to be $3000m_e$. For $g_{II}^{\text{effective}}$, we put $\epsilon_0 \approx \epsilon$ in (5), which is not sensitive to the value listed.
$(g_I^{\text{effective}}/g)^4$	0.52	0.40	0.28	0.17	
$(g_{II}^{\text{effective}}/g)^4$	1.32	1.39	1.46	1.51	

considerably in its magnitude, but the above simple arguments permits us to discuss roughly their angular distributions as follows; the normally scattered meson has angular distribution which is nearly the same as in reference 2, because the effect of V' -particle is only to change the coupling constant. But as to the charge exchange scattering, the angular distribution is more like that of Process II, because this scattering is composed of process I and II and, exact evaluation shows that the process II is predominant⁴⁾. Since the angular distribution of scattered meson given in reference 2 is nearly the same for process I and II, and we may roughly expect almost the same angular distribution for normal- and exchange-scattering.

In conclusion, the writer wishes to express his sincere thanks to Prof. M. Kobayasi and to Mr. S. Takagi for their kind interest taken in this work.

- 1) For summary of references, see A. Pais, preprint.
- 2) J. Ashkin, A. Simon and R. E. Marshak, *Prog. Theor. Phys.* **5** (1950), 634.
- 3) P. J. Issacs, A. M. Sachs and J. Steinberger, *Phys. Rev.* **85** (1952), 803; Fermi *et al.* *Phys. Rev.* **85** (1952), 934, 935, 936.
- 4) Owing to the choice of coupling as given in Fig. 3, for $\pi^- + P \rightarrow \pi^0 + N$, we have as effective coupling constant, in (4) and (5)

$$\begin{aligned}
 g_I^{\text{2eff}} &= g_{\pi N} + \frac{[m+m_1][(\mathbf{p}_0+\mathbf{q})^2-m^2]}{2m[(\mathbf{p}_0+\mathbf{q})^2-m_1^2]} g_1 g_{N1}, \\
 g_{II}^{\text{2eff}} &= g_{\pi P} + \frac{[m+m_1][(\mathbf{p}_0-\mathbf{q})^2-m^2]}{2m[(\mathbf{p}_0-\mathbf{q})^2-m_1^2]} g_1 g_{P1} \\
 &= -\left(g_{\pi N} - \frac{[m+m_1][(\mathbf{p}_0-\mathbf{q})^2-m^2]}{2m[(\mathbf{p}_0-\mathbf{q})^2-m_1^2]} g_1 g_{N1} \right)
 \end{aligned}$$

and thus, $(g_I^{\text{2eff}}/g_{\pi N})^2$ is just corresponding quantity in Table I, $((g_1 g_N)^{1/2}=0.8(g_{\pi N})^{1/2})$, but $(g_{II}^{\text{2eff}}/g_{\pi N})^2$ is

40	80	120	160
0.72	0.67	0.62	0.59

An Empirical Mass Spectrum of Elementary Particles

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May 14, 1952

It seems to be a general conviction of current physicists that the theory of elementary particles in its ultimate form could or should give the mass spectrum of these particles just in the same way as quantum mechanics has succeeded in accounting for the regularity of atomic spectra. Even if we disregard any philosophical background in such a postulation of theoretical physics, the recent discovery of many unstable, apparently elementary particles drives us to the efforts towards a systematic comprehension of the variety of elementary particles. With the present undoubtedly insufficient accumulation of our knowledge, however, it may perhaps be too ambitious and rather unsound to look for an empirical "Balmer's law". Nevertheless we should like here to present one such attempt because it

happens to be extremely simple, and because the significance and utility, if any, of this kind of attempt could best be appreciated at the stage where it awaits more experimental data to prove or disprove itself by its own predictions.

The nature of V_0 particles¹⁾ and τ -mesons²⁾ has been investigated by several authors. Among other things, we note that their decay Q -values are rather uniform, i.e. of the same order of magnitude of the rest mass of the daughter π -mesons. This gives us a hint that some regularity might be found if the masses were measured in a unit of the order of the π -meson mass. The π -meson mass, being $\sim 274 = 137 \times 2$ electron masses (m_e), gives us a second, rather fanciful hint that $137 m_e$ could be chosen as the unit. The ensuing result is given in the accompanying table. We see

particle	mass no. n	$137 \times n$	experimental mass
lepton	0	0	~ 0
photon	0	0	0
μ	$1\frac{1}{2}$	206	$210 \pm 3 m_e$
π	2	274	$276 \pm 3 (\pi^\pm)$
V_{02}	6	822	800 ± 30
τ	7	959	966 ± 10
α			1000~1500
nucleon	$13\frac{1}{2}$	1849	1837, 1839
V_{01}	16, $16\frac{1}{2}$	$Q=35, 70 \text{ Mev}$	$35 \pm 5, 75 \pm 3 \text{ Mev}$
V^*	$17\frac{1}{2}$	$Q=280 \text{ Mev}$	$\sim 280 \text{ Mev}$

that the "mass number" of the observed particles is either integer or half-odd, which is generally valid within a deviation of about $\sim \pm 15 m_e$, or $\sim \pm 1/10$ mass unit, for those cases in which the experimental error is also of this order of magnitude. In the above table, we have adopted the view that the heavy V_0 particles have two kinds of Q -values, namely $\sim 35 \text{ Mev}$ ($1/2$ mass unit) and $\sim 70 \text{ Mev}$ ($1 m.u.$)³⁾, decaying into a proton and a π -meson. V^* means the nucleon isobar whose existence is being con-

jectured from γ - π reaction and π -proton scattering,⁵⁾ with an excitation of roughly about 280 Mev (4 m.u.).

We can make a few comments on the result. 1) As was pointed out by Enatsu⁴⁾, the adopted mass unit incidentally agrees with Heisenberg's natural unit. 2) Bosons seem to have integral, while fermions half-integral, mass numbers. 3) The small mass value of the electron cannot be explained by the above rule. But we can take the view that this as well as the proton-neutron and π^\pm - π^0 mass differences correspond to a kind of fine structure. Indeed, their magnitude is just of the order of $1/137 m.u.$

It goes without saying that this rule is purely of an empirical nature, and might turn out to be entirely illusory or accidental in the event of getting more reliable data or establishing the true theory of mass spectrum. But the rather strange distribution of the observed mass numbers might simply mean the lack of our knowledge. Indeed, only those particles which have favorable lives as well as abundances for detection have so far been observed, and we have no grounds at all to exclude the possibility that there exist other particles which are liable to escape direct observation. At any rate, an effective and close-by test of this rule may be provided by more accurate determination of the masses of the observed particles. In particular, the α -meson may be predicted to have any of $\sim 1030, \sim 1100, \sim 1160, \sim 1230, \sim 1300, \dots$ electron masses ($7\frac{1}{2}, 8, 8\frac{1}{2}, 9, 9\frac{1}{2}, \dots m.u.$).

- 1) E. g., R. Armenteros et al., Phil. Mag. **42** (1951), 1113.
- 2) P. H. Fowler et al., Phil. Mag. **42** (1951), 1040.
- 3) S. D. Wanlass et al., Bull. Amer. Phys. Soc. **27** (1952), No. 3, 7.
- 4) Remarks by H. Enatsu at the Tokyo meeting of the Physical Society of Japan. April 1-3, 1952.
- 5) K. A. Brueckner, Bull. Amer. Phys. Soc. **27** (1952), No. 1, 50.

On the New Description of Quantum-mechanics^{*)}

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May 20, 1952

The conventional quantum-mechanics has been framed from the foundation of the theory of the "instantaneous observation", we can prepare at any moment such quantum state of an individual system as we wish to prepare; and we can describe the time variations of the quantum state by the Schroedinger equation. And a particle can be described by the usual wave function as an unsharply defined individual within finite space-time regions.

On the contrary, Feynman¹⁾ has recently built up the quantum theory on the foundation viewed from a point entirely different from the conventional one. We should like to emphasize that his essential idea throughout his works is one of the so-called "particle path".

According to the conventional quantum theory, it has been well known that the geometrical and kinematical description of a particle can not be attained.

However, the particle "path" involves such an idea against the conventional one, that the particle has a distinct position, and moreover suggests us a possibility of the so-called "continuous observation".

Therefore, if we take the path as a physical quantity.....a quantity measured by a sort of so-called "continuous observation", a sort of the accidental uncertainty, further, enters into the conventional quantum-mechanical description. This new probability amplitude may be a kind of one of random choice of the position, because the phenomenon is a random one that a particle continuously chooses any path and moreover

the phenomenon that a particle continuously chooses any path is a kind of the "stochastic phenomena."

Then, as for the probability amplitude specified by the position of the particle, from the requirement of the methodical description of the microscopic phenomena, we should further take into account the probability amplitude of random choice of the position.

On these lines, we shall rewrite the Feynman's equation, as follows:

$$\Phi(x_{k+1}, t + \epsilon) = \int \exp\left[\frac{i}{\hbar} S(x_{k+1}, x_k)\right] \times \zeta(x_{k+1} - x_k) \psi(x_k, t) \frac{dx_k}{A}, \quad (1)$$

where $\zeta(x_{k+1} - x_k)$ is a random probability amplitude with which the particle chooses the position x_{k+1} from the position x_k along the path; and we may also reinterpret that the function $\zeta(x_{k+1} - x_k)$ is a kind of the correlation function between the two points (x_{k+1}, x_k) .

In the above-mentioned formula, we may call Φ the "super wave function,

$$\begin{aligned} \Psi(x, t) &= \int \Phi(x - \xi, t) d\xi \\ &= \int \zeta(\xi) \psi(x - \xi, t) d\xi \\ &= \int \Phi(x + \xi, t) d\xi \\ &= \int \zeta(-\xi) \psi(x + \xi, t) d\xi. \end{aligned} \quad (2)$$

If we may reinterpret ξ in the formula (2), the unknown coordinate, or the "hidden variable" which describes the internal behaviours of the particle, Ψ means the state which involves all effects of the internal influences of the particle; and we may call Ψ a "super probability amplitude".

Next, we may consider the motion of the particle in the three dimensional potential $V(X, Y, Z, x, y, z)$, where X, Y and Z mean the representative position of the particle, and x, y and z are the "hidden variables" of the particle.

Searching for the equations of motion of ψ, Φ and Ψ , we obtain the following formulas :

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} \left(\frac{\hbar}{i} \right)^2 \nabla^2_{(X, Y, Z)} \psi + V\psi, \quad (3)$$

$$i\hbar \frac{\partial \Phi}{\partial t} = \frac{1}{2m} \left(\frac{\hbar}{i} \right)^2 \nabla^2_{(X, Y, Z)} \Phi + V\Phi, \quad (4)$$

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{1}{2m} \left(\frac{\hbar}{i} \right)^2 \nabla^2_{(X, Y, Z)} \Psi + V\Psi, \quad (5)$$

$$i\hbar \frac{\partial \Phi}{\partial t} = \frac{1}{2m} \left(\frac{\hbar}{i} \right)^2 \nabla^2_{(x, y, z)} \Phi$$

$$+ [V(X, Y, Z, x, y, z) + U(x, y, z)] \Phi, \quad (6)$$

$$U(x, y, z) = \frac{\hbar^2}{2m} \sum_{(x, y, z)} \left[2 \frac{\partial(\log \zeta)}{\partial x} \frac{\partial}{\partial x} - \left(\frac{\partial(\log \zeta)}{\partial x} \right)^2 + \frac{\partial^2(\log \zeta)}{\partial x^2} \right]. \quad (7)$$

U -operator is analogous to the so-called "quantum-mechanical potential" which was introduced into the theory by Bohm²⁾. But U -potential operator is essentially different from the Bohm's potential.

At the present stage, it seems to be very difficult to give the function type of ζ explicitly.

However, to carry our discussions somewhat concretely and clearly, we may roughly assume ζ -function as the Gaussian distribution, as follows :

$$\zeta(x, y, z) = (s\sqrt{2\pi})^{-\frac{3}{2}} e^{-\frac{x^2+y^2+z^2}{4s^2}} \times e^{\frac{i}{\hbar}(p_x x + p_y y + p_z z)} \quad (8)$$

where s means the effective length of the

particle.

Hence, it follows that the mean co-ordinate of "hidden variables" is always the representative point of the particle and a priori weighted factor $(2s\sqrt{2\pi})^{-\frac{3}{2}} \times e^{-\frac{P_x^2 + P_y^2 + P_z^2}{P_0^2}}$ appears in Ψ , where P_x, P_y, P_z and P_0 mean the components of the momentum of the particle and \hbar/s respectively.

Moreover, as the degrees of uncertainty in the space of hidden variables, we have

$$(\Delta x)^2 = s^2, \quad (9)$$

$$(\Delta p_x)^2 = \frac{\hbar^2}{2s^2}. \quad (10)$$

Hence, we obtain

$$(\Delta x)^2 (\Delta p_x)^2 = \frac{\hbar^2}{2}. \quad (11)$$

Next, the physical space utilized to describe our new theory seems to be however, different from the one used to describe the conventional theory.

In other words, this physical space may be speculated to be such a space as describes the probability amplitude for finding the positions of the representative points of the particle and at the same time, accompanies such a local space at each world point that is described by the "hidden variables" of the particle.

The article with the detailed contents on this subject will soon appear in this Journal.

Further researches are now being developed.

* More detailed content than that of this letter was read before the meetings of the Physical Society of Japan held at Tokyo on April 1, 1952.

- 1) R. P. Feynman, Rev. Mod. Phys. **20** (1948), 367; Phys. Rev. **76** (1946), 749; **76** (1949), 749; **76** (1949), 769; **80** (1950), 440.
- 2) D. Bohm, Phys. Rev. **85** (1952), 166; **85** (1952), 180.

Excited States of Elementary Particle and the Quantization of Elementary Particles

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May 24, 1952

Recent many experiments upon V -particles are not sufficient to deduce a definite clarification concerning the nature of V -particles, however, it seems quite certain¹⁾ that there are two modes of decay scheme of a V^0 -particle:

$$V^0 \rightarrow P + \pi^-, \quad (1)$$

$$V^0 \rightarrow \pi^+ + \pi^-. \quad (2)$$

If in the above two processes there are accompanied no other neutral particles we must conclude that the V^0 -particle which decays according to (1) and the one which decays according to (2) are different in nature. On the other hand, there is a strong evidence¹⁾ that the processes (1) and (2) must be written correctly as follows:

$$V^0 \rightarrow P + \pi^- + \pi^0, \quad (1)'$$

$$V^0 \rightarrow N + \pi^+ + \pi^-. \quad (2)'$$

If this is the case, the two V^0 -particles which decay according to (1)' and (2)' turn out to be the same kind of particles. We assume that the latter possibility is right and further we propose a possible model in which a V^0 -particle is nothing but an excited state of a nucleon. This excited state, we assume, is a consequence of a new kind of quantization of a nucleon which is surrounded by the π -meson field. Then the V^0 -particle decays (1)' and (2)' can be explained as the quantum jump from the excited state of a nucleon to the ground state of the nucleon. The energy liberated at this quantum jump is emitted in the

form of quanta of the π -meson field which induces the excited state of a nucleon.

According to the above model, one of the decay products of a V^0 -particle must be always a proton or a neutron, therefore we must assume that two π -mesons are emitted at the decay of a V^0 -particle if the decay scheme (2) is true. In order to understand this decay mode into three particles, we assume a certain selection rule which forbids the transition from an excited to a ground state of a nucleon accompanied by a single quantum emission, but allows the transition accompanied by two π -meson emission. In the case of pseudoscalar π -meson field, it seems possible to obtain such a selection rule by assigning suitable properties to the excited state of a nucleon, namely a V^0 -particle. If there are charged V -particles in nature, they are perhaps also excited states of nucleons.

It is the most important purpose of this short note to remark that we can also apply the above considerations to electrons and neutrinos. For this purpose, we return to the original Yukawa theory, according to which the π -meson field which interacts with protons and neutrons also interacts with electrons and neutrinos. Therefore if it is permitted for the π -meson field to produce certain excited states of nucleons, it is natural to assume that there are also some excited states of an electron or neutrino. We decidedly assume that μ -meson is nothing but an excited state of an electron or a neutrino.

This assumption is supported by the very fact that in the μ -decay process there appears one electron and two neutrinos²⁾. From the above assumed excited state hypothesis, it is easy to understand the μ -decay process. If we consider a μ -meson as an excited state of an electron, the μ -decay can be explained as the quantum transition from the excited to the ground state of an electron accompanied by the neutrino pair

creation, which can be shown by the formula

$$\mu^\pm \rightarrow e^\pm + \nu + \nu. \quad (3)$$

In this case it is forbidden to emit a quantum of the field which produces such an excited state and the quantum jump between two states, therefore the μ -decay process results in the pair creation. On the other hand if we consider a μ -meson as an excited state of a neutrino, the μ -decay can be explained as the quantum jump accompanied by the electron-neutrino pair creation, which can be shown by the formula

$$\mu^\pm \rightarrow \nu + e^\pm + \nu. \quad (4)$$

In the former case, there is another possible mode of μ -decay besides that given by (3), namely the decay process which is the quantum jump accompanied by the electron pair creation, the three electron decay of a μ -meson. In order to forbid this three electron decay of a μ -meson, we assume the latter possibility (4), namely *we assume that a μ -meson is an excited state of a neutrino*. From the usual standpoint based on the analogy between β -decay and μ -decay, the μ -decay process cannot be explained so naturally owing to the fact that there are two neutrinos in the final stage of the decay process. It is to be remarked here that the decay mode (3) can be well approximated as the charge retention case in the Tiomno, Wheeler and Rau's investigations³⁾ and the decay mode (4) can be well approximated as antisymmetric or simple charge exchange case in the same investigations. It is shown²⁾⁴⁾ that the experimental energy spectrum of a decay electron can be very well explained both in the case of the charge retention with scalar coupling and in the case of the antisymmetric charge exchange with tensor coupling. If we recall that the tensor interaction is also very hopeful in nuclear β -processes⁵⁾, it is reasonable to take the latter case or the assump-

tion which considers a μ -meson as an excited state of a neutrino.

The most difficult point in this picture is how to understand the π -meson decay process into a μ -meson and a neutrino. According to the experiment⁶⁾, the π - e decay rate is smaller than 1/1400 times the π - μ decay rate. As the present excited state hypothesis is based on the original Yuhawa meson theory, in which the π -meson field interacts with both nucleons and electron-neutrinos, we can reasonably expect that the π -meson decays into an electron and a neutrino with a mean life of the order of 10^{-8} sec in order to get the agreement with experiments in nuclear β -processes. This value of the life time for the π - e decay process is the same order of magnitude as that observed for the π - μ decay process. Therefore the π - e decay will compete with π - μ decay and this contradicts with the experiment.

In order to overcome this difficulty, we must assign some properties to the μ -meson, the excited state of a neutrino, just as the assignment of some properties to a V^0 -particle in order to avoid the two particle decay scheme of a V^0 -particle. Thus because of some differences in properties between an excited state and a ground state of a neutrino, it will be the case that the π - e decay is forbidden or nearly forbidden, making the π - μ decay rate far larger than the π - e decay rate. It is here remarkable that the same selection rule considerations are very successful in both cases of excited states. That the π - μ decay occurs with the life time of the order of 10^{-8} sec is quite reasonable because the excited state of a neutrino will interact with the π -meson field at least as strong as the ground state does.

In this last paragraph we discuss a little about the mechanism which permits the existence of such an excited state of an elementary particle. The present author

considers that the present strong coupling theoretical considerations would not be the correct picture. Rather it seems that *we must consider the quantization of an elementary particle itself which produces some kind of boson field around it.* In order to solve this quantization problem it seems essential to have a deep going understanding of the nature of elementary particles interacting with their surrounding fields.

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W. B. Fretter, Phys. Rev. **83** (1951), 1053.
G. Cocconi and A. Silberman, Phys. Rev. **84** (1951), 1062.
- 2) R. Sagane, W. L. Gardner and H. W. Hubbard, Phys. Rev. **82** (1951), 557.
- 3) J. Tiomno, J. A. Wheeler and R. R. Rau, Rev. Mod. Phys. **21** (1949), 144.
- 4) G. E. Uhlenbeck and C. S. Wang Chang, Phys. Rev. **85** (1952), 684.
- 5) E. J. Konopinski, Rev. Mod. Phys. **15** (1945), 209.
D. R. Hamilton, Phys. Rev. **71** (1947), 545.
C. W. Sherwin, Phys. Rev. **73** (1948), 1174.
J. C. Jacobsen and O. Kofoed-Hansen, Phys. Rev. **73** (1948), 657.

ever, as was recently pointed out by Dyson,¹⁾ this problem seems to be not so simple as ever considered.

Though it seems to be impossible to deduce the final decision of this problem at the present stage of the theory, we think that we can at least offer some materials which somehow support to his opinion.

As the estimation taken in this letter is rather rough, and the physical meaning of approximations used in it is not so clear, it is doubtful whether the essential natures of the solution of quantum electrodynamics are represented sufficiently. But we believe that it would by some means indicate the leality. In this consideration, we neglect all effects of vacuum polarization and assume that each term in each order has a finite contribution after the subtraction treatments and then we regard the contribution of each diagram in the each order to be equal. Using these assumptions, we can find the solution of quantum electrodynamics formally as the function of the expansion parameter e^2 .

According to Feynman, the Green function of one electron interacting with the electromagnetic field is given by

$$K_+ = i \int_0^\infty \exp \left[-is \int_0^1 du [\gamma_u (P_u - B_u) + m] \right. \\ \left. + \frac{ie^2}{2} s^2 \int_0^1 du \int_0^1 du' \gamma_u \gamma_{u'} D_+ (S_{uu'}^2) \right], \quad (1)$$

where $B = eA^{\text{ext}}$. By using this equation, we obtain formally

$$K_+ = i e^{i \frac{L^2}{2Mc^2}} \int_0^\infty ds e^{-\frac{Mc^2}{2} \left(s + \frac{L}{Mc^2} \right)^2}, \quad (2)$$

where

$$\left. \begin{aligned} L &= i \int_0^1 du [\gamma_u (P_u - B_u) + m] \\ \text{and} \\ M &= \frac{1}{i} \int_0^1 du \int_0^1 du' \gamma_u \gamma_{u'} D_+ (S_{uu'}^2) \end{aligned} \right\} \quad (3)$$

The equation (2) is nothing but the gaussian type integral, and then we have

Convergence Problem in Quantum Electrodynamics

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May 30, 1952

It has generally been believed that if we could reduce each order term to be finite in virtue of the appropriate renormalization procedures and if the expansion parameter would be very small, the validity of the perturbation method in quantum field theory would be partially guaranteed. How-

to make $R_e(Me^2) > 0$ to gain the finite value from this integral. In virtue of this condition, the domain of validity of this integral is divided into two half-sides in e^2 -plane according to $R_e(M) > 0$ or < 0 . If $R_e(M)$ is positive, the domain of validity does not include $e^2 < 0$, and vice versa. This domain also varies according to the magnitude and the sign of $I_m(M)$. In this discussion, if we confine ourselves to the first case $-a < \arg(M) < a$ and $0 < a < \frac{\pi}{2}$, the domain of validity would be the one indicated by Fig. 1.

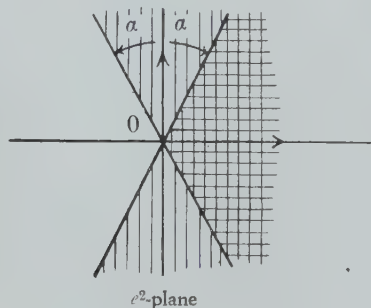


Fig. 1

The unshaded area is not valid and the doubly shaded one is absolutely valid. The shaded one is questionable.

Next, we will examine the behaviours of the function (2) in this domain. Making the suitable transformation of variable in (2), we have

$$K_+ = \frac{i}{L} \int_{+0}^{\infty} dv \left(1 + \frac{2Me^2}{L^2} v \right)^{-\frac{1}{2}} e^{-v}, \quad (4)$$

where the path of integral is taken appropriately not to round the origin. This integral is closely connected with one of the integral solutions of so-called Whittaker's function $W_{-\frac{1}{4}, \frac{1}{4}}(L^2/2Me^2)^{1/2}$ and is regular at the neighborhood of $e^2 \rightarrow \infty$.

Meanwhile, this function is irregular

at the neighborhood of $e^2=0$, and with the condition $|\arg(L^2/2Me^2)| < \pi$, we get only the asymptotic expansion

$$\frac{i}{L} \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n} n!} \left(\frac{2Me^2}{L^2} \right)^n. \quad (5)$$

That is to say, we can conclude that these series are not convergent and if we take $e^{-2}=137$, there does exist the value n at which the n -th term exceeds the foregoing $(n-1)$ -th term.

If the above procedure represent some reality, we would be able to expect that, although the perturbation method can give the approximately correct answer, there does not exist any formal justifications for it, and if that is true, we would be obliged to content ourselves with the only approximately sufficient answers.

Besides this difficulty, it is very suggestive that the expansion of above function with inverse powers of e^2 is convergent. Even if this does not indicate the validity of the strong coupling treatment immediately, we would be able to anticipate that there would exist some other correct approaches.

The discussions made in this letter are very rough, and so it is very doubtful whether the difficulty of quantum electrodynamics really exists or not. We hope that this problem will be clarified more rigorously.

In conclusion, we thank to Prof. M. Kobayasi and Asist. Prof. T. Inoue for their kind guidances and Mr. K. Sawada for his valuable discussions. One of these authors (Y.K.) also wishes to express his gratitude for the financial aids from Yukawa Fellowship of Osaka University.

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Errata

S. Emonomoto, Y. Fujimoto, S. Horie and Y. Tsuzuki, Disintegration of Light Nuclei by σ -Meson Capture (Vol. 7, No. 4, p. 353).

The scales of Figs. 1, 2 and 3 are to be read as follows :

Number of
protons

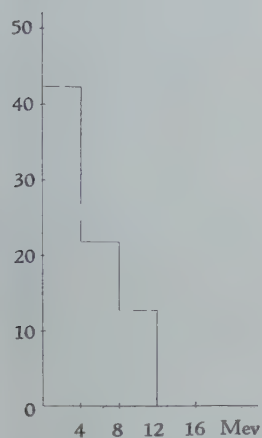


Fig. 1. Energy distribution of protons.

Number of
 α -particles

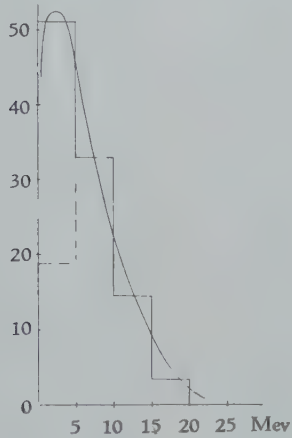


Fig. 2. Energy distribution of α -particles. The broken line indicates the result excluding "probable" α 's. The solid curve is an approximate analytic one, i. e.

$$\sqrt{E} \exp(-E/\epsilon) \text{ with } \epsilon = 4.8 \text{ Mev.}$$

Number of
heavier fragments

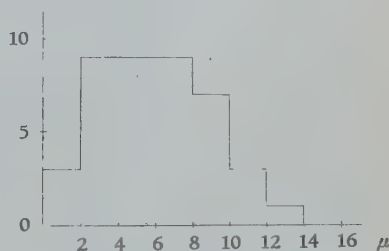


Fig. 3. Range distribution of heavier fragments.

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